Kanat K. Shakenov*, Shaharzat N. Kuttykozhaeva, Nargul A. Issabekova****

SOLUTION OF BOUNDARY PROBLEM FOR MUSKET–LEVERETT EQUATIONS BY MONTE CARLO METHODS

1. MUSKET–LEVERETT MODEL

Musket–Leverett model of a biphase filtration of incompressible liquids (ρ_i is constant) in the porous environment is characterized by system of the equations relative to a saturation $s(x, t)$ and the "reduced" pressure $p(x, t)$

$$m\frac{\partial s}{\partial t} = \operatorname{div}\left(K_0 \nabla s + K_1 \nabla p + \vec{f}_0\right) \equiv -\operatorname{div}\vec{V}_1(s, p) \tag{1}$$

$$\operatorname{div}\left(K \nabla p + \vec{f}\right) \equiv -\operatorname{div}\vec{V}(s, p) = 0 \tag{2}$$

where K_0 is filtration tensor for homogeneous liquid, $K_i = k_{0i}(s) \cdot K_0$ is symmetric tensor of the phase permeability, $K = k \cdot K_0, k = k_{01} + k_{02}, k_{0i}$ is a phase permeability for homogeneous isotropic soil,

$$\vec{f}_0 = K_1 \cdot \int_s^1 \nabla \cdot \frac{\partial p_c}{\partial s} \cdot \frac{k_{02}}{k} d\xi, \quad p_c(x, s) = p_2 - p_1,$$

$$\vec{f} = K \int_s^1 \nabla \cdot \frac{\partial p_c}{\partial s} \cdot \frac{k_{02}}{k} d\xi + K_2 \nabla p_c + K_2(\rho_2 - \rho_1)\vec{g}, \rho_i$$

is density, $\vec{g}$ is gravitational acceleration vector [1].

* Al-Farabi Kazakh National University
** Kokshetau State University named after Sh. Ualikhanov

Let's assume that filtration tensor $K_0(x)$ is symmetric and positive definite, that is $k_{0ij} = k_{0ji}, \nu \sum_i \xi_i^2 \le (K_0\xi, \xi) = \sum_{i,j} k_{0ij}\xi_i\xi_j \le \nu^{-1} \sum_i \xi_i^2, \nu > 0$. Capillary pressure and relative phase permeability have properties $\frac{\partial p_c}{\partial s} < 0, \quad k = k_{01} + k_{02} > 0$. Using properties $k_{0i}(s) > 0, k_{01}(0) = k_{02}(1) = 0$ we get $a(x,s) \equiv -\frac{\partial p_c}{\partial s} \cdot \frac{k_{01}k_{02}}{k_{01}+k_{02}} > 0$ for $s \in (0,1)$ and $a(x,0) = a(x,1) = 0$. From these assumptions and properties we get that equations (1) and (2) form quasilinear system which consist of uniformly elliptic equation for $p(x, t)$ and the parabolic equation for $s(x, t)$ degenerating for $s = 0$ and $s = 1$.

2. INITIAL BOUNDARY VALUE PROBLEM

Let us consider filtrational flow in the given finite domain Ω with the piecewise smooth border $\partial\Omega$. Let $\vec{n}$ is outer normal to $\partial\Omega, Q = \Omega \times [0,T], \partial Q_i = \partial\Omega_i \times [0,T]$. The border $\partial\Omega$ can be divided into a few connected components $\partial\Omega_i$. The border $\partial\Omega$ consist of an impenetrable surface $\partial\Omega_0$ and surface $\partial\Omega_1$ corresponding to supercharging and production wells and given borders $\partial\Omega_2 \subset \partial\Omega_1$ with a non uniform stationary liquid.

Condition of impenetrability on surface $\partial\Omega_0$ for both phases

$$\vec{V} \cdot \vec{n} = 0, \quad \vec{V}_1 \cdot \vec{n} = 0, \quad (x,t) \in \partial Q_0 \tag{3}$$

Pressure in the moistening phase coinciding with hydrostatic pressure in a stationary liquid and value of a saturation are given on sections $\partial\Omega_2 \subset \partial\Omega_1$

$$p = p_0(x,t), \quad s = s_0(x,t), \quad (x,t) \in \partial Q_2 \tag{4}$$

We assume that the flow of a mix (debit of well) on $\partial\Omega_1$ is known

$$\vec{V} \cdot \vec{n} = R(x,t), \quad (x,t) \in \partial\Omega_1 \tag{5}$$

$$\vec{V}_1 \cdot \vec{n} = b \cdot R(x,t), \quad (x,t) \in \partial\Omega_1 \tag{6}$$

