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Optimal Control of an Infinite Order Hyperbolic System with Multiple Time-Varying Lags

1. Introduction

Distributed parameter systems with delays can be used to describe many phenomena in the real world. As is well known, heat conduction, properties of elastic-plastic material, fluid dynamics, diffusion-reaction processes, transmission of the signals at the certain distance by using electric long lines, etc., all lie within this area. The object that we are studying (temperature, displacement, concentration, velocity, etc.) is usually referred to as the state.

We are interested in the case where the state satisfies proper differential equations that are derived from basic physical laws, such as Newton's law, Fourier's law etc. The space in which the state exists is called the state space, and the equation that the state satisfies is called the state equation. In particular, we are interested in the cases where the state equations are one of the following types: partial differential equations, integro-differential equations, or abstract evolution equations.

Equations with deviating arguments appeared in the Euler's works. However , systematic research of such equations began only in the 20th century, as a result of the development of applied sciences and particularly automatic control theory. Consequently, equations with deviating arguments are a well-known mathematical tool for representing many physical problems . Historically, they have achieved great popularity among mathematicians, physists and engineers.

During the recent twenty years, equations with deviating arguments have been applied not only in applied mathematics, physics and automatic control, but also in some problems of economy and biology. Currently, the theory of the equations with deviating arguments, constitutes a very important subfield of mathematical control theory.

Consequently, the equations with deviating arguments are widely applied in the optimal control problems of distributed parameter systems with time delays.

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Various optimization problems associated with the optimal control of distributed parameter systems with time delays appearing in the boundary conditions have been studied recently by Wang (1975), Knowles (1978), Kowalewski (1987, 1993, 1995, 1998) and El-Saify (2005, 2006).

In Wang (1975), optimal control problems for second order parabolic systems with the Neumann boundary conditions involving constant time delays were considered. Such systems constitute in a linear approximation, a universal mathematical model for many diffusion processes in which time-delayed feedback signals are introduced at the boundary of a system spatial domain. For example in the area of plasma control, it is of interest to confine a plasma in a given bounded spatial domain Ω by introducing a finite electric potential barrier or "magnetic mirror" surrounding Ω. For a collision-dominated plasma (Kowalewski and Duda 1992), its particle density is describable by second order parabolic equation.

Due to particle inertia and finiteness of electric potential barrier or the magnetic mirror field strength, the particle reflection at the domain boundary is not instantaneous. Consequently, the particle flux at the boundary of Ω at any time depends on the flux of particles which escaped earlier and reflected back into Ω at a later time. This leads to the Neumann boundary conditions involving time delays. Necessary and sufficient conditions which the optimal controls must satisfy were derived. Estimates and a sufficient condition for the boundedness of solutions were obtained for second order parabolic systems with specified forms of feedback controls.

Subsequently, in Knowles (1978), the time-optimal control problems of linear second order parabolic systems with the Neumann boundary conditions involving constant time delays were considered. Using the results of Wang (1975), the existence of a unique solution of such parabolic systems were discussed. A characterization of the optimal control in terms of the adjoint system is given. This characterization was used to derive specific properties of the optimal control (bang-bangness, uniqueness, etc.). These results were also extended to certain cases of nonlinear control without convexity and to certain fixed time problems.

Consequently, in Kowalewski (1987, 1993, 1995) linear quadratic problems for second order hyperbolic systems with time delays given in the different form (constant time delays, time-varying delays, etc.) were presented.

Finally, in El-Saify (2005, 2006) linear quadratic optimal distributed and boundary control problems for $n \times n$ second order and $n \times n$ infinite order parabolic time-varying lag systems were considered.

In particular, in Kowalewski (1998) the optimal distributed control problem for second order hyperbolic system with multiple time-varying time delays appearing both in the state equation and in the Neumann boundary condition was considered. The presented optimal distributed control problem can be generalized onto the case of an infinite order time delay hyperbolic system.

For this reason in this paper, we consider an optimal distributed control problem for a linear infinite order hyperbolic system in which different multiple time-varying lags appear both in the state equation and in the boundary condition.

Sufficient conditions for the existence of a unique solution of such hyperbolic equation with the Neumann boundary condition are proved. The performance functional has the quadratic form. The time horizon is fixed.

Finally, we impose some constraints on the distributed control. Necessary and sufficient conditions of optimality with the quadratic performance functional and constrained control are derived for the Neumann problem.

2. Preliminaries

Let Ω be a bounded open set of R^n with smooth boundary Γ.

