Górnictwo i Geoinżynieria • Rok 30 • Zeszyt 2 • 2006

*Bogusław Ptaszyński**

MODELS OF AIR MIXING IN A MINE WORKING

1. Introduction and aim of the paper

Mixing is a physical process aimed at obtaining homogeneity of a mixture. Homogeneity is understood here as a state in which the concentrations of the mixture components are equal in the cross-section of the working or in the entire system.

Mixing may occur on two levels:

1) macroscopic — in an area defined by dimensions several times larger than the molecular dimensions of the components constituting the mixture;
2) molecular — when the state of homogeneity concerns an area defined by dimensions equal to the molecular dimensions of the mixture components.

It is essential to notice that macroscopic mixing is a stage in the course of molecular mixing. In mine ventilation, the process of mixing accompanies the mass transfer effect in underground excavations.

In theoretical dissertations on air flow in workings, usually three models of transfer are assumed. Two of these namely:

1) the ideal mixing model,
2) the piston flow model.

are the extreme, ideal theoretical models of mass transfer. The third transfer model, the so-called diffusion model, takes into account longitudinal diffusion. In some research of air flow in chamber workings other models may be applied [1], including those that take into consideration transverse diffusion. This article is not concerned with such cases.

The actual mass flow model in a mine working may be different from the aforementioned models, but at the same time it falls between the ideal mixing model and the piston flow model. In certain issues it is important to be able to point out which of these models used to represent the actual mass flow in the excavation yields lesser errors.

* Wydział Górnictwa i Geoinżynierii, Akademia Górniczo-Hutnicza, Kraków

The aim of the article is to present the form of the transient function Γ and the weight function E for various mass transfer models in a mine working and to consider their usefulness in research mapping the actual effect of mass transfer in the working.

2. The air dwell time in the working

The dwell time of air molecules in the working is a parameter related to the nature of its flow.

In case of turbulent flow in the mine working, vortices and stream disturbances of the flowing air result in the fact that the distance covered by particular elements of air volume in the area between the beginning and the end of the working is always different. In consequence, the dwell times of these air volume elements in the working are also different. Taking into account that it is a process of stochastic nature, it is impossible to determine the dwell time of a given air volume element in the working.

The value referred to as dwell time has the properties of a chance (random) variable and has distribution functions.

One distinguishes the function $E(t)$, called the dwell time spectrum. It is determined by interpreting the product of $E(t)\,dt$ as a fraction of air volume leaving the working, characterized by the fact that its dwell time falls within the time range from t to $t + dt$. It is to be inferred from this definition that the function $E(t)$ is the so-called dwell time probability density and the following relation occurs:

$$\int_0^\infty E(t)\,dt = 1$$

In physical system dynamics, the function $E(t)$ with above properties is a weight function. Another function characterizing the dynamics of air flow in a working is the function $F(t)$, called the distribution function or the cumulative distribution function of dwell time. The value of this function is determined by the fraction of air volume in the stream flowing out of the working, the dwell time of which falls within the time range from $t = 0$ to t. A relation known from statistics exists between these two functions:

$$F(t) = \int_0^t E(t)\,dt.$$

The function $F(t)$ can assume values from the range:

$$0 \le F(t) \le t.$$

One can also consider the function $J(t)$ being a complement of the function $F(t)$ to unity:

$$J(t) = 1 - F(t).$$

This function determines the fraction of molecules whose dwell time is longer than t in the outlet air stream. In order to portray the physical sense of the functions E(t), $F(t)$ and $J(t)$ the following example could be used [1].

Let us imagine an abrupt change of marker concentration in the air stream at the beginning of the working, from the value of 0 to C_1 at the moment $\tau = 0$. After the time τ at the end of the working, according to the definition of the value of $F(\tau)$, the volume fraction of the outlet air stream where the marker is located, i.e. whose molecules have stayed in the working for a period shorter than τ, amounts to $F(\tau)$. It can thus be reasoned that the remaining part of the stream (i.e. without the marker), equal to $(1 - F(\tau)) = J(\tau)$, incorporates the remaining fraction of the volume stream comprised of molecules that have stayed in the working for a period longer than τ. Hence the equation for the concentration of the marker C in the outlet air stream in the working could be stated as follows:

$$C = C_1 F(\tau) + 0\left[1 - F(\tau)\right],$$

therefore:

$$F(\tau) = F(t) = \frac{C}{C_1}.$$

Thus the function $F(t)$ corresponds to the relative response of the working as a system to the abrupt change of the concentration at its inlet (at zero initial conditions). This property grants possibilities of practical use of the function in researching actual air flow in mine workings.

