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An Algorithm for Uniform Scanning of Coating Buildings Modeled with Spline Functions

1. Introduction

This paper presents an automatic measurement method which will facilitate the creation of surfaces using spline functions. Current methods of constructing spline surfaces based on rectangular lattices were presented in [3, 5, 8]. These require the input of relatively evenly spaced points, which is not an easy task in the case of measurements of buildings which do not have any characteristic details. Insightful analyses of the errors which occur due to distortion of the uniformity of the distribution of points can be found in [5, 6]. It suffices to note here that there will be a detrimental impact on the creation of curves when the distances between consecutive points differ by about twofold, whereas for creating surfaces smaller differences in distance (about one and a half-fold) already cause detrimental undulations of the function. The undulations also depend strongly on the geometry of the object itself, and therefore these figures are only indicative.

The problem can be solved in one of two ways. The more promising option is connected to the elaboration of a method for creating spline surfaces using triangular lattices. Theoretically, such lattices would permit the modeling of objects measured with less regularity. Current research related to their development, yield moderate results. Therefore, the appearance of effective and widely available software using these algorithms should not be expected within the next couple of years.

The second approach consists in automatic measurement which would provide a sufficiently regular data set. Instruments used for these tasks include laser scanners, reflectorless automatic total stations and scanning total stations. Regardless of the type of instrument used, the technique for distributing the points on the object is an essential aspect. The measurements are usually made by creating

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a regular vertical point lattice with set distances between rows and columns. After projecting it on the spatial, non-planar and non-vertical object, we receive an irregular set of observations. Deformations will be greater, the greater the differences are in the shape of the building in the normal direction to the point lattice. For example, projecting a flat lattice on an object with a spherical geometry will create observed points on its edges which are several times more sparsely distributed then in its center. This is not a problem when using laser scanners. In this case the set of points is very dense and a sufficiently large and regular number of representatives of the object can be selected to be used for creating the spline surface. The situation becomes more complicated when automatic total stations are used, in which case the point lattice is usually not very dense due to the slow scanning speed. Furthermore, limiting this set of points to only the set of regularly spaced observations carries the risk of losing information about the object's characteristic details.

Considering the current methods for creating spline surfaces as well as the measuring techniques, often connected with usage of automatic total stations due to financial restrictions, an algorithm has been prepared for the uniform spacing of observations on measured objects depending on their shape. Taking into account that the majority of buildings approximated with spline surfaces are shells in the form of quadrics, stress has been put on developing an observation method for such surfaces.

The method presented here has a second and equally important application related to automatic dedensifying of data sets obtained through laser scanning. The analysis of spline surfaces created using a very large set of points is a task which often exceeds the capabilities of even modern, multi-processor computers. It is therefore desirable to reduce the full set to a limited quantity (several thousand) of uniformly spaced observations which are sufficiently representative of the shape of the object. By using a slightly modified uniform measurement method, it is possible, through comparison, to select a set of observations with the desired distribution from the entire cloud of points.

2. Algorithm for the Uniform Spacing of Measurement Points on Quadrics

The main advantage of spline surfaces is that they provide a good representation of the shape of any non-linear objects used in technology and construction (especially quadrics). The domain of spline applications is the precise determination of any local deformities of such objects. The quality of the approximation for these objects is strongly dependent on the uniformity of the point distribution obtained from the measurement. Making use of the fact that scanning will pertain to surfaces which are easy to recognize and describe mathematically (such as spheres, cylinders, hyperboloids, etc.) an algorithm can be proposed for an automatic and uniform spacing of points on their frame.

Preliminary to measuring the shell, the reference network must be stabilized as well as measured and adjusted in order to orient the observation stations relative to each other and to the object. A scan of a selected group of points will be made from each one of these positions thus creating a uniform cloud of observations covering the entire building.