We define the infinite order Sobolev space $H^\infty\{a_\alpha,2\}(\Omega)$ of functions $\Phi(x)$ defined on Ω (Dubinskii 1975, 1976) as follows

$$H^\infty\{a_\alpha,2\}(\Omega) = \left\{\Phi(x)\in C^\infty(\Omega) : \sum_{|\alpha|=0}^{\infty} a_\alpha \parallel \mathscr{D}^\alpha\Phi \parallel_2^2 < \infty\right\} \tag{1}$$

where: $C^\infty(\Omega)$ is a space of infinite differentiable functions, $a_\alpha \geq 0$ is a numerical sequence and $\|\cdot\|_2$ is a norm in the space $L^2(\Omega)$, and

$$\mathscr{D}^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}\dots(\partial x_n)^{\alpha_n}}\ , \tag{2}$$

where: $\alpha = (\alpha_1,\dots,\alpha_n)$ is a multi-index for differentiation, $|\alpha| = \sum_{i=1}^{n}\alpha_i$.

The space $H^{-\infty}\{a_\alpha,2\}(\Omega)$ (Dubinskii 1975, 1976) is defined as the formal conjugate space to the space $H^\infty\{a_\alpha,2\}(\Omega)$, namely:

$$H^{-\infty}\{a_\alpha,2\}(\Omega) = \left\{\Psi(x) : \Psi(x) = \sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha\mathscr{D}^\alpha\Psi_\alpha(x)\right\} \tag{3}$$

where: $\Psi_\alpha \in L^2(\Omega)$ and $\sum_{|\alpha|=0}^{\infty} a_\alpha \parallel \Psi_\alpha \parallel_2^2 < \infty$.

The duality pairing of the spaces $H^\infty\{a_\alpha,2\}(\Omega)$ and $H^{-\infty}\{a_\alpha,2\}(\Omega)$ is postulated by the formula

$$\langle\Phi,\Psi\rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega \Psi_\alpha(x)\mathscr{D}^\alpha\Phi(x)\,dx, \tag{4}$$

where: $\Phi \in H^\infty\{a_\alpha,2\}(\Omega)$, $\Psi \in H^{-\infty}\{a_\alpha,2\}(\Omega)$.

From above, $H^\infty\{a_\alpha,2\}(\Omega)$ is everywhere dense in $L^2(\Omega)$ with topological inclusions and $H^{-\infty}\{a_\alpha,2\}(\Omega)$ denotes the topological dual space with respect to $L^2(\Omega)$ so we have the following chain:

$$H^{\infty}\{a_\alpha,2\}(\Omega) \subseteq L^2(\Omega) \subseteq H^{-\infty}\{a_\alpha,2\}(\Omega)\ . \tag{5}$$

3. The infinite order hyperbolic equation

Let us consider the following linear infinite order hyperbolic equation

$$\frac{\partial^2 y}{\partial t^2} + Ay = u \qquad (x,t) \in \Omega \times (0,T) \tag{6}$$

$$y(x,0) = y_0(x) \qquad x \in \Omega \tag{7}$$

$$y'(x,0) = y_I(x) \qquad x \in \Omega \tag{8}$$

$$\frac{\partial y}{\partial \eta_A}(x,t) = q \qquad (x,t) \in \Gamma \times (0,T) \tag{9}$$

where : Ω has the same properties as in the Section 2.

$$Q = \Omega \times (0,T), \qquad \bar{Q} = \bar{\Omega} \times [0,T], \qquad \Sigma = \Gamma \times (0,T)$$

The operator $\frac{\partial^2}{\partial t^2} + A$ in the state equation (6) is an infinite order hyperbolic operator and A (Dubinskii (1986) and El-Saify and Bahaa (2002)) is given by

$$Ay = \left(\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \mathscr{D}^{2\alpha} + 1 \right) y \tag{10}$$

and

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \mathscr{D}^{2\alpha} \tag{11}$$

is an infinite order elliptic partial differential operator.

The operator A is a mapping of $H^{\infty}\{a_\alpha,2\}$ onto $H^{-\infty}\{a_\alpha,2\}$. For this operator the bilinear form $\Pi(t;y,\varphi) = (Ay,\varphi)_{L^2(\Omega)}$ is coercive on $H^{\infty}\{a_\alpha,2\}$ i.e. there exists $\lambda > 0, \lambda \in I\!R$ such that $\Pi(t;y,\varphi) \geq \lambda \parallel y \parallel^2_{H^{\infty}\{a_\alpha,2\}}$. Moreover, we assume that $\forall\ y,\varphi \in H^{\infty}\{a_\alpha,2\}$ the function $t \to \Pi(t;y,\varphi)$ is continuously differentiable in $[0,T]$ and $\Pi(t;y,\varphi) = \Pi(t;\varphi,y)$.