2.1. The Ideal Mixing Model

It is assumed in this model that the stream flowing into the working immediately propagates in its entire volume, and the concentrations at any point of the working and at any given moment are identical and equal to the concentration in the stream flowing out of the working. The mass balance equation for the component in question, with the assumption that the mass source of that component is absent from the examined volume, takes the following form

$$LS\frac{\mathrm{d}C}{\mathrm{d}t} = QC_1 - QC \tag{1}$$

where:

- L — the length of the working [m],
- S — the cross-sectional area of the working [m^2],
- Q — the air volume expenditure in the working [m^3/s],
- C_1 — the concentration of the examined component at the inlet of the working,
- C — the concentration of the examined component in the working.

The solution of this equation at the initial condition of $C(t=0) = 0$ is given by the function

$$C(t) = C_1\left(1 - e^{-\frac{Q}{LS}t}\right). \tag{2}$$

From the equation (2) and the interpretation of the function $F(t)$ it can be inferred that the following relation occurs for the ideal mixing model

$$F(t) = \frac{C(t)}{C_1} = 1 - \exp\left(-\frac{t}{t_0}\right) \tag{3}$$

where $t_0 = \dfrac{LS}{Q}$ — is the so-called average dwell time of the air in the working.

Utilizing the relation between $F(t)$ and $E(t)$, the formula for the function $E(t)$ can be derived for the ideal mixing model in the form

$$E(t) = \frac{1}{t}\exp\left(-\frac{t}{t_0}\right) \tag{4}$$

2.2. The Piston Flow Model

The piston flow model presumes that during the air flow through the working, mixing does not occur, i.e. a particular gas component is transferred to the control volume only through convection in the entire mass.

If it is assumed that the gas component in question flows in only at the inlet of the working with a constant cross-sectional area (which means that there are no other sources of mass inflow to the working), one could state [2, 3]:

— the mass balance equation for the total transient flow of the mass stream in the working

$$\frac{\partial \rho}{\partial t} = -\frac{\partial (v\rho)}{\partial x} \tag{5}$$

— the mass balance equation for the gas marker

$$\frac{\partial C}{\partial t} = -\frac{\partial (vC)}{\partial x} \tag{6}$$

If one assumes that air density is the same in the working and does not vary in time, then it can be inferred from the equation (5) that:

$$0 = -\left(\frac{\partial v}{\partial x}\rho + \frac{\partial \rho}{\partial x} v\right) = -\frac{\partial v}{\partial x}\rho + 0v = -\frac{\partial v}{\partial x}\rho = -\frac{\partial v}{\partial x},$$

which indicates that air velocity in the working as a position function is constant.

Thus the equation (6) can be expressed in the form

$$\frac{\partial C}{\partial t} = -v\frac{\partial C}{\partial x} \tag{7}$$

This is a partial differential equation of the first degree, and the initial and boundary conditions of the problem in question are as follows

$$C(t=0,x) = C_0(x) = 0 \tag{8}$$

$$C(t=0,x) = C_1 \tag{9}$$

Applying the Laplace transformation to the problem described by the equations (7–9), the following ordinary differential equation in the image plane was obtained

$$v\frac{\mathrm{d}C(s)}{\mathrm{d}x} + sC(s) = 0 \tag{10}$$

with the boundary condition in the image plane

$$C(s,x=0) = C_1\frac{1}{s} \tag{11}$$

The solution of the problem in the complex number plane is given by the function

$$C(s,x) = C_1\frac{1}{s}e^{-\left(\frac{x}{v}s\right)} \tag{12}$$

After the inverse Laplace transformation of the equation (12), the following relation was obtained

$$C(t,x) = C_1 1\left(t-\frac{x}{v}\right) \tag{13}$$

where the function $1\left(t-\frac{x}{v}\right)$ — is a Heaviside unit step function.

