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Refined Astrogravimetric Geoid in Poland – Part II**

1. Preamble

In the part I of this paper theoretical background of astronomic levelling and least squares collocation methods were given. Furthermore both accuracy and weights of astrogeodetic and astrogravimetric deflections of the vertical were determined. In the present part II the astrogeodetic and astrogravimetric geoid models are developed from improved deflections of the vertical with the use of astronomical levelling and least squares collocation. In order to assess their accuracy all four computed models are tested on sites of satellite POLREF network connected to precise levelling network.

2. Determination of the Astrogeodetic Geoid Model

Astronomical levelling provides geoid height differences ΔN between the points of surveyed astrogeodetic deflections of the vertical ξ, η. Calculated geoid height differences between the astrogeodetic points can further be adjusted in the network adjustment. For the ABC triangle the condition of loop closure [3]:

$$\Delta N_{AB} + \Delta N_{BC} + \Delta N_{CA} = 0 \tag{1}$$

must be fulfilled. The adjusted geoid height differences in the network can be obtained using least squares adjustment based on condition of equation).

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For the computation of geoid height differences $\Delta N_{P_iP_J}$ between two investigated astrogeodetic points P_i and P_j using the method of astronomical levelling the following formula was applied [5]:

$$\Delta N_{P_iP_j} = -\frac{s_{P_iP_j}}{2}[(\xi_{P_i} + \xi_{P_j})\cos\alpha_{P_iP_j} + (\eta_{P_i} + \eta_{P_j})\sin\alpha_{P_iP_j}] \tag{2}$$

where $s_{P_iP_j}$ is a distance between points P_i and P_j, ξ_{P_i}, η_{P_i}, ξ_{P_j}, η_{P_j} are deflections of the vertical in points P_i and P_j respectively and $\alpha_{P_iP_j}$ is an azimuth.

In order to compute geoid height differences $\Delta N_{P_iP_j}$ by astronomic levelling, a program creating a network based on automatic selection of shortest connections between the points was applied. A resulting network of simple geometric, basically triangular structure consists of 158 points (Fig. 1). It should be noted that the selection of connections does not affect substantially the results of adjustment. Differences between geoid heights computed with different selection algorithms of shorters lines do not exceed mean square errors of adjusted $\Delta N_{P_iP_j}$ [5].

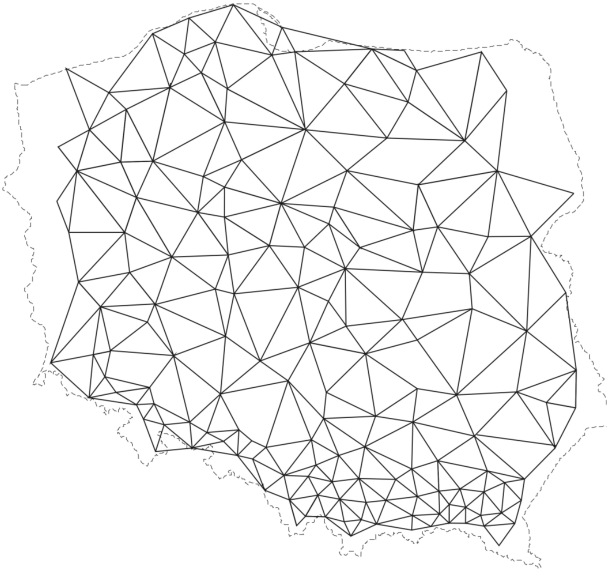

Fig. 1. Astrogeodetic network created by program

Mean square errors of geoid height differences $\Delta N_{P_iP_j}$ similarly to mean square errors of height differences in adjustment of levelling network were determined:

$$m_{\Delta N_{P_iP_j}} = 0.027\sqrt{s_{P_iP_j}}\text{, in metres} \tag{3}$$

where $s_{P_iP_j}$ is given in kilometres.

From the analysis of positiveness of the statistical test for the *a posteriori* variance factor, a scaling factor in equation (3) equal to 0.027 was determined. Mean square errors calculated with equation (3) were used for the determination of weights of "observed" geoid height differences. The same method of weight determination of geoid height differences was applied in [5].