The equations (6)–(9) constitute a Neumann problem. The left-hand side of (9) is written in the following form

$$\frac{\partial y}{\partial \eta_A} = \sum_{|w|=0}^{\infty} (\mathscr{D}^w y(u)) \cos(n,x_i) = q(x,t) \qquad x \in \Gamma,\ t \in (0,T) \tag{12}$$

where: $\dfrac{\partial y}{\partial \eta_A}$ is a normal derivative at Γ, directed towards the exterior of Ω, $\cos(n,x_i)$ is an i-th direction cosine of n, with n being the normal at Γ exterior to Ω.

We shall formulate sufficient conditions for the existence of a unique solution of the mixed initial-boundary value problem (6)–(9) for the cases where the function u is a element of the space $H^{0,1}(Q)$ (i.e. $u \in L^2(0,T;H^0(\Omega)) = L^2(Q)$ and $u' = \frac{\partial u}{\partial t} \in L^2(0,T;H^0(\Omega))$).

For this purpose for $r = \infty$ and $s = 2$, we introduce the Sobolev space $H^{\infty,2}(Q)$ (Lions and Magenes 1972, Vol. 2, p.6) defined by

$$\left.\begin{array}{l} H^{\infty,2}(Q) = H^0(0,T;H^\infty\{a_\alpha,2\}(\Omega)) \cap H^2(0,T;H^0(\Omega)) \\ \text{which is a Hilbert space normed by} \\ \left(\displaystyle\int_0^T \| y(t) \|^2_{H^\infty\{a_\alpha,2\}(\Omega)}\, dt + \| y \|^2_{H^2(0,T;H^0(\Omega))} \right)^{1/2} \end{array}\right\} \tag{13}$$

where: $H^2(0,T;H^0(\Omega))$ denotes the Sobolev space of second order of functions defined on $(0,T)$ and taking values in $H^0(\Omega)$.

Consequently, the starting point for our considerations will be the following theorems about the existence of a unique solution for the Neumann problem (6)–(9) which can be found in Lions and Magenes (1972, Vol. 2, p. 103).

Theorem 1. *Let y_0, y_I, q and u be given with $y_0 \in H^\infty\{a_\alpha,2\}(\Omega)$, $y_I \in H^\infty\{a_\alpha,2\}(\Omega)$, $q \in H^{\infty,3}(\Sigma)$, $u \in H^{0,1}(Q)$ and the following compatibility relations*

$$\frac{\partial w_0}{\partial \eta_A}(x,0) = q(x,0) \qquad \text{on } \Gamma \tag{14}$$

$$\frac{\partial w_I}{\partial \eta_A}(x,0) + \left(\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \eta_A}\right)\right) w_0(x,0) = \frac{\partial}{\partial t} q(x,0) \qquad \text{on } \Gamma \tag{15}$$

are fulfilled.

Moreover, the function w (Proposition 3.1 of Lions and Magenes (1972, Vol.2, p.100)) has the following properties:

$$w \in L^2(0,T;H^\infty\{a_\alpha,2\}(\Omega)), \quad w' \in L^2(0,T;H^\infty\{a_\alpha,2\}(\Omega)), \quad w''' \in L^2(0,T;H^0(\Omega)) \tag{16}$$

with $w(x,0) = w_0 \in H^\infty\{a_\alpha,2\}(\Omega)$ and $w'(x,0) = w_I \in H^\infty\{a_\alpha,2\}(\Omega)$ (Proposition 3.2 of Lions and Magenes (1972, Vol.2, p.100)). Then, there exists a unique solution $y \in H^{\infty,2}(Q)$ for the mixed initial-boundary value problem (6)–(9).

4. The infinite order hyperbolic delay equation

Consider now the distributed-parameter system described by the following infinite order hyperbolic delay equation

$$\frac{\partial^2 y}{\partial t^2}+Ay+\sum_{i=1}^{m} b_i(x,t)y(x,t-h_i(t))=u, \qquad x\in\Omega,\ t\in(0,T) \tag{17}$$

$$y(x,t')=\Phi_0(x,t') \qquad x\in\Omega,\ t'\in[-\Delta(0),0) \tag{18}$$

$$y(x,0)=y_0(x) \qquad x\in\Omega \tag{19}$$

$$y'(x,0)=y_I(x) \qquad x\in\Omega \tag{20}$$

$$\frac{\partial y}{\partial \eta_A}=\sum_{s=1}^{l} c_s(x,t)y(x,t-k_s(t))+v \qquad x\in\Gamma,\ t\in(0,T) \tag{21}$$

$$y(x,t')=\Psi_0(x,t') \qquad x\in\Gamma,\ t'\in[-\Delta(0),0) \tag{22}$$

where: Ω has the same properties as in the Section 2.