At last, it is enough to give initial condition only for a saturation

$$s(x,0) = s_0(x,0), \quad x \in \Omega \tag{7}$$

Let's consider a case when $\partial\Omega \equiv \partial\Omega_1$ or $\partial\Omega \equiv \partial\Omega_2$. If $\partial\Omega \equiv \partial\Omega_1$, using the law of conservation of mass for mix in a domain Ω we obtain the necessary condition

$$\int_\Omega p(x,t)dx = \int_{\partial\Omega} R(x,t) = 0, \quad t \in [0,T] \tag{8}$$

Let's assume that summary velocity of filtration does not depend on a saturation, that is coefficients $K = K_0(x) \cdot k(s)$ and $\vec{f}(x,s)$ of the equation (2) do not depend on s. Then the system of equations (1) and (2) disintegrates and assumes sequential definition of velocity field $\vec{V}$ and phase saturation $s_i(x,t)$.

These conditions in terms of functional parameters of Musket–Leverett model look like

1) $k = k_{01}(s) + k_{02}(s) = \text{const.}$

2) $\frac{1}{\bar{m}(x)} \det K_0(x) = \text{const.} \quad \Rightarrow \quad p_c = p_c(s) \quad \Rightarrow \frac{\partial p_c}{\partial x_i} = 0.$

3) Gravities are not taken into account or liquids have identical density $\rho_1 = \rho_2$.

Realization of the condition $\frac{\partial \vec{f}}{\partial s} = 0$ follows from assumption 2) and 3) [1].

We shall solve the problem (1)–(7) after digitization only with respect to t and with above described assumptions by algorithms of Monte Carlo methods.

3. MONTE CARLO METHODS

The equation (2) can be written as

$$\operatorname{div}\left(K(x)\nabla p(x,t) + \vec{f}(x,t)\right) = 0 \quad \text{in} \quad Q \tag{9}$$

Boundary conditions:

$$\left(K(x)\nabla p(x,t) + \vec{f}(x,t)\right) \cdot \vec{n} = 0, \quad (x,t) \in \partial Q_0 \tag{10}$$

$$p = p_0(x,t), \quad (x,t) \in \partial Q_1 \tag{11}$$

$$\left(K(x)\nabla p(x,t) + \vec{f}\right) \cdot \vec{n} = R(x,t), \quad (x,t) \in \partial Q_1 \tag{12}$$

The equation (9) can by written in the form

$$\Delta p(x,t) + \sum_{i=1}^{3} B_i(x) \frac{\partial p(x,t)}{\partial x_i} = F(x,t) \quad \text{in} \quad Q \tag{13}$$

Boundary conditions (10) and (12) are a boundary conditions of Neumann type, the boundary condition (11) is a boundary condition of Dirichlet type. The problem (13), (10)–(12) is solved by algorithms “random walk by spheres” and “random walk by lattices” of Monte Carlo methods. [2, 3, 4].

$\nabla p(x,t) = \left(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3} \right)$ is estimated by Monte Carlo methods [5]. We shall substitute ∇p in (1) and obtain equation for definition of $s(x,t)$

$$m\frac{\partial s(x,t)}{\partial t} = \mathrm{div}\left(K_0(x)\nabla s(x,t) + K_1(x)\nabla p(x,t) + \vec{f}_0(x,t)\right) \quad \text{in} \quad Q \tag{14}$$

Initial condition (7) and boundary conditions (3), (4) and (6) can by written

$$s(x,0) = s_0(x,0), \quad x \in \Omega \tag{15}$$

$$\mathrm{div}\left(K_0(x)\nabla s(x,t) + K_1(x)\nabla p(x,t) + \vec{f}_0(x,t)\right)\cdot \vec{n} = 0, \quad (x,t) \in \partial Q_0 \tag{16}$$

$$s = s_0(x,t), \quad (x,t) \in \partial Q_1 \tag{17}$$

$$\mathrm{div}\left(K_0(x)\nabla s(x,t) + K_1(x)\nabla p(x,t) + \vec{f}_0(x,t)\right)\cdot \vec{n} = b \cdot R(x,t), \quad (x,t) \in \partial Q_1 \tag{18}$$

The problem (14)–(18) after digitization only with respect to *t* is solved by algorithms "random walk by spheres" and "random walk by lattices" of Monte Carlo methods [2, 3, 4].

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