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Statistical Algorithms for Modelling the Results of Geodetic Observations

1. Introduction

The observation of the characteristic values describing the transformation processes of the earth's crust, as well as atmospheric and natural phenomena, often requires extensive analysis and modelling. This analysis is most often based on the mutual functional relations which result from the research methods used in geodesy. The relations between the characteristic parameters of the process or phenomenon being considered are usually non-linear, which means that their modelling first requires the transformation of functional relations into linear form. The easiest method for linearizing complex functional relations is by expanding them into a Taylor series and then accounting for just the first few partial derivatives.

If we have a nonlinear functional relation which can be written in the following symbolic form:

$$f_n(b_1x_1, b_2x_2, \dots, b_nx_n) = B \quad (1)$$

then the linear expression of this function can be written as:

$$\left(\frac{\partial f}{\partial X_1}\right)_0 dX_1 + \left(\frac{\partial f}{\partial X_2}\right)_0 dX_2 + \dots + \left(\frac{\partial f}{\partial X_n}\right)_0 dX_n = B - f(b_1\bar{X}_1, b_2\bar{X}_2, \dots, b_n\bar{X}_n) \quad (2)$$

The values of the partial derivatives calculated for the approximate values of the parameters (unknowns), i.e. for $(\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n)$, are usually symbolized as the coefficients a_i , and the right side of this equation, which gives the value of the functional relation for the approximate parameters, is a free term and is symbolized as l_i .

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That is:

$$a_i = \left(\frac{\partial f}{\partial x_i} \right)_0 \quad \text{and} \quad l = B - f(b_1 \bar{X}_1, b_2 \bar{X}_2, \dots, b_n \bar{X}_n).$$

Using the symbols described above, every set of observational equations with m unknowns $dX_i = x_i$, with set variable coefficients (a_{ij}) and free terms (l_j) can be written in the following analytical form:

$$\begin{aligned} a_{11}x_1 + \dots + a_{1m}x_m &= l_1 \\ \dots \quad \dots \quad \dots \quad \dots &\quad \dots \quad \dots \quad \dots \\ a_{n1} + \dots + a_{nm}x_m &= l_n \end{aligned} \tag{3}$$

If we define the following matrices:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \quad \text{– matrix of the variable coefficients with the dimensions } (n \times m),$$

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_m \end{bmatrix} \quad \text{– single column matrix of the estimated unknowns,}$$

$$\mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \\ \dots \\ l_n \end{bmatrix} \quad \text{– single column matrix of the free terms,}$$

then the set of equations (3) can be written in the following matrix form:

$$\mathbf{A} \times \mathbf{X} = \mathbf{L} \tag{4}$$

If the system (1) has more equations than unknowns, i.e. ($n > m$), then \mathbf{A} is a vertical rectangular matrix and the set of equations under consideration is an overdetermined system of equations.

If the number of equations is less than the number of unknowns, i.e. ($n < m$), then \mathbf{A} is a horizontal rectangular matrix and the set of equations is an underdetermined system of equations.

Both overdetermined and underdetermined systems of equations do not possess uniquely defined values of the unknowns because they are internally inconsistent. Therefore, solving such systems requires the formulation of additional restrictions. These restrictions can be placed on either the modelled deviations (residues) $\delta = \mathbf{L} - \mathbf{A} \times \mathbf{X}$ or on the values of the estimated parameters (\mathbf{X}).

Below we will only consider algorithms for solving overdetermined systems of equations based on Gauss–Markov models.

2. Algorithm for Solving Overdetermined Systems of Equations

Let us suppose that the system of equations (4) has more equations than unknowns, i.e. ($n > m$). A Gauss–Markov model will then be used to estimate the parameters \mathbf{X} . This model imposes an additional restriction, in that the sum of the squares of the model deviations $\delta = \mathbf{L} - \mathbf{A} \times \mathbf{X}$ should have the least possible value:

$$F = (\mathbf{L} - \mathbf{A} \times \mathbf{X})^T (\mathbf{L} - \mathbf{A} \times \mathbf{X}) = \min \quad (5)$$

A necessary condition for the existence of a minimum of function (5) is obtained by setting the first derivative of the matrix expression equal to zero:

$$\frac{\partial F}{\partial \mathbf{X}} = 2 \mathbf{A}^T \mathbf{A} \mathbf{X} - 2 \mathbf{A}^T \mathbf{L} = 0 \quad \Leftrightarrow \quad (\mathbf{A}^T \mathbf{A}) \times \mathbf{X} = \mathbf{A}^T \mathbf{L} \quad (6)$$