$$y\equiv y(x,t;u), \qquad u\equiv u(x,t), \qquad v\equiv v(x,t)$$

$$Q=\Omega\times(0,T), \quad \bar{Q}=\bar{\Omega}\times[0,T], \quad Q_0=\Omega\times[-\Delta(0),0),$$

$$\Sigma=\Gamma\times(0,T), \quad \Sigma_0=\Gamma\times[-\Delta(0),0)$$

b_i are given real C^∞ functions defined on $\bar{Q}$,
c_s are given real C^∞ functions defined on Σ ,
$h_i(t)$ and $k_s(t)$ are functions representing multiple time-varying lags,
Φ_0, Ψ_0 are initial functions defined on Q_0 and Σ_0 respectively.

Moreover,

$$\Delta(0)=max\{h_1(0),h_2(0),...,h_m(0),k_1(0),k_2(0),...,k_l(0)\} \tag{23}$$

The operator A is given by (10)
It is easy to notice that equations (17)–(22) constitute the Neumann problem. The left-hand side of the Neumann boundary condition (21) may be written in the following form

$$\frac{\partial y}{\partial \eta_A}=q(x,t) \qquad x\in\Gamma,\ t\in(0,T) \tag{24}$$

where:

$$q(x,t)=\sum_{s=1}^{l} c_s(x,t)y(x,t-k_s(t))+v(x,t) \tag{25}$$

Let $t-h_i(t)$ and $t-k_s(t)$ be strictly increasing functions, $h_i(t)$ and $k_s(t)$ being non-negative in $[0,T]$ and also being C^1 functions. Then, there exist the inverse functions of $t-h_i(t)$ and $t-k_s(t)$ respectively.

Let us denote $r_i(t) \overset{df}{=} t-h_i(t)$ and $\lambda_s(t) \overset{df}{=} t-k_s(t)$, then the inverse functions of $r_i(t)$ and $\lambda_s(t)$ have the following forms $t=f_i(r_i)=r_i+s_i(r_i)$ and $t=\varepsilon_s(r_s)=r_s+q_s(r_s)$, where $s_i(r_i)$ and $q_s(r_s)$ are time-varying predictions. Let $f_i(t)$ and $\varepsilon_s(t)$ be the inverse functions of $t-h_i(t)$ and $t-k_s(t)$ respectively.

Thus, we define the following iterations:

$$\left.\begin{array}{l}
\hat{t}_o = 0 \\
\hat{t}_1 = \min\{f_1(0), f_2(0), ..., f_m(0), \varepsilon_1(0), \varepsilon_2(0), ..., \varepsilon_l(0)\} \\
\hat{t}_2 = \min\{f_1(\hat{t}_1), f_2(\hat{t}_1), ..., f_m(\hat{t}_1), \varepsilon_1(\hat{t}_1), \varepsilon_2(\hat{t}_1), ..., \varepsilon_l(\hat{t}_1)\} \\
\vdots \\
\hat{t}_j = \min\left\{f_1(\hat{t}_{j-1}), f_2(\hat{t}_{j-1}), ..., f_m(\hat{t}_{j-1}), \varepsilon_1(\hat{t}_{j-1}), \varepsilon_2(\hat{t}_{j-1}), ..., \varepsilon_l(\hat{t}_{j-1})\right\}
\end{array}\right\} \tag{26}$$

First we shall prove the existence of a unique solution of the mixed initial-boundary value problem (17)–(22). We shall consider the case where the distributed control u belongs to $H^{0,1}(Q)$.

The existence of a unique solution for the mixed initial-boundary value problem (17)–(22) on the cylinder Q can be proved using a constructive method, i.e. solving at first equations (17)–(22) on the subcylinder Q_1 and in turn on Q_2 , etc. until the procedure covers the whole cylinder Q. In this way, the solution in the previous step determines the next one.

For simplicity, we introduce the following notations:

$$\begin{array}{lll}
E_j \overset{\triangle}{=} (\hat{t}_{j-1}, \hat{t}_j), & Q_j = \Omega \times E_j, & Q_0 = \Omega \times [-\Delta(0), 0) \\
\Sigma_j = \Gamma \times E_j, & \Sigma_o = \Gamma \times [-\Delta(0), 0) & \text{for } j = 1, \ldots .
\end{array} \tag{27}$$

Using the Theorem 1, the following lemma can be proved.