The astrogeodetic network (Fig. 1) was adjusted with the *GeoLab* software. That software is extremely useful for wide variety of project types, in particular for least squares adjustment of horizontal and vertical networks, and is used in numerous institutions in many countries all over the world.

The input data to the GeoLab consisted of 413 "observed" geoid height differences and fixed geoid height of Borowa Góra equal to 30.760 m that is a central point of the network. The adjustment provided geoid heights of 157 points that determine new astrogeodetic geoid model as well as standard deviations of adjusted geoid heights. Average standard deviation of adjusted geoid height equals 0.18 m while the maximum adjusted geoid height equals 0.29 m.

Quality of the developed astrogeodetic geoid model was then estimated by comparing it with GPS/levelling geoid spanned on the sites of the POLREF network. Differences between geoid heights of the POLREF sites and respective geoid heights interpolated from the astrogeodetic geoid model were calculated and analysed. Least squares collocation was applied to minimise numerical errors of interpolation of the astrogeodetic geoid. It required the determination of empirical covariance function of residual geoid heights N^{res}. Geoid heights N^{EGM96} were calculated from the global geopotential model EGM96 in the points of the astrogeodetic network of geoid height N^{astr}, and residual geoid heights N^{res} were formed as follows:

$$N^{res} = N^{astr} - N^{EGM96} \tag{4}$$

Then the parameters: variance C_0 = 0.7044 m^2 and correlation length ν = 217.3 km of the empirical covariance function for the residual geoid heights N^{res} were determined (Fig. 2).

Least squares collocation was then applied to interpolate residual geoid heights in the sites of the POLREF network. After calculating at those sites geoid heights N^{EGM96} from the global geopotential model EGM96 the interpolated geoid heights of the astrogeodetic geoid model were obtained in the sites of the POLREF network with the use of rearranged equation (4). Differences between GPS/levelling geoid heights $N_{Q_i}^{GPS/lev}$ and interpolated heights $N_{Q_i}^{astr}$ of the astrogeodetic geoid model at the POLREF sites:

$$\Delta_{Q_i}^{a} = N_{Q_i}^{GPS/lev} - N_{Q_i}^{astr} \tag{5}$$

were then calculated and analysed.

The standard deviation of the differences $\Delta_{Q_i}^{a}$ equals 0.60 m. It reflects real accuracy of the astrogeodetic geoid model in Poland determined using astronomical levelling.

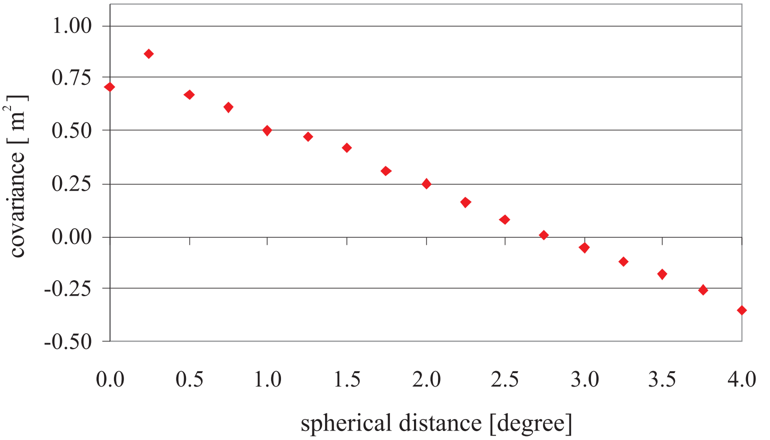

Fig. 2. Empirical covariance function of differences $N^{astr} - N^{EGM96}$

3. Determination of the Astrogravimetric Geoid Model by Helmert Method

Next astrogeodetic geoid model was developed using all deflections of the vertical available in Poland, i.e. 171 archival astrogeodetic deflections of the vertical, 370 archival astrogravimetric deflections of the vertical and 23 astrogeodetic deflections of the vertical surveyed in 2003–2004.