The system of equations (6) contains the square symmetrical matrix $(\mathbf{A}^T \mathbf{A})$ multiplied by the unknowns \mathbf{X} and is called a normal (Gaussian) system of equations. If the matrix $(\mathbf{A}^T \mathbf{A})$ is nonsingular, i.e. $\det(\mathbf{A}^T \mathbf{A}) \neq 0$, then the solution to the system of equations (4) can be written as:

$$\hat{\mathbf{X}} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \times \mathbf{L} = \mathbf{A}^+ \times \mathbf{L} \quad (7)$$

The matrix \mathbf{A}^+ defined by equation (7) is called the pseudoinverse of matrix \mathbf{A} , and the calculated estimations of the unknowns $\hat{\mathbf{X}}$ fulfil the restriction (5) and give the most probable values of the parameters sought for the given process or phenomenon.

In many geophysical problems, we encounter systems of equations in which $\det(\mathbf{A}^T \mathbf{A}) = 0$, which means that these systems contain mutually dependent equations. In this case, matrix \mathbf{A} has a rank defect which is defined as the difference between the smaller dimension of matrix \mathbf{A} and the rank of the matrix, i.e. $d = m - R[\mathbf{A}]$. The value of the defect signifies that matrix \mathbf{A} contains “ d ” columns

whose elements are a linear combination of the elements of other columns. In this case, the matrix $(\mathbf{A}^T \mathbf{A})^{-1}$ is undetermined; therefore additional restrictions, which use the generalized inverse of the matrix $(\mathbf{A}^T \mathbf{A})$, must be considered. There are many types of restrictions of the estimated parameters and, consequently, many ways to calculate the generalized inverse of the matrix $(\mathbf{A}^T \mathbf{A})$, which we will call \mathbf{N} , i.e. $(\mathbf{A}^T \mathbf{A}) = \mathbf{N}$ from here on.

For a unique selection of a generalized inverse of a symmetrical square matrix \mathbf{M} , i.e. for a unique solution to an overdetermined system of equations with a defect, Moore and Penrose defined the following restrictions:

$$\begin{aligned} \mathbf{M}\mathbf{M}^{-1}\mathbf{M} &= \mathbf{M} & \mathbf{M}^{-1}\mathbf{M}\mathbf{M}^{-1} &= \mathbf{M}^{-1} \\ (\mathbf{M}\mathbf{M}^{-1})^T &= \mathbf{M}\mathbf{M}^{-1} & (\mathbf{M}^{-1}\mathbf{M})^T &= \mathbf{M}^{-1}\mathbf{M} \end{aligned} \quad (8)$$

The geometrical interpretation of these restrictions is that they ensure the minimization of the length of the estimated vector of unknowns, i.e. $\hat{\mathbf{X}}^T \hat{\mathbf{X}} = \min$.

Following the formula for the general inverse, the solution of the system of equations, as per relation (7), assumes the following form:

$$\hat{\mathbf{X}} = \mathbf{N}(\mathbf{N}\mathbf{N})^{-1} \mathbf{A}^T \times \mathbf{L} = \mathbf{A}^+ \times \mathbf{L} \quad (9)$$

It follows from formula (9) that the pseudoinverse of a rank deficient matrix \mathbf{A} is expressed by the following equation:

$$\mathbf{A}^+ = \mathbf{N}/(\mathbf{N}\mathbf{N})^{-1} \mathbf{A}^T \quad (10)$$

Equation (10) shows that the basis for solving overdetermined systems of equations in which matrix \mathbf{A} is deficient with defect $d = m - R[\mathbf{A}]$ is the generalized inverse of the matrix $(\mathbf{N} \times \mathbf{N})$, i.e. $(\mathbf{N} \times \mathbf{N})^{-1}$.

To determine \mathbf{A}^+ and, consequently, to calculate $(\mathbf{N} \times \mathbf{N})^{-1}$, we propose the following efficient algorithm. First, the matrix $\mathbf{N} = (\mathbf{A}^T \mathbf{A})$ with dimensions $(m \times m)$ and defect d must be divided into four submatrices (Fig. 1):

- \mathbf{N}_n – nonsingular matrix with dimensions,
- \mathbf{N}_d – horizontal rectangular matrix with dimensions,
- \mathbf{N}_d^T – vertical rectangular matrix with dimensions,
- \mathbf{N}_0 – square matrix with dimensions the.