The second stage is linked to the settinng of quadric's general equation (1), from which the points to be observed will be determined

$$F(x_i, y_i, z_i) = \\ = a_{11}x_i^2 + a_{22}y_i^2 + a_{33}z_i^2 + 2a_{12}x_iy_i + 2a_{13}x_iz_i + 2a_{23}x_iz_i + 2a_{41}x_i + 2a_{42}y_i + 2a_{43}z_i + a_{44} = 0 \qquad (1)$$

To achieve this, a certain group of points should initially be measured in order to approximate the surface. The least-squares method (LSM) approximation can be made using one of two approaches: by minimizing geometric distances or algebraic distances [1, 7]. The minimization of the geometric distances d_i is based on the spatial distance of a point to the given surface equation. It has a clear geometric interpretation and gives good approximate results. However, it leads to a system of non-linear equations which specify the parameters of the quadric and which must be solved by an iteration method, such as the Gauss–Newton method [2]

$$(J^T J)\Delta A = -J^T D \qquad (2)$$

where:

$D = [d_i \quad d_2 \quad \dots \quad d_N]^T - d_i$ – geometric distances vector,

$J = \frac{\partial D}{\partial \mathrm{A}}$ – Jacobian matrix,

$A = [a_{ij} \quad a_{ik} \quad \dots \quad a_{NM}]$ – the vector of the desired quadric parameters (1),

$\Delta A = A_i - A_{i-1}, A_{i-1}$ – vector of the solutions from the previous iteration.

The method requires that relatively precise starting parameters A_0 be provided. This is why the Levenberg–Marquardt [2] method is usually used which combines the Gauss–Newton method with the gradient descent method which has a broad range of A_0 determination. Geometrical distance minimization requires significant processing power and can therefore hardly be recommended for the relatively simple computers of automatic total stations.

Because determining a uniform distribution of points does not require a very precise determination of the quadric, the simpler method of the minimization of algebraic distances l_i can be used for this purpose. It is based on the assumption that if a point is located exactly on the surface then this distance is zero. Otherwise $l_i = F(x_i, y_i, z_i) \neq 0$ is calculated. Next the squares of these distances are minimized

$$\sum (l_i)^2 = \sum (F(x_i, y_i z_i))^2 \to 0 \tag{3}$$

This leads to a system of linear equations which is easily solved. However, because the algebraic distance does not have a geometric interpretation, it is difficult to determine the quality of the quadric coefficients calculated in this way. Practical tests indicate that when the object is covered with measurement points on the greater part of the mathematical surface, the formula of whose is used for approximation, then this method gives good results, which are comparable to the results form the LSM for geometric distances. On the other hand, this method will give erroneous results when only a small part of, for example, a spherical or a cylindrical surface was measured.

In order to carry out the approximation to minimize the squares of l_i, equation (1) should first be divided by one of the terms a_{ij}, for example by a_{44} thus obtaining

$$\begin{aligned} &F(x_i, y_i, z_i) = \\ &= b_{11}x_i^2 + b_{22}y_i^2 + b_{33}z_i^2 + 2b_{12}x_i y_i + 2b_{13}x_i z_i + 2b_{23}x_i z_i + 2b_{41}x_i + 2b_{42}y_i + 2b_{43}z_i + 1 = 0 \end{aligned} \tag{4}$$

where

$$b_{ij} = a_{ij} / a_{44}.$$

Next expressing condition (3) in matrix form

$$L^T L \to 0$$

where

$$L = [l_1 \quad l_2 \quad \dots \quad l_N]^T$$

and equations (4) calculated for each point

$$L = AX + B$$

where:

$$Z = [b_{11} \quad b_{22} \quad \dots \quad b_{42} \quad b_{43}]^T,$$

$$A = \begin{bmatrix} x_i^2 & y_i^2 & \dots & 2y_i & 2z_i \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix},$$

$$B = [1 \quad 1 \quad \dots \quad 1 \quad 1]^T$$

the following system of equations (5) can be created, whose solution will provide us with the desired parameters b_{ij} of the quadric

$$X = -(A^T A)^{-1} A^T B \tag{5}$$

For the calculations it is good to use a set of about 15–20 points measured on as long as possible a part of the shell and representative of its shape. If frequent deformations occur on the surface, it is good to measure a slightly larger set of points.

Theoretically, the nine parameter equation (4) can be solved by interpolation using only nine measured points. However, in this case any surface deformation or measurement error which might occur for any one point could significantly change the shape and location of the quadric.

One should keep in mind that when the observations are made to the exterior or interior surface of some thin-walled masonry structures, the measured points do not represent a quadric. This occurs due to the varying thickness of the wall of the building which fulfills the quadric equation only for the central surface. However, this does not impact the efficacy of the algorithm described here, because it will not change the spatial localization of the model. On the other hand, the changes in its shape will lead to a slight decrease of the uniformity of the relative locations of the measurement profiles, which has a negligible effect on the construction of the spline surfaces.