Thus, the function $F(t)$ for the piston air flow model in the working takes the form

$$F(t) = \frac{C(t,x)}{C_1} = 1\left(t - \frac{x}{v}\right) \tag{14}$$

The function $E(t)$ for the piston flow model therefore takes the form

$$E(t) = \sigma\left(t - \frac{x}{v}\right) \tag{15}$$

where the function $\sigma\left(t - \frac{x}{v}\right)$ — is a Dirac pseudo-function.

2.3. The Longitudinal Diffusion Model

In the longitudinal diffusion model it is presumed that apart from the convection effect, an important role in mass transfer is played by the diffusion effect. This most often pertains to turbulent diffusion, which, aside from molecular diffusion, involves the effects of the nonideality of flow laws applied to actual conditions.

Assuming the applicability of Fick's law, the mass balance of the examined gas component (marker) can be expressed in the form

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial x}\left(vC - D\frac{\partial C}{\partial x}\right) \tag{16}$$

If it is presupposed that the turbulent longitudinal diffusion factor D and the velocity v are independent of the variable x, then the equation (16) can be stated as

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} - v\frac{\partial C}{\partial x} \tag{17}$$

The initial and boundary conditions for the examined problem are as follows:

$$C(t = 0, x) = C_0 = 0 \tag{18}$$

for $x = 0$:

$$vC - D\frac{\partial C}{\partial x} = vC_1 \tag{19}$$

for $x = L$:

$$D\frac{\partial C}{\partial x} = 0 \tag{20}$$

Introducing new variables in the equation (17)

$$\Gamma = \frac{C}{C1}; \quad \xi = \frac{x}{L}; \quad \theta = \frac{t}{\overline{\tau}} \tag{21}$$

one derives

$$\frac{\partial(\Gamma C_1)}{\partial(\theta\overline{\tau})} = D\frac{\partial^2(\Gamma C_1)}{\partial\left[(\xi L)^2\right]} - v\frac{\partial(\Gamma C_1)}{\partial(\xi L)} \tag{22}$$

which after transformations takes the form

$$\frac{C_1}{\overline{\tau}}\frac{\partial\Gamma}{\partial\theta} = D\frac{C_1}{L^2}\frac{\partial^2\Gamma}{\partial\xi^2} - v\frac{C_1}{L}\frac{\partial\Gamma}{\partial\xi} \tag{23}$$

Using the substitution

$$Pe = \frac{Lv}{D} \tag{24}$$

and the fact that

$$\overline{\tau} = \frac{L}{v} \tag{25}$$

the relation (23) can ultimately be expressed as

$$\frac{\partial\Gamma}{\partial\theta} = \frac{1}{Pe}\frac{\partial^2\Gamma}{\partial\xi^2} - \frac{\partial\Gamma}{\partial\xi} \tag{26}$$

with boundary and initial conditions in the form:

for $\xi = 0$ $\qquad \Gamma = 1 + \frac{1}{Pe}\frac{\partial\Gamma}{\partial\xi}$ (27)

for $\xi = 1$ $\qquad \frac{\partial\Gamma}{\partial\xi} = 0$ (28)

for $\theta = 0$ and for $1 \geq \xi > 0$ $\qquad \Gamma = 0$ (29)

Assuming for simplification that $Pe > 50$, the boundary condition (27) can be approximated as

$$\text{for } \xi = 0 \qquad \Gamma = 1 \tag{30}$$

After applying the substitution

$$\Gamma(\theta,\xi) = \Gamma_1(\theta,\xi)e^{\left(\frac{Pe}{2}\xi\right)} + 1 \tag{31}$$

the examined problem has the form

$$\frac{\partial\Gamma_1(\theta,\xi)}{\partial\theta} = \frac{1}{Pe}\frac{\partial^2\Gamma_1(\theta,\xi)}{\partial\xi^2} - \frac{Pe}{4}\Gamma_1(\theta,\xi) \tag{32}$$