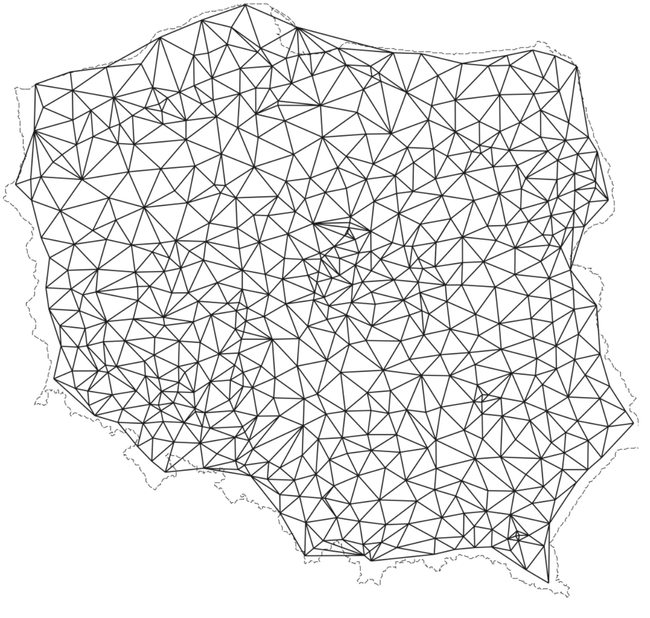

Fig. 3. Triangular astrogravimetric network created by program

The network of triangles (Fig. 3) based on 564 points and with 1597 "observed" geoid height differences was formed with the use of Delauney algorithm. The network deficiency was again removed by using geoid height of Borowa Góra equal to 30.760 m.

Geoid height differences $\Delta N_{P_iP_j}$ between the points of the network have been calculated with (2). Mean square errors of geoid height differences $\Delta N_{P_iP_j}$, were determined as follows:

$$m_{\Delta N_{P_iP_j}} = 0.015\sqrt{s_{P_iP_j}}, \text{ in metres} \tag{6}$$

where $s_{P_iP_j}$ is given in kilometres. As previously, a scaling factor in (6) equal to 0.015 was determined. from the analysis of positiveness of the statistical test for the *a posteriori* variance factor.

The adjustment provided geoid heights of 550 points that determine new astrogravimetric geoid model as well as standard deviations of adjusted geoid heights. Average standard deviation of adjusted geoid height equals 0.07 m while the maximum adjusted geoid height equals 0.10 m. The external accuracy estimate of the new astrogravimetric geoid model was conducted by comparing it with GPS/levelling geoid spanned on the sites of the POLREF network. Following the same procedure as applied for quality estimate of the new astrogeodetic geoid model, the empirical covariance function of the residual geoid heights N^{res1}:

$$N^{res1} = N^{astr/grav} - N^{EGM96} \tag{7}$$

obtained in the points of the astrogravimetric network of geoid height $N^{astr/grav}$ was determined. The parameters of the empirical covariance function are as follows: variance C_0 = 0.4411 m^2 and correlation length ν = 135.4 km (Fig. 4).

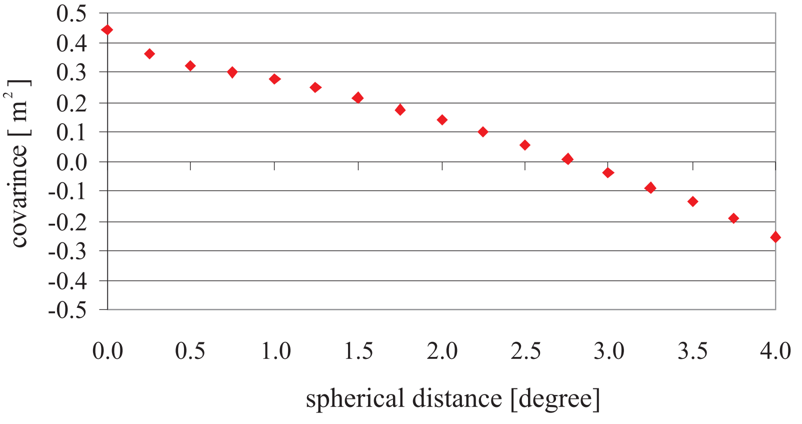

Fig. 4. Empirical covariance function of differences $N^{astr/grav} - N^{EGM96}$

Least squares collocation was then applied to interpolate residual geoid heights $N_{Q_i}^{astr/grav}$ in the sites of the POLREF network.