The next step of the algorithm consists in calculating the spatial location of the object, and then in projecting the lattice of uniformly spaced profiles on it, which will determine the scanning points. These profiles will be set according to the method presented in figure 1. The so-called fundamental axis *P* will be associated with one of the principle axes of the object. It is best for this axis to correspond to the object's axis of rotation. It should pass through the center of the quadric or through the point of the pseudo-center in the case of noncentral quadrics. Thanks to axis *P* being fixed in this way, the algorithm works for any type of spatially placed surface and not just for the most popular surfaces with a vertical axis of rotation. The fundamental axis is the base for designating the cutting planes, which determine the location of the scanning points. The first group of these will be the set of planes perpendicular to axis P and equidistant from each other by a constant distance G. The second group will be created by rotating a plane containing axis P around this axis by intervals of H degrees. The points of intersection of these planes and the quadric equations will define the set of points to be measured. The detailed execution of this algorithm is presented below.

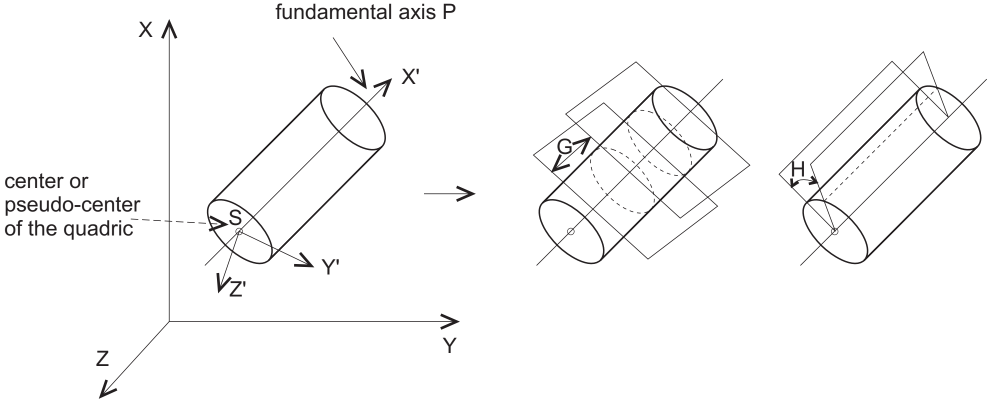

Fig. 1. Method for determining the quadric profiles based on the approximated model. XYZ – coordinate system for the reference network, $X'Y'Z'$ – coordinate system associated with the principle directions of the quadric

The elements which describe the spatial location of the approximated model of the quadric and which are necessary to determine the cutting planes are the directions of the principle semi-axes and the location of the center (or the pseudo-center of symmetry). Defining the principle directions in the reference network coordinate system, requires the previous calculation of the eigenvalues of the characteristic equation derived from the reduced determinant of the quadric [4]

$$\begin{vmatrix} b_{11}-\lambda_1 & b_{12} & b_{13} \\ b_{12} & b_{22}-\lambda_2 & b_{23} \\ b_{13} & b_{23} & 1-\lambda_3 \end{vmatrix} \tag{6}$$

In theory, depending on the type of quadric, this equation can have three different roots, two roots (with one double root) or one triple root. The algorithm uses the general equation for second degree surfaces. For the initial measurements used to calculate the surface, about twenty points lying on the physical surface, which is always slightly deformed in comparison to the project, are used.

The measurement itself is also carried out with a specified precision. Because of this, the quadric equation determined in the process of the approximation will practically always provide three different eigenvalues. For solids of revolution or spheres, the roots can have very similar values, however the probability of obtaining two or three which are exactly the same is very small.

The principle directions are defined by calculating the values of the cosines of the angles of the directional semi-axes in the reference network coordinate system. Every semi-axis is defined in relation to all of the coordinate axes XYZ of the reference network:

$$\begin{array}{cccc} & X' & Y' & Z' \\ X & \cos(XX') & \cos(XY') & \cos(XZ') \\ Y & \cos(YX') & \cos(YY') & \cos(YZ') \\ Z & \cos(ZX') & \cos(ZY') & \cos(ZZ') \end{array} \tag{7}$$

In order to determine the direction of a given semi-axis, the following system of equations must be solved (for example, for X'):

$$\begin{aligned} &(b_{11}-\lambda_1)\cos(XX')+b_{12}\cos(YX')+b_{13}\cos(ZX')=0 \\ &b_{12}\cos(XX')+(b_{22}-\lambda_1)\cos(YX')+b_{23}\cos(ZX')=0 \\ &b_{13}\cos(XX')+b_{23}\cos(YX')+(1-\lambda_1)\cos(ZX')=0 \end{aligned} \tag{8}$$

using the condition

$$\cos^2(XX')+\cos^2(YX')+\cos^2(ZX')=1,$$

for the previously calculated eigenvalue λ_i corresponding to this semi-axis.