In the equation (32) the term with the first derivative $\frac{\partial\Gamma_1}{\partial\xi}$ has already been dropped, and the boundary conditions take the form:

— for $\xi = 0$

$$\begin{aligned} &\Gamma(\theta,\xi=0) = 1 = \Gamma_1(\theta,\xi=0)e^0 + 1 = \Gamma_{1_0}(\theta,\xi=0) + 1 \\ &\Rightarrow \Gamma_{1_0}(\theta,\xi=0) = 0 \end{aligned} \tag{33}$$

— for $\xi = 1$

$$\begin{aligned} &\frac{\partial\,\Gamma(\theta,\xi=1)}{\partial\xi} = 0 = e^{\left(\frac{Pe}{2}\xi\right)}\left(\frac{\partial\Gamma_1}{\partial\xi} + \frac{Pe}{2}\Gamma_1\right)\Bigg|_{\xi=1} = e^{\left(\frac{Pe}{2}\right)}\left(\frac{\partial\Gamma_1}{\partial\xi} + \frac{Pe}{2}\Gamma_1\right)\Bigg|_{\xi=1} \\ &\Rightarrow \left(\frac{\partial\Gamma_1(\theta,\xi)}{\partial\xi} + \frac{Pe}{2}\Gamma_1(\theta,\xi)\right)\Bigg|_{\xi=1} = 0 \end{aligned} \tag{34}$$

— for $\theta = 0$

$$\begin{aligned} &\Gamma(\theta=0,\xi) = 0 = \Gamma_1(\theta=0,\xi)e^{\left(\frac{Pe}{2}\xi\right)} + 1 \\ &\Rightarrow \Gamma_1(\theta=0,\xi) = -e^{-\left(\frac{Pe}{2}\xi\right)} = \Gamma_1^0(\xi) \end{aligned} \tag{35}$$

Thus, after the above transformations, the initial problem described by the formulae (26–29) has been turned into the problem determined by the equations (32–35). This problem is homogeneous.

A non-trivial solution in the following form is sought

$$\Gamma_1(\theta,\xi) = X(\xi)\cdot T(\theta) \tag{36}$$

Substituting (36) in the equation (32), one obtains after reduction

$$\frac{X''(\xi)}{X(\xi)} = \frac{Pe\cdot T'(\theta) + \frac{Pe^2}{4}T(\theta)}{T(\theta)} \tag{37}$$

The boundary conditions have the form:

$$X(0) = 0 \tag{38}$$

$$\left. \left(X'(\xi) + \frac{Pe}{2}X(\xi) \right) \right|_{\xi=1} = 0 \tag{39}$$

Both sides of the equation (37) must be constants. Considering that constant as $\left(-\lambda_i^2\right)$, the equation (37) could be replaced by the following system:

$$\begin{cases} X''(\xi) + \lambda^2 X(\xi) = 0, \qquad 0 \le \xi \le 1 & (40) \\ T'(\theta) + \left(\frac{Pe}{4} + \frac{\lambda^2}{Pe} \right)\cdot T(\theta) = 0 & (41) \end{cases}$$

The solution of the equation (40) is the function

$$X(\xi) = C_1\cdot\cos(\lambda\xi) + C_2\sin(\lambda\xi) \tag{42}$$

The first boundary condition will be satisfied when $C_1 = 0$, which means that the form of the function (42) is as follows

$$X(\xi) = C_2\sin(\lambda\xi) \tag{43}$$

The second boundary condition will be satisfied for the roots of the following equation

$$\tan\lambda = -\frac{2}{Pe}\cdot\lambda \tag{44}$$

If the values λ_i satisfy the equation (44), then the solution of the equation (40) along with boundary conditions takes the form

$$X(\xi) = C_2 \sin(\lambda_i \cdot \xi) \qquad i = 1, 2, ..., n \tag{45}$$

The solution of the equation (41) is given by the following function

$$T(\theta) = C_3 \cdot \exp\left[-\left(\frac{Pe}{4} + \frac{\lambda_i^2}{Pe}\right)\cdot\theta\right] \tag{46}$$

The function Γ_1 has the form

$$\Gamma_1(\theta,\xi) = A_i \cdot \exp\left[-\left(\frac{Pe}{4} + \frac{\lambda_i^2}{Pe}\right)\cdot\theta\right]\cdot\sin(\lambda_i \cdot \xi) \tag{47}$$

The factor A_i, equal to the product of $(C_2 \cdot C_3)$, has a value attributed to the value i resulting from the equation (45).