Differences:

$$\Delta_{Q_i}^{a/g} = N_{Q_i}^{GPS/lev} - N_{Q_i}^{astr/grav} \tag{8}$$

were then calculated and analysed. The standard deviation of the differences $\Delta_{Q_i}^{a/g}$ equals 0.51 m. It reflects real accuracy of astrogravimetric geoid model in Poland determined using astronomical levelling.

4. Determination of the Astrogeodetic Geoid Model by Least Squares Collocation

Modelling geoid from the deflections of the vertical by least squares collocation, by contrast to the use of integral approach is not widely documented in literature. Therefore the study of the potentiality of the determination of the astrogeodetic geoid model by least squares collocation was undertaken by the author.

The procedure of modelling astrogeodetic geoid in Poland using least squares collocation was the following. Geoid heights N^{col} were represented as the sum:

$$N^{col} = N_{GM} N_{\xi\eta\, res} \tag{9}$$

of the low frequency component N_{GM} obtained from the global geopotential model, and the component $N_{\xi\eta\, res}$ determined by least squares collocation with the use of residual deflections of the vertical ξ^{res}, η^{res}:

$$\begin{aligned} \xi^{res} &= \xi^{astr} - \xi_{GM} \\ \eta^{res} &= \eta^{astr} - \eta_{GM} \end{aligned} \tag{10}$$

Global geopotential model EGM96 was used for the determination of the astrogeodetic geoid by least squares collocation to be consistent with previously developed geoid models. The functionals N_{GM}, Δg_{GM}, and ξGM, ηGM were calculated on 5′ × 10′ grid (approximate 10 km × 10 km).

The residual geoid heights $N_{\xi\eta\, res}$ were obtained from:

$$\mathbf{N}_{\xi\eta\, res} = \mathbf{C}_{sl}\mathbf{C}_{ll}^{-1}\mathbf{l} \tag{11}$$

where $\mathbf{C}_{sl}$ is the vector of covariance between the signals to be predicted and the measurements $\mathbf{l}$ and $\mathbf{C}_{ll}$ is autocovariance matrix between the observations (see eq. (11), part I).

Computations were conducted with the use of the *gpcol* program of the *GRAVSOFT* package [6]. The program uses the logarithmic model of covariance function [2]. That model is defined by three parameters: variance C_0 of gravity anomalies, and parameters D and T that determine the degree of damping high and low frequencies of the gravity signal, respectively.

Suitable choice of the parameters of the covariance function is obtained by its proper fit to the empirical data. Practically, parameters C_0, D and T can be determined using gravity anomalies from the area of interest, e.g. with the use of the *gpfit* program of the *GRAVSOFT* package [6].

Mean 5′ × 5′ Faye gravity anomalies from Poland [13] were used as input data to the *gpfit* program. Empirical covariance function of mean residual gravity anomalies as well as fitted logarithmic analytical covariance function were determined (Fig. 5). The parameters of the covariance function are as follows: $\sqrt{C_0}$ = 7.93 mGal, D = 6 km, T = 30 km.