The choice of one of the principle axes as the fundamental axis will now be of essential importance for determining the correct cutting planes. The user can make this choice manually, or it can be done automatically. For this purpose, in addition to the measurements for the preliminary calculation of the quadric parameters, two additional points should be measured to create a line in space close to the desired fundamental axis. It is not necessary for these points to lie on the surface being measured. They will be used to determine a line whose slope compared to the previously calculated principle axes of the quadric will make it possible to chose the axis most similar to the direction of the line as the fundamental axis P. Therefore, the algorithm, based on the coordinates of these two points, will calculate the vector W whose components W_x, W_y, W_z divided by the vector's length are the directional cosines of the line in the reference network coordinate system.

The method taken to determine the two groups of section planes (planes perpendicular to the direction of the fundamental axis and planes containing this axis and rotated around it) provides a better distribution of scanning points than if the points were designated by the intersection of two perpendicular groups of mutually parallel planes. However, in this case, the correct choice of one of the principle axes as the fundamental axis is of utmost importance. If, for example, in the case shown in figure 1 a direction corresponding to Y' or Z' were to be chosen as the fundamental axis, then the distribution of points consequently obtained would not be uniform.

Equally important is the choice of the point through which the fundamental axis will pass. An incorrect determination of this point will result in a deterioration of the uniformity of the profiles obtained by rotation around this axis. The easiest case will be for quadrics which contain one center of symmetry S (for example spheres, ellipsoids, hyperboloids). The directional form of the equation of the fundamental axis passing through center S has the following form

$$\frac{x - x_S}{W_x} = \frac{y - y_S}{W_y} = \frac{z - z_S}{W_z} \tag{9}$$

The coordinates of the center of the quadric are calculated from the following system of equations:

$$\begin{aligned} b_{11} x_S + b_{12} y_S + b_{13} z_S + b_{41} &= 0 \\ b_{12} x_S + b_{22} y_S + b_{23} z_S + b_{42} &= 0 \\ b_{13} x_S + b_{23} y_S + b_{33} z_S + b_{43} &= 0 \end{aligned} \tag{10}$$

Similar calculations can be made for surfaces which contain an infinite number of centers (for example, elliptical cylinders). Here we should once again repeat that we are considering the general equation of a quadric calculated based on an approximation process from a set of points measured on a non-ideal physical object. Therefore, regardless of the type of quadric, the coordinates of some pseudo--center of the surface will always be determined. As an example, for an elliptical cylinder, this will be one of the points lying on its axis, and this is sufficient for defining the fundamental axis correctly. A bit more problematic will be the case of noncentral quadrics which do not have a center S (for example elliptical paraboloids). In this case, we will also be able to calculate a pseudo-center from equation (10), but this point may be shifted sideways in relation to the line normal to the apex of the surface which would be optimal for designating as the attached fundamental axis. The determination of the correct pseudo-center can be done in several ways. The first requires the construction of a plane perpendicular to the direction of the fundamental axis and passing through any given point such that the plane sections the paraboloid. The equations of the plane and the quadric together designate an ellipse whose center will be the point S. An easier method will be the calculation of the center *S* using the equation of the line normal to the surface and parallel to the fundamental axis. For example, for an elliptical paraboloid and for a fundamental axis direction determined as described above, both of these conditions are fulfilled only by the apex point. The equations of the normal line of the quadric (4) and passing through a set point *S* are:

$$\frac{x-x_S}{F'_{1x}(x_S,y_S,z_S)}=\frac{y-y_S}{F'_{1y}(x_S,y_S,z_S)}=\frac{z-z_S}{F'_{1z}(x_S,y_S,z_S)} \tag{11}$$