Lemma 1. *Let*

$$u \in H^{0,1}(Q) \tag{28}$$

$$l_j \in H^{0,1}(Q_j) \tag{29}$$

where

$$l_j(x,t) = u(x,t) - \sum_{i=1}^{m} b_i(x,t) y_{j-1}(x, t-h_i(t))$$

$$q_j \in H^{\infty,3}\left(\sum\nolimits_j\right) \tag{30}$$

where

$$q_j(x,t) = \sum_{s=1}^{l} c_s(x,t) y_{j-1}(x, t - k_s(t)) + v(x,t)$$

$$w_{j-1}(\cdot, \hat{t}_{j-1}) = y_{j-1}(\cdot, \hat{t}_{j-1}) \in H^{\infty}\{a_\alpha, 2\}(\Omega) \tag{31}$$

$$w'_{j-1}(\cdot, \hat{t}_{j-1}) = y'_{j-1}(\cdot, \hat{t}_{j-1}) \in H^{\infty}\{a_\alpha, 2\}(\Omega) \tag{32}$$

and the following compatibility relations are fulfilled

$$\frac{\partial w_{j-1}}{\partial \eta_A}(x, \hat{t}_{j-1}) = q_j(x, \hat{t}_{j-1}) \qquad \text{on } \Gamma \tag{33}$$

$$\frac{\partial w'_{j-1}}{\partial \eta_A}(x, \hat{t}_{j-1}) + \left(\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \eta_A}\right)\right) w_{j-1}(x, \hat{t}_{j-1}) = \frac{\partial}{\partial t} q_j(x, \hat{t}_{j-1}) \qquad \text{on } \Gamma \tag{34}$$

Then, there exists a unique solution $y_j \in H^{\infty,2}(Q_j)$ for the mixed initial-boundary value problem (17), (21), (31), (32). ■

Proof: For $j = 1, \sum_{i=1}^{m} y_{j-1}\big|_{Q_0}(x, t - h_i(t)) = \sum_{i=1}^{m} \Phi_0(x, t - h_i(t))$ and $\sum_{s=1}^{l} y_{j-1}\big|_{\Sigma_0}(x, t - k_s(t)) = \sum_{s=1}^{l} \Psi_0(x, t - k_s(t))$ respectively. Then the assumptions (29), (30), (31) and (32) are fulfilled if we assume that $\Phi_0 \in H^{\infty,2}(Q_0)$, $v \in H^{\infty,3}(\Sigma)$ and $\Psi_0 \in H^{\infty,3}(\Sigma_0)$. These assumptions are sufficient to ensure the existence of a unique solution $y_l \in H^{\infty,2}(Q_1)$ if $y_0 \in H^{\infty}\{a_\alpha, 2\}(\Omega)$, $y_I \in H^{\infty}\{a_\alpha, 2\}(\Omega)$ and the following compatibility conditions are satisfied:

$$\frac{\partial w_0}{\partial \eta_A}(x, 0) = q_1(x, 0) \qquad \text{on } \Gamma \tag{35}$$

$$\frac{\partial w_I}{\partial \eta_A}(x, 0) + \left(\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \eta_A}\right)\right) w_0(x, 0) = \frac{\partial}{\partial t} q_1(x, 0) \qquad \text{on } \Gamma \tag{36}$$

In order to extend the result to Q_2 it is necessary to impose the compatibility relations

$$\frac{\partial w_1}{\partial \eta_A}(x, \hat{t}_1) = q_2(x, \hat{t}_1) \qquad \text{on } \Gamma \tag{37}$$

$$\frac{\partial w'_1}{\partial \eta_A}(x, \hat{t}_1) + \left(\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \eta_A}\right)\right) w_0(x, \hat{t}_1) = \frac{\partial}{\partial t} q_2(x, \hat{t}_1) \qquad \text{on } \Gamma \tag{38}$$

and it is sufficient to verify that

$$l_2 \in H^{0,1}(Q_2) \tag{39}$$

$$w_1(\cdot, \hat{t}_1) = y_1(\cdot, \hat{t}_1) \in H^{\infty}\{a_\alpha, 2\}(\Omega) \tag{40}$$

$$w_1'(\cdot,\hat{t}_1) = y_1'(\cdot,\hat{t}_1) \in H^\infty\{a_\alpha,2\}(\Omega) \tag{41}$$

$$q_2 \in H^{\infty,3}(\Sigma_2) \tag{42}$$

First, using the solution in the previous step and the condition (28) we can prove immediately the condition (39).