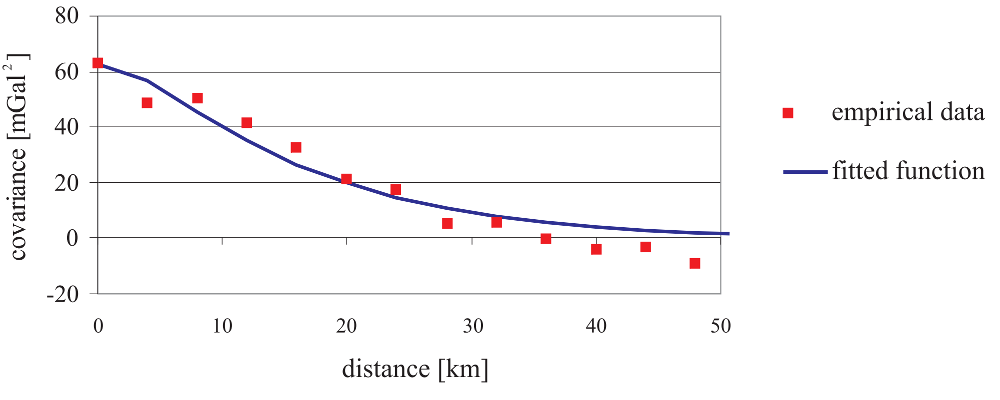

Fig. 5. Empirical covariance function of mean 5′ × 5′ residual gravity anomalies (dots) and fitted logarithmic analytical covariance function (line)

The major stage of the determination of the astrogeodetic geoid was computation of residual components $N_{\xi\eta\, res}$ on 5′ × 10′ grid. Then, using (9) the astrogeodetic geoid model was determined by least squares collocation. Standard deviations of adjusted geoid heights do not exceed 0.20 m and their average equals 0.19 m

The external accuracy of the astrogeodetic geoid model was estimated by comparing it with GPS/levelling geoid spanned on the sites of the POLREF network. Differences between GPS/levelling geoid heights $N_{Q_i}^{GPS/lev}$ and heights $N_{Q_i}^{a/col}$ of the astrogeodetic geoid model at the POLREF sites:

$$\Delta_{Q_i}^{a/col} = N_{Q_i}^{GPS/lev} - N_{Q_i}^{a/col} \tag{12}$$

were then calculated and analysed. The statistics of obtained differences are given in table 1.

Table 1. Statistics of differences $N^{GPS/lev} - N^{a/col}$

Mean	Std dev.	Min	Max
–0.50	0.12	–0.83	–0.08

The results obtained indicate that the accuracy of the astrogeodetic geoid model determined by least squares collocation is much improved, i.e. 5 times better in the case investigated, with respect to the classical astronomical levelling solution when using the same data, i.e. deflections of the vertical.

The observed improvement of the accuracy stimulated the author to do similar comparison using astrogravimetric geoid models.

5. Determination of the Astrogravimetric Geoid Model by Least Squares Collocation

New astrogravimetric geoid model was developed by least squares collocation using archival astrogeodetic 171 deflections of the vertical and about 6000 mean 5′ × 5′ gravity anomalies [4]. The data used approximately corresponded to those applied for the determination of the first astrogravimetric geoid in Poland [1].

The astrogravimetric geoid model was calculated using the *gpcol* program. Due to large number of data causing numerical problem with inverting of the matrix of normal equations, the area of Poland has been subdivided onto 24 blocks (Fig. 6) and the calculations were performed in each block separately. The geoid model was obtained by combining the sets of geoid heights determined in individual blocks.

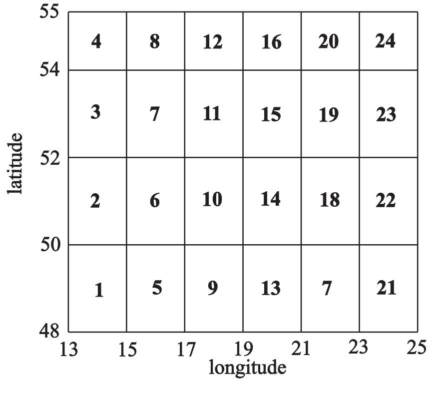

Fig. 6. Blocks used for individual least squares collocation solution for the astrogravimetric geoid model

Calculations were conducted using equation (9) following the procedure applied for developing the astrogeodetic geoid model. The only difference is that besides residual deflections of the vertical the residual gravity anomalies were used as input data.