The direction of the fundamental axis is then designated by equations (9) where the coordinates of point *S* in the numerators of the equation are not important in this case. The condition of the fundamental axis *P* being parallel to the quadratics normal line will, after transformations, have the following form

$$\frac{b_{11}x_S+b_{12}y_S+b_{13}z_S+b_{41}}{W_x}=\frac{b_{12}x_S+b_{22}y_S+b_{23}z_S+b_{42}}{W_y}=\frac{b_{13}x_S+b_{23}y_S+b_{33}z_S+b_{43}}{W_z} \tag{12}$$

It will permit the calculation of the desired coordinates of the pseudo-center *S* from the system of equations. Instead of analytically calculating point *S* by one of these two methods, the points can be measured on the object. The elliptical (rotational) paraboloids occurring in actual technological applications are mostly the bowls of radio telescopes whose apex point can usually be measured. One can also measure the center of the detector which gathers the radio waves reflected from the surface of the bowl and which is sometimes easier to see.

Determination of the direction and the attachment of the fundamental axis allows for the calculation of both groups of cutting planes. The first is the group of planes perpendicular to the fundamental axis and equidistant from each other by an interval G. First, the range of the fundamental axis in which these planes will be created should be specified. This can be done by measuring two points on the building corresponding to the building's extreme dimensions in the direction of the fundamental axis. Next, lines perpendicular to the fundamental axis and passing through it should be drawn through these points. These lines will set the limit range for creating the cutting planes. Starting with one of these, points should be set on the fundamental axis spaced apart by the distance G. The set of planes passing through these points can now be determined using the condition that they must be perpendicular to the fundamental axis. The second group of planes are the planes which contain the fundamental axis and are rotated around it by an angle which results from the set number of profiles. The first is constructed as containing the two points on the fundamental axis whose coordinates were previously calculated and any other third point in space. The successive planes will be the planes which are simultaneously tilted by the set angle from the first plane (the angle successively increased by the set interval *H*), parallel to the fundamental axis *P* and passing through the center point S. The interval *H* will be incremented up to the value of the semi-total angle for closed solids. For open solids, two points should be measured which will specify the range of the created planes and the direction in which they will be created should be chosen.

The locations of the scanning points are calculated from the intersections of the approximated quadric with both groups of cutting planes

$$b_{11}x_i^2 + b_{22}y_i^2 + b_{33}z_i^2 + 2b_{12}x_iy_i + 2b_{13}x_iz_i + 2b_{23}y_iz_i + 2b_{41}x_i + 2b_{42}y_i + 2b_{43}z_i + 1 = 0 \quad (13)$$

where:

G, *H* – plane groups,

l, *k* – numbers of the plane in a given group.

The last step is the process of automatic scanning of the object carried out from the set observation stations of the reference network. Based on the coordinates of the observation stations in reference network, horizontal and vertical angles will be calculated to set the total station sight direction during the measurement of the distances of the points calculated from the system of equations (13). Before starting scanning, the problem of which groups of points will be observed from each observation station should be solved. The correct choice can be made by measuring three points on the surface from the given observation station, which determines the cutting plane and determining the concavity/convexity of the building. For surfaces concave in relation to the observer (for example, the bowl of a radio telescope) scanning will be carried out for points located behind the section plane. For buildings convex towards the observer (for example a cooling tower) the points located between the plane and the observation station should be scanned. As a result, the unnecessary scanning of elements of the building which are not visible from the chosen point of the reference network will be avoided. By properly choosing the cutting planes, it is also possible to avoid observations of points which lie at very sharp angles to the given observation station which usually increases measurement error. All points bounded by the given plane can be scanned from the observation station or only those which were not yet measured from other points of the reference network.

3. Summary

To summarize, the sequence of actions carried out by the user for the presented algorithm is as follows: measurement and adjustment of the reference network, measurement of about 20 points to specify the location of the quadric, measurement of 2 points to specify the direction of the fundamental *P* axis, possibly a measurement of the center *S*, measurement of 2 or 4 points to determine the range of section planes, measurement of 3 points at each observation station and determination of whether the object is concave or complex, which will determine the range of scanning.

As was mentioned in the introduction, the algorithm presented here can also be used to uniformly dedensify the cloud of points obtained from classical laser scanning. For this, it is sufficient to calculate the parameters of the quadric, project groups of cutting planes onto it, and then to calculate the desired locations of the points. A comparison of these with the scanning data will permit the selection of uniformly spaced points from the cloud.

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