To verify (40) and (41) we use the fact (by Proposition 3.1 of Lions and Magenes 1972: Vol.2, p. 100) that the function w_1 has the following properties:

$$w_1 \in L^2(E_1;H^\infty\{a_\alpha,2\}(\Omega)), \qquad w_1' \in L^2(E_1;H^\infty\{a_\alpha,2\}(\Omega)), \qquad w_1''' \in L^2(E_1;H^0(\Omega))$$

Then, from the Theorem 3.1 of Lions and Magenes (1972, Vol. 1, p. 19), it follows that the mappings $t \to w_1(\cdot,t)$ and $t \to w_1'(\cdot,t)$ are continuous from $[0,\hat{t}_1] \to H^\infty\{a_\alpha,2\}(\Omega)$ and $[0,\hat{t}_1] \to H^\infty\{a_\alpha,2\}(\Omega)$ respectively. Hence $w_1(\cdot,\hat{t}_1) \in H^\infty\{a_\alpha,2\}(\Omega)$ and $w_1'(\cdot,\hat{t}_1) \in H^\infty\{a_\alpha,2\}(\Omega)$. But, from the Section 3 of Lions and Magenes (1972: Vol.2, p. 99), it follows that $w_1(\cdot,\hat{t}_1) = y_1(\cdot,\hat{t}_1)$ and $w_1'(\cdot,\hat{t}_1) = y_1'(\cdot,\hat{t}_1)$.

From the proceding results we can deduce that $y_1(\cdot,\hat{t}_1) \in H^\infty\{a_\alpha,2\}(\Omega)$ and $y_1'(\cdot,\hat{t}_1) \in H^\infty\{a_\alpha,2\}(\Omega)$. Again, from the the Trace Theorem (Lions and Magenes 1972, Vol. 2, p. 9) $y_1 \in H^{\infty/2}(Q_1)$ implies that $y_1 \to y_1|_{\Sigma_1}$ is a linear continuous mapping of $H^{\infty,2}(Q_1) \to H^{\infty,2}(\sum_1) \subset H^{\infty,3}(\sum_1)$. Thus $y_1|_{\sum_1} \in H^{\infty,3}(\Sigma_1)$. Assuming that c_s are C^∞ functions and $v \in H^{\infty,3}(\Sigma)$, the condition (42) is fulfilled. Then, there exists a unique solution $y_2 \in H^{\infty,2}(Q_2)$. We shall now extend our result to any $Q_j, j = 3,\dots$.

Theorem 2. *Let $y_0, y_I, \Phi_0, \Psi_0, v$ and u be given with $y_0 \in H^\infty\{a_\alpha,2\}(\Omega)$, $y_I \in H^\infty\{a_\alpha,2\}(\Omega)$, $\Phi_0 \in H^{\infty,2}(Q_0)$, $\Psi_0 \in H^{\infty,3}(\Sigma_0)$, $v \in H^{\infty,3}(\sum)$, $u \in H^{0,1}(Q)$ and the compatibility relations* (35)*,* (36) *are fulfilled. Then, there exists a unique solution $y \in H^{\infty,2}(Q)$ for time delay infinite order hyperbolic equation* (17)–(22) *with $y(\cdot,\hat{t}_j) \in H^\infty\{a_\alpha,2\}(\Omega)$ and $y'(\cdot,\hat{t}_j) \in H^\infty\{a_\alpha,2\}(\Omega)$ for $j = 1,\dots$.* ∎

5. Problem formulation. Optimization Theorem

We shall formulate the optimal control problem in the context of the case where $u \in H^{0,1}(Q)$. Let us denote by $U = H^{0,1}(Q)$ the space of controls. The time horizon T is fixed in our problem. The performance functional is given by

$$I(u) = \lambda_1 \int_Q |\, y(x,t;u) - z_d \,|^2 \, dxdt + \lambda_2 \parallel u \parallel^2_{H^{0,1}(Q)} \tag{43}$$

where: $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 > 0$; z_d is a given element in $L^2(Q)$.

Using the formula (13), the second term on the right-hand side of (43) can be written as

$$\| u \|^2_{H^{0,1}(Q)} = 2\int_0^T \langle u(t), u(t)\rangle_{L^2(\Omega)} dt + \int_0^T \left\langle \frac{\partial u(t)}{\partial t}, \frac{\partial u(t)}{\partial t} \right\rangle_{L^2(\Omega)} dt = \int_Q \left[\left(2 - \frac{\partial^2}{\partial t^2} \right) u \right] u dx dt \tag{44}$$

Moreover, $u(x,0) = u(x,T) = 0, x \in \Omega$.

Finally, we assume the following constraint on controls:

$$u \in U_{ad} \text{ is a closed, convex subset of } U. \tag{45}$$

Let $y(x,t;u)$ denote the solution of (17)–(22) at (x,t) corresponding to a given control $u \in U_{ad}$. We note from the Theorem 2 that for any $u \in U_{ad}$ performance functional (43) is well-defined since $y(u) \in H^{\infty,2}(Q) \subset L^2(Q)$. The solving of the stated optimal control problem is equivalent to a seeking an $u_0 \in U_{ad}$ such that $I(u_0) \leq I(u) \;\; \forall u \in U_{ad}$.