Using the principle of superposition applicable to homogeneous equations, the solution can be expressed in the form

$$\Gamma_1(\theta,\xi) = \sum_{i=1}^{\infty} A_i \cdot \exp\left[-\left(\frac{Pe}{4} + \frac{\lambda_i^2}{Pe}\right)\cdot\theta\right]\cdot\sin(\lambda_i \cdot \xi) \tag{48}$$

For the function $\Gamma_1(\theta,\xi)$ to satisfy the initial condition determined by the equation (35), the following relation must occur

$$\sum_{i=1}^{\infty} A_i \cdot \sin(\lambda_i \cdot \xi) = -\exp\left(-\frac{Pe}{2}\cdot\xi\right) \tag{49}$$

The factors in this series must satisfy the relation

$$A_i = 2\int_0^1 \left[-\exp\left(-\frac{Pe}{2}\cdot\xi\right)\right]\cdot\sin(\lambda_i \cdot \xi)\,\mathrm{d}\xi \tag{50}$$

Once the above integral has been calculated, the factors in the series are expressed by the relation

$$A_i = \frac{-2\lambda_i}{\left(\frac{Pe}{2}\right)^2 + (\lambda_i)^2} \tag{51}$$

Substituting the above relation in the formula (48), one obtains

$$\Gamma_1(\theta,\xi) = \exp\left(-\frac{Pe}{4}\cdot\theta\right)\times\sum_{i=1}^{\infty}\left[\frac{-2\lambda_i}{\left(\frac{Pe}{2}\right)^2+(\lambda_i)^2}\cdot\exp\left(\frac{-\lambda_i^2}{Pe}\cdot\theta\right)\cdot\sin(\lambda_i\cdot\xi)\right] \tag{52}$$

Utilizing the substitution described by the formula (31), the ultimate formula can be presented

$$\Gamma(\theta,\xi) = 1 - e^{\left(\frac{Pe}{2}\cdot\xi\right)}\times\sum_{i=1}^{\infty}\left[\frac{-2\lambda_i}{\left(\frac{Pe}{2}\right)^2+(\lambda_i)^2}\cdot e^{-\left(\frac{Pe}{4}+\frac{\lambda_i^2}{Pe}\right)\cdot\theta}\cdot\sin(\lambda_i\cdot\xi)\right] \tag{53}$$

In the examined problem of the longitudinal diffusion model, (for the value $Pe \geq 50$) the transient function $F(t)$ has the form

$$F(t) = \Gamma(t,\xi=1) = 1 - e^{\left(\frac{Pe}{2}\right)}\times\sum_{i=1}^{\infty}\left[\frac{-2\lambda_i}{\left(\frac{Pe}{2}\right)^2+(\lambda_i)^2}\cdot e^{-\left(\frac{Pe}{4}+\frac{\lambda_i^2}{Pe}\right)\cdot\frac{t}{t_0}}\cdot\sin(\lambda_i)\right] \tag{54}$$

The weight function ***E*(*t*)** for this model is as follows

$$E(t) = \left(\frac{\mathrm{d}\Gamma(\theta,\xi)}{\mathrm{d}\theta}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\Bigg|_{\xi=1} = -e^{\left(\frac{Pe}{2}\right)}\cdot\frac{2}{Pe\cdot t_0}\times\sum_{i=1}^{\infty}\left[\frac{\lambda_i}{e^{\left(\frac{Pe}{4}+\frac{\lambda_i^2}{Pe}\right)\cdot\frac{t}{t_0}}}\cdot\sin\lambda_i\right] \tag{55}$$