The vector of unknowns ($\hat{\mathbf{X}}$) estimated with equation (9) fulfils condition (5) and simultaneously leads to minimizing the length of this vector, i.e. $\hat{\mathbf{X}}^T \hat{\mathbf{X}}$.

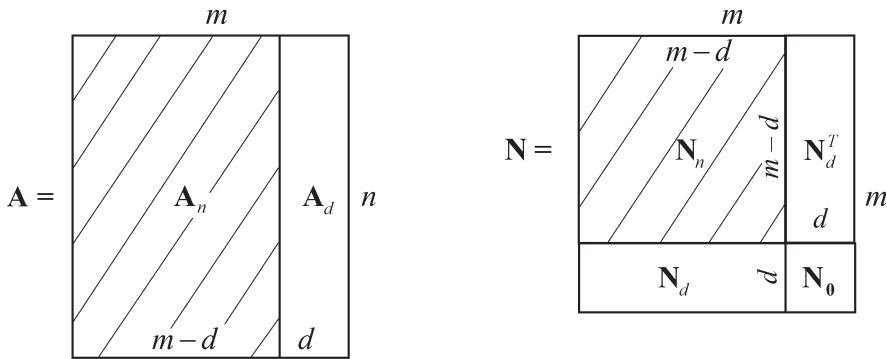


Fig. 1. Lines of division of the rows and columns of matrix \mathbf{N} are determined using the defect of matrix \mathbf{A} or $(\mathbf{A}^T \mathbf{A})$

It results from the above descriptions of the submatrices that the lines of division of the rows and columns of matrix \mathbf{N} are determined using the defect of matrix \mathbf{A} or $(\mathbf{A}^T \mathbf{A})$.

Based on the submatrices defined above, and using matrix transformations, the following formula for calculating the pseudoinverse of matrix \mathbf{A} was derived:

$$\mathbf{A}^+ = \begin{bmatrix} \mathbf{N}_n \\ \mathbf{N}_d \end{bmatrix} \times (\mathbf{N}_n \mathbf{N}_n + \mathbf{N}_d^T \mathbf{N}_d)^{-1} \times \mathbf{A}_n^T \quad (11)$$

where matrix \mathbf{A}_n is the submatrix composed of the $(n-d)$ first columns of matrix \mathbf{A} , as illustrated in figure 1.

The estimators of the parameters sought for the rank deficient systems of equations can now be expressed with the formula:

$$\hat{\mathbf{X}} = \mathbf{A}^+ \times \mathbf{L} = \begin{bmatrix} \mathbf{N}_n \\ \mathbf{N}_d \end{bmatrix} \times (\mathbf{N}_n \mathbf{N}_n + \mathbf{N}_d^T \mathbf{N}_d)^{-1} \times \mathbf{A}_n^T \times \mathbf{L} \quad (12)$$

In order to evaluate the accuracy of observation result modelling as per equation (3), it is necessary to estimate the variance calculated from random residues, which is expressed as:

$$\hat{\sigma}_0^2 = \frac{(\mathbf{L} - \mathbf{A}\hat{\mathbf{X}})^T (\mathbf{L} - \mathbf{A}\hat{\mathbf{X}})}{n - R(\mathbf{A})} \quad (13)$$

After performing the variance analysis in equation (12), we obtain a formula for the covariance matrix of the estimated vector of unknowns:

$$\text{Cov}[\hat{\mathbf{X}}] = \hat{\sigma}_0^2 \mathbf{A}^+ (\mathbf{A}^+)^T \quad (14)$$

The evaluation of observation result modelling using equation (3) can be obtained using the indicator λ , which is defined as the ratio of the part of the variance explained by the model to the total variance of the observed parameters, i.e.:

$$\lambda = 1 - \frac{(\mathbf{L} - \mathbf{A}\hat{\mathbf{X}})^T (\mathbf{L} - \mathbf{A}\hat{\mathbf{X}})}{\sum (l_i - \hat{l})^2} \quad (15)$$

where \hat{l} is the average value of the free term in the system being observed (3).

3. Numerical Example of the Modelling of Survey Measurement Results

To illustrate the algorithm described above and the use of the generalized inverse of the normal equations matrix, the results of survey measurements of the elevation of a water table at four ground points were analysed and modelled. A diagram of the locations of the ground points measured and the results are presented in figure 2. Point A is a national survey benchmark, with a determined elevation within the assumed reference system. Points 1 and 4 are located on one side of the river and points 2 and 3 on the other side. The distance between the measurement points ranges from 450 to 650 meters.