The external accuracy of the astrogravimetric geoid model was also estimated by comparing it with GPS/levelling geoid spanned on the sites of the POLREF network. Differences between GPS/levelling geoid heights $N_{Q_i}^{GPS/lev}$ and heights $N_{Q_i}^{ag/col}$ of the astrogravimetric geoid model at the POLREF sites:

$$\Delta_{Q_i}^{ag/col} = N_{Q_i}^{GPS/lev} - N_{Q_i}^{ag/col} \tag{13}$$

were then calculated and analysed. The statistics of obtained differences are given in table 2.

Table 2. Statistics of differences $N^{GPS/lev} - N^{ag/col}$

Mean	Std dev.	Min	Max
–0.45	0.07	–0.81	–0.33

The results obtained indicate that the accuracy of the astrogravimetric geoid model determined by least squares collocation is much improved, i.e. 7 times better in the case investigated, with respect to the classical astrogravimetric levelling solution when using the same data. The standard deviation of geoid height got reduced from 0.51 m to 0.07 m (Tab. 2).

6. Conclusions

Improved archival and newly determined deflections of the vertical were used to develop geoid models for Poland (see part I of this paper). Then four geoid models were calculated.

First two geoid models were calculated using classical astronomical levelling approach. The input data used for computing the first model consisted of 171 archival astrogeodetic deflections of the vertical while data for computing the second model – consisted of the same 171 astrogeodetic deflections of the vertical with added 370 archival astrogravimetric deflections of the vertical. Accuracy of those models estimated at the stations of the POLREF network, represented by standard deviations equals σ_1 = 0.60 m and σ_2 = 0.51 m, respectively (Fig. 7).

Third and fourth geoid models were calculated using least squares collocation approach. The input data used for computing the third model consisted of 171 archival astrogeodetic deflections of the vertical while data for computing the fourth model – consisted of same 171 astrogeodetic deflections of the vertical with added about 6000 mean 5′ × 5′ gravity anomalies. Accuracy of those models estimated at the stations of the POLREF network, represented by standard deviations equals σ_3 = 0.19 m and σ_4 = 0.07 m, respectively (Fig. 7).

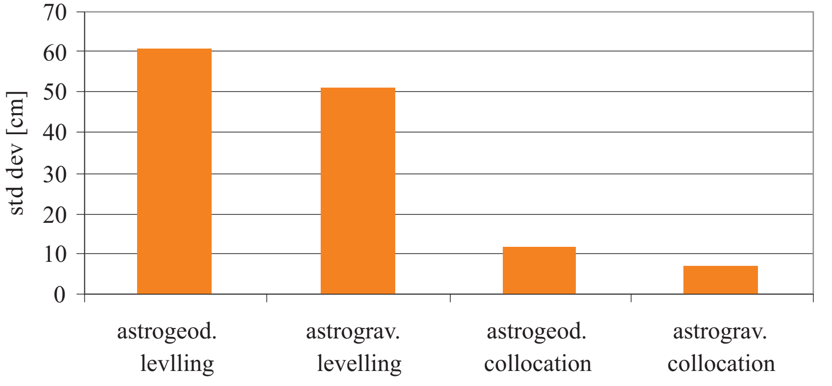

Fig. 7. External accuracy of geoid models developed, represented by the standard deviation of their fit to the GPS/levelling geoid at the sites of the POLREF network

Astrogeodetic geoid models – first and third are equivalent in terms of the input data used; they differ in terms of geoid determination method applied. Astrogravimetric geoid models – second and fourth differ in the same way in terms of geoid determination method used; the input data used for calculating those models can be considered identical, since the archival astrogravimetric deflections of the vertical were determined from mean 5′ × 5′ gravity anomalies.

The results obtained (Fig. 7) indicate that both astrogeodetic geoid and astrogravimetric geoid determined from the same input data using least squares collocation approach is by factor 5 to 7 more accurate than the ones obtained using classical astronomical levelling. Both astrogeodetic and astrogravimetric geoid models developed with the use of least squares collocation approach are thus substantially improved as compared with the existing models developed on the basis of astronomical levelling algorithm.

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