The starting point for our considerations will be the following theorem which can be found in (Lions 1971, p. 10):

Theorem 3. *Assume that the function $u \to I(u)$ is strictly convex, differentiable such that $I(u) \to +\infty$ as $\| u \| \to +\infty$, $u \in U_{ad}$ (the last hypothesis may be omitted if U_{ad} is bounded). Then, the unique element u_0 in U_{ad} satisfying $I(u_0) = inf_{u \in U_{ad}} I(u)$ is characterized by*

$$I'(u_0)(u - u_0) \geq 0 \qquad \forall u \in U_{ad} \tag{46}$$

■

For the above control problem, from the Theorem 3, it follows that for $\lambda_2 > 0$ a unique optimal control u_0 exists; moreover u_0 is characterized by the condition (46).

For the performance functional of form (44), the relation (46) can be expressed as

$$\lambda_1 \int_Q (y(u_0) - z_d)(y(u) - y(u_0)) dx dt + \lambda_2 \langle u_0, u - u_0 \rangle_{H^{0,1}(Q)} \geq 0 \qquad \forall u \in U_{ad} \tag{47}$$

In order to simplify (47), we introduce the adjoint equation and for every $u \in U_{ad}$ we define the adjoint variable $p = p(u) = p(x,t;u)$ as the solution of the following infinite order hyperbolic equation

$$\frac{\partial^2 p(u)}{\partial t^2} + Ap(u) + \sum_{i=1}^m b_i(x, t + s_i(t)) p(x, t + s_i(t); u) \left[1 + s_i'(t)\right] = \lambda_1 (y(u) - z_d) \qquad x \in \Omega,\; t \in (0, T - \Delta(T)) \tag{48}$$

$$\frac{\partial^2 p(u)}{\partial t^2} + Ap(u) = \lambda_1(y(u) - z_d) \qquad x \in \Omega,\ t \in (T - \Delta(T), T) \tag{49}$$

$$p(x,T;u) = 0 \qquad x \in \Omega \tag{50}$$

$$p'(x,T;u) = 0 \qquad x \in \Omega \tag{51}$$

$$\frac{\partial p(u)}{\partial \eta_A}(x,t) = \sum_{s=1}^{l} c_s(x, t + q_s(t)) p(x, t + q_s(t); u) \left[1 + q_s'(t)\right] \qquad x \in \Gamma,\ t \in (0, T - \Delta(T)) \tag{52}$$

$$\frac{\partial p(u)}{\partial \eta_A}(x,t) = 0 \qquad x \in \Gamma,\ t \in (T - \Delta(T), T) \tag{53}$$

where

$$\begin{cases} \Delta(T) = max\{h_1(T), h_2(T), ..., h_m(T), k_1(T), k_2(T), ..., k_l(T)\} \\ \dfrac{\partial p(u)}{\partial \eta_A}(x,t) = \displaystyle\sum_{|w|=0}^{\infty} (\mathscr{D}^w p(u)) \cos(n, x_i) \end{cases} \tag{54}$$

and the operator A is given by (10).

Using the Theorem 2, one may prove the following result.

Lemma 2. *Let the hypothesis of Theorem 2 be satisfied. Then, for given $z_d \in L^2(Q)$ and any $u \in H^{0,1}(Q)$, there exists a unique solution $p(u) \in H^{\infty,2}(Q)$ for the adjoint problem* (48)–(53).

Using the adjoint equation (48)–(53), we simplify the first component of the left-hand side of (47). Consequently, after transformations we get

$$\lambda_1 \int_Q (y(u_0) - z_d)(y(u) - y(u_0)) dxdt = \\ = \int_Q p(u_0)(u - u_0) dxdt \tag{55}$$

Using the formula (44) and substituting (55) into (47) gives

$$\int_Q \left[p(u_0) + \lambda_2 \left(2 - \frac{\partial^2}{\partial t^2} \right) u_0 \right] (u - u_0) dxdt \geq 0 \qquad \forall u \in U_{ad}, \tag{56}$$

Theorem 4. *For the problem* (17)–(22) *with the cost function* (43) *with $z_d \in L^2(Q)$ and $\lambda_2 > 0$ and with constraints on controls* (45)*, there exists a unique optimal control u_0 which satisfies the maximum condition* (56). ■

We must notice that the conditions of optimality derived above (Theorem 4) do not provide any analytical formula for the optimal control. Thus, we turn from the exact determination of the optimal control and we have to use approximation methods.