Following equation (1) the relations between the observed survey lines and the elevations of the measurement points assume the form:

$$Z_k - Z_p = h \quad (16)$$

where Z_k represents the elevation of the end Z_p point of a given survey line and represents the elevation of the initial point of that same line. After expansion into a Taylor series, these relations (despite always being linear in this particular case, owing to the nature of the problem) assume the following form:

$$dZ_k - dZ_p = h - Z_{k0} - Z_{p0} = l_h \quad (17)$$

The differential form of the observation equations (17) always leads to small values for the free terms, which is very beneficial for ease of calculation.

Based on the observed values, the approximate elevations of the measured points were calculated:

$$\begin{aligned} Z_{10} &= 121.250 \text{ m} \\ Z_{20} &= 119.830 \text{ m} \\ Z_{30} &= 120.405 \text{ m} \\ Z_{40} &= 121.540 \text{ m} \end{aligned} \quad (18)$$

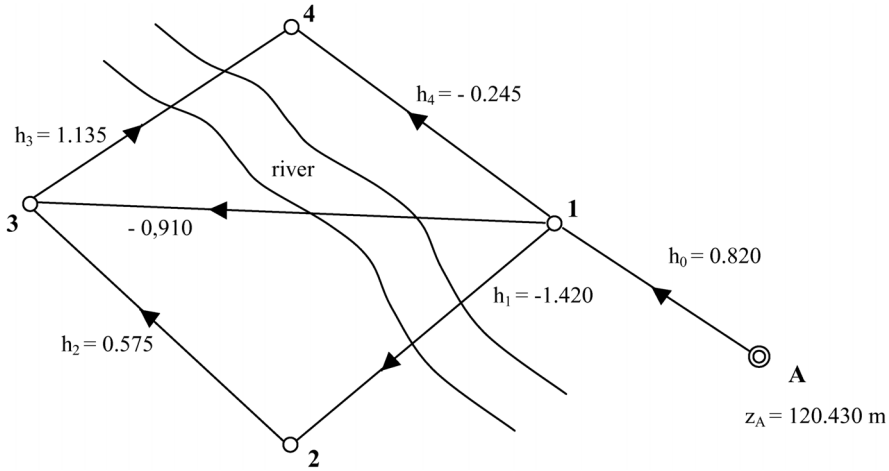


Fig. 2. A diagram of the locations of the ground points measured and the results

The observation equations for the five survey lines are expressed in the form:

$$\begin{aligned}
 dZ_2 - dZ_1 &= -1.420 - (119.830 - 121.250) = 0 \\
 dZ_3 - dZ_2 &= 0.575 - (120.405 - 119.830) = 0 \\
 dZ_4 - dZ_3 &= 1.135 - (121.540 - 120.405) = 0 \\
 dZ_1 - dZ_4 &= -0.245 - (121.250 - 121.540) = 0.045 \\
 dZ_3 - dZ_1 &= -0.910 - (120.405 - 121.250) = -0.065
 \end{aligned}
 \tag{19}$$

Matrix **A** will be composed of 4 columns corresponding to dZ_1, dZ_2, dZ_3, dZ_4 and 5 rows corresponding to the 5 observation equations, i.e.:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix} .$$

L and **X** will be single column matrices, with the following elements:

$$\mathbf{L} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.045 \\ -0.065 \end{bmatrix} , \quad \mathbf{X} = \begin{bmatrix} dZ_1 \\ dZ_2 \\ dZ_3 \\ dZ_4 \end{bmatrix} .$$

The coefficient matrix of the system of normal (Gauss) equations following formula (6) will be:

$$(\mathbf{A}^T \mathbf{A}) = \mathbf{N} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad (20)$$

The determinant of this matrix is equal to zero, i.e. $\det(\mathbf{A}^T \mathbf{A}) = 0$, from which it follows that the matrix is rank deficient with defect $d = m - R[\mathbf{A}^T \mathbf{A}] = 3 - 2 = 1$, which means that this matrix does not have a common inverse. Following the scheme for dividing the matrix presented in Figure 1 for the matrix \mathbf{N} with $d = 1$, one distinguishes the matrices \mathbf{N}_n and \mathbf{N}_d and then calculates the generalized inverse matrix $\mathbf{N}\mathbf{N}$, which consequently leads to the pseudoinverse of matrix \mathbf{A} , as defined by equation (11).

After determining the matrices:

$$\mathbf{N}_n = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \quad \mathbf{N}_d = [-1 \ 0 \ -1].$$

the pseudoinverse of matrix is:

$$\mathbf{A}^+ = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \\ -1 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} 0.219 & 0.25 & 0.156 \\ 0.25 & 0.50 & 0.25 \\ 0.156 & 0.25 & 0.219 \end{bmatrix} \times \begin{bmatrix} -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix}.$$

Thus:

$$\mathbf{A}^+ = \begin{bmatrix} -0.25 & 0 & 0 & 0.25 & -0.25 \\ 0.375 & -0.375 & -0.125 & 0.125 & 0 \\ 0 & 0.25 & -0.25 & 0 & 0.25 \\ -0.125 & 0.125 & 0.375 & -0.375 & 0 \end{bmatrix} \quad (21)$$

After calculating equation (12), we obtain the estimated increments of the elevations of the points to their estimated values:

$$\hat{\mathbf{A}} = \mathbf{A}^+ \times \mathbf{L} = \begin{bmatrix} -0.25 & 0 & 0 & 0.25 & -0.25 \\ 0.375 & -0.375 & -0.125 & 0.125 & 0 \\ 0 & 0.25 & -0.25 & 0 & 0.25 \\ -0.125 & 0.125 & 0.375 & -0.375 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.045 \\ -0.065 \end{bmatrix}^T = \begin{bmatrix} 0.028 \\ 0.006 \\ -0.016 \\ -0.017 \end{bmatrix} \quad (22)$$

The modelled elevations of the measured points, which represent the most probable values, are the sums of the approximated elevations and the estimated increments of these elevations:

$$\begin{aligned} Z_1 &= Z_{10} + dZ_1 = 121.278 \text{ m}, \\ Z_2 &= Z_{20} + dZ_2 = 119.836 \text{ m}, \\ Z_3 &= Z_{30} + dZ_3 = 120.839 \text{ m}, \\ Z_4 &= Z_{40} + dZ_4 = 121.523 \text{ m}. \end{aligned}$$

The random deviations of the estimated model of measurement point elevations will be calculated using the formula:

$$\delta = \mathbf{L} - \mathbf{A} \times \mathbf{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.045 \\ -0.065 \end{bmatrix} - \begin{bmatrix} -0.022 \\ -0.022 \\ -0.001 \\ 0.044 \\ 0.044 \end{bmatrix} = \begin{bmatrix} 0.022 \\ 0.022 \\ 0.001 \\ 0.001 \\ -0.021 \end{bmatrix}.$$

Thus, the residue variance is:

$$\hat{\sigma}_0^2 = \frac{(\mathbf{L} - \mathbf{A}\hat{\mathbf{X}})^T (\mathbf{L} - \mathbf{A}\hat{\mathbf{X}})}{n - R(\mathbf{A})} = \frac{0.0001409}{5 - 3} = 0.000705.$$

The standard deviation from the random residues will be approximately $\sigma = 0.027$ m. Following equation (14) the covariance matrix for the modelled elevations of the measurement points is:

$$\text{Cov}[\hat{\mathbf{X}}] = 0.000705 \times \begin{bmatrix} 0.1875 & -0.0625 & -0.0625 & -0.0625 \\ /- / & 0.3125 & -0.0625 & 0.1875 \\ /- / & /- / & 0.1875 & -0.0625 \\ /- / & /- / & /- / & 0.3125 \end{bmatrix} \quad (23)$$

Based on the diagonal terms of the covariance matrix (23), we can calculate the standard deviations of the modelled values of the measured points:

$$\begin{aligned} \sigma[Z_1] &= 0.012 \text{ m}, \\ \sigma[Z_2] &= 0.015 \text{ m}, \\ \sigma[Z_3] &= 0.012 \text{ m}, \\ \sigma[Z_4] &= 0.015 \text{ m}. \end{aligned}$$

This analysis and these calculations allow us to formulate the final conclusion that the modelled elevations of the measured points have been determined with an error of between 0.012 m and 0.015 m.

The credibility of the model of observation results calculated following equation (3) is given by the indicator λ , the value of which in this case is:

$$\lambda = 1 - \frac{(\mathbf{L} - \mathbf{A}\hat{\mathbf{X}})^T (\mathbf{L} - \mathbf{A}\hat{\mathbf{X}})}{\sum (l_i - \hat{l})^2} = 1 - \frac{0.000705}{0.006170} = 1 - 0.11 = 0.89.$$

This shows that the reconciliation of the observation results provided by this model of observation equations yields a credibility of 89%.

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