In the case of performance functional (43) with $\lambda_1 > 0$ and $\lambda_2 = 0$, the optimal control problem reduces to minimization of the functional on a closed and convex subset in a Hilbert space. Then the optimization problem is equivalent to a quadratic programming one which can be solved by the use of the well-known algorithms, e.g. Gilbert's (1966).

A practical application of Gilbert's algorithm to an optimal control problem for time delay parabolic system was presented in Kowalewski and Duda (1992). Using Gilbert's algorithm, a one-dimensional numerical example of the plasma control process was solved.

6. Conclusions

The results presented in the paper can be treated as a generalization of the results obtained by Kowalewski (1987, 1993, 1995, 1998) and El-Saify (2006) to the case of a distributed infinite order hyperbolic system with different multiple time-varying lags appearing both in the state equations and in the boundary conditions.

Sufficient conditions for the existence of a unique solution of such hyperbolic equation are proved – Lemma 1 and Theorem 2.

The optimal distributed control was characterized by the adjoint equation – Lemma 2. By using this characterization, necessary and sufficient conditions of optimality were proved – Theorem 4.

In this paper we have considered the optimal infinite order hyperbolic system where different multiple time-varying lags appear both in the state equation and in the Neumann boundary condition. We can also derived conditions of optimality for a more complex case of such distributed infinite order hyperbolic system with the Dirichlet boundary condition. Finally, we can consider a more complex case of optimal boundary control for a distributed infinite order hyperbolic system in which different multiple time lags appear in the state equation and in the boundary condition simultaneously.

The ideas mentioned above will be developed in forthcoming papers.

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References

[1] Dubinskii J.A. (1975). Sobolev spaces of infinite order and behavior of solution of some boundary value problems with unbounded increase of the order of the equation. *Matiematiczeskii Sbornik* **98**: 163–184.

[2] Dubinskii J.A. (1976). Non-trivality of Sobolev spaces of infinite order for a full euclidean space and a tour's. *Matiematiczeskii Sbornik* **100**: 436–446.

[3] Dubinskii J.A. (1986). *Sobolev Spaces of Infinite Order and Differential Equations*. Teubner-Texte zur Mathematik **87**, Teubner-Verlag, Leipzig.

[4] El-Saify H.A., Bahaa G.M. (2002). Optimal control for $n \times n$ hyperbolic systems involving operators of infinite order. *Mathematica Slovaca* **52**: 409–422.

[5] El-Saify H.A. (2005). Optimal control of $n \times n$ parabolic system involving time lag. *IMA Journal of Mathematical Control and Information* **22**: 240–250.

[6] El-Saify H.A. (2006). Optimal boundary control problem for $n \times n$ infinite order parabolic lag system. *IMA Journal of Mathematical Control and Information* **23**: 433–445.

[7] Gilbert E.S. (1966). An iterative procedure for computing the minimum of a quadratic form on a convex set. *SIAM Journal of Control* **4**: 61–80.

[8] Knowles G. (1978). Time optimal control of parabolic systems with boundary conditions involving time delays. *Journal of Optimization Theory and Applications* **25**: 563–574.

[9] Kowalewski A. (1987). Optimal control of hyperbolic system with boundary condition involving a time-varying lag. *Proceedings of IMACS/IFAC International Symposium of DPS, Hiroshima, Japan*, pp. 461–467.

[10] Kowalewski A. (1988). Boundary control of distributed parabolic system with boundary condition involving a time-varying lag. *International Journal of Control* **48**: 2233–2248.

[11] Kowalewski A. (1993). Optimal control of hyperbolic system with time lags. *Applied Mathematics and Computer Science* **3**: 687–697.

[12] Kowalewski A. (1995). Optimal control of a hyperbolic system with time-varying lags. *IMA Journal of Mathematical Control and Information* **12**: 261–272.

[13] Kowalewski A. (1998). Optimal control of a distributed hyperbolic system with multiple time-varying lags. *International Journal of Control* **71**: 419–435.

[14] Kowalewski A., Duda J. (1992). On some optimal control problem for a parabolic system with boundary condition involving a time-varying lag. *IMA Journal of Mathematical Control and Information* **9**: 131–146.

[15] Lions J. (1971). *Optimal Control of Systems Governed by Partial Differential Equations*. Springer-Verlag, Berlin-Heidelberg.

[16] Lions J., Magenes E. (1972). *Non-Homogeneous Boundary Value Problems and Applications, Vols.* **1** *and* **2**. Springer-Verlag, Berlin-Heidelberg.

[17] Wang P.K.C. (1975). Optimal control of parabolic systems with boundary conditions involving time delays. *SIAM Journal of Control* **13**: 274–293.