Journal of Casting & Materials Engineering Vol. 5 No. 4 (2021) 53–60
https://doi.org/10.7494/jcme.2021.5.4.53

Derivation of Equations for a Size Distribution of Spherical Particles in Non-Transparent Materials

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Received: 25 May 2021/ Accepted: 9 September 2021/ Published online: 20 September 2021.
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Abstract

This paper presents a new proposition on how to derive mathematical formulas that describe an unknown Probability Density Function (PDF_3) of the spherical radii (r_3) of particles randomly placed in non-transparent materials. We have presented two attempts here, both of which are based on data collected from a random planar cross-section passed through space containing three-dimensional nodules. The first attempt uses a Probability Density Function (PDF_2) the form of which is experimentally obtained on the basis of a set containing two-dimensional radii (r_2). These radii are produced by an intersection of the space by a random plane. In turn, the second solution also uses an experimentally obtained Probability Density Function (PDF_1). But the form of PDF_1 has been created on the basis of a set containing chord lengths collected from a cross-section. The most important finding presented in this paper is the conclusion that if the PDF_1 has proportional scopes, the PDF_3 must have a constant value in these scopes. This fact allows stating that there are no nodules in the sample space that have particular radii belonging to the proportional ranges the PDF_1.

Keywords:

planimetric analysis, linear analysis, estimation of diameter sizes distribution, probability density function

1. INTRODUCTION

The problem of mapping an unknown probability distribution of spherical particles size, randomly placed in non-transparent materials, is one of the classical stereological tasks. The data (a set of mark radii, an area of intersections or chord lengths) used in such an analysis, is obtained from an examination of flat cross-sections (random cutting planes). Doing such research, it is important to be aware of two facts:

1. generally, a random cutting plane intersects three-dimensional particles outside their centre points. Hence, a mean value of mark radii, an area of intersections or chord lengths differs from the mean value of the linear sizes of nodules,
2. a random cutting plane intersects bigger particles more often than smaller ones, rather than this results from the true count of the particles.

To solve this stereological problem, many attempts have been proposed and those solutions can be categorised into three groups. The first group includes methods that use mark radii as input data [1–3]. The second group uses areas of intersected nodules [4]. The last one employs a set of random chords [5–7].

The methods from the first group are currently most frequently used for an analysis of the graphite size distribution in ductile iron [8–10]. The accuracy of these methods has been improved in [11].

Our paper develops the method from the third group. New aspects which distinguish our work from others are different form of a constant parameter in the final equation and the statement of the fact that if PDF_1 has proportional ranges, the sample space does not have nodules with a particular size belonging to these proportional scopes.

This article is also an extension of our previous work [12] which deals with the same problem. However, the information included there is too concise and required an extension and unification of variables and symbols. We decided here to add more explanations and figures that help understand the derivation of the formulas for readers who are unfamiliar with probability theory and mathematical statistics. We have also proposed another way for the estimation of the expected value of the outer particles' area.

A list of variables used in the text can be found at the end of the article.

The data, i.e. a set of chords, creates the function $f_1(x)$ in a form of a histogram which is not a continuous function. But, in a computer image analysis this is not a big problem. We can get a large enough collection of chords to estimate the function $f_1(x)$ with practically any small step and hence to produce the function $f_3(x)$ with satisfactory accuracy. Equation (36) includes also an unknown value of $\overline{S}$. We will propose now how to estimate this value.

Let us integrate both sides of Equation (36) in the range from 0 to R_{max}:

$$\int_0^{R_{max}} f_3(x)\mathrm{d}x = \frac{\overline{S}}{8\pi}\int_0^{R_{max}}\left(\frac{f_1(x)}{x^2} - \frac{1}{x}\frac{\mathrm{d}f_1(x)}{\mathrm{d}x}\right)\mathrm{d}x \tag{38}$$

According to Equation (4) the left hand side integral of the above formula is equal to 1. After rearranging we get an equation that allows us to estimate the value of $\overline{S}$ on the basis of experimental data:

$$\overline{S} = \frac{8\pi}{\int_0^{R_{max}}\left(\frac{f_1(x)}{x^2} + \frac{1}{x}\frac{\mathrm{d}f_1(x)}{\mathrm{d}x}\right)\mathrm{d}x} \tag{39}$$

An analysis of Equation (34) allows us to detect intervals in which there are no nodules with the radius belonging to those ranges. Suppose, we have a range from x_1 to x_2. If this range does not contain any radius r_3, this means that the function $F_3(x)$ has a constant value. This fact concludes that the function $f_1(x)$ depends proportionally on the variable x, i.e. $f_1(x) = c \cdot x$. The proportional coefficient c is equal:

$$c = \frac{8\pi}{\overline{S}}\left(1 - F_3(x)\right) \tag{40}$$

The above coefficient has a sense of a tangent of the angle between the X axis and the proportional range of the function $f_1(x)$. This tells us that not only each interval with no particles is proportional, but also its direction passes through the value of 0 on the X axis.

Let us consider a particular situation when the sample space is bounded from the bottom – it contains nodules with a minimal radius R_{min}. An example of such a sample is depicted in Figure 4 and corresponding histograms of PDF_1 and PDF_3 are shown in Figure 5.

Notice, even if in the sample there are no nodules with a radius of less than R_{min}, the collected data contains chords with a length less than the value of R_{min}. The shorter chords can be obtained not only by piercing the minimal particles, but also from larger nodules that are pierced by a random secant at a large enough distance from their centres.

As it is schematically presented in Figure 5b the $f_3(x)$ is equal to 0 for $x < R_{min}$. The zero value of the $f_3(x)$ also results a zero value of its $F_3(x)$. Because of this reason, the function (34) and its histogram have a proportional character in a range from 0 to x_a:

$$f_1(x) = \frac{8\pi}{\overline{S}}x \tag{41}$$

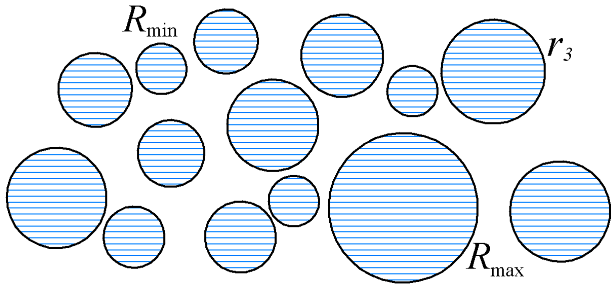

Fig. 4. Set of chords (blue lines) obtained from a sample that contains nodules with a minimal radius R_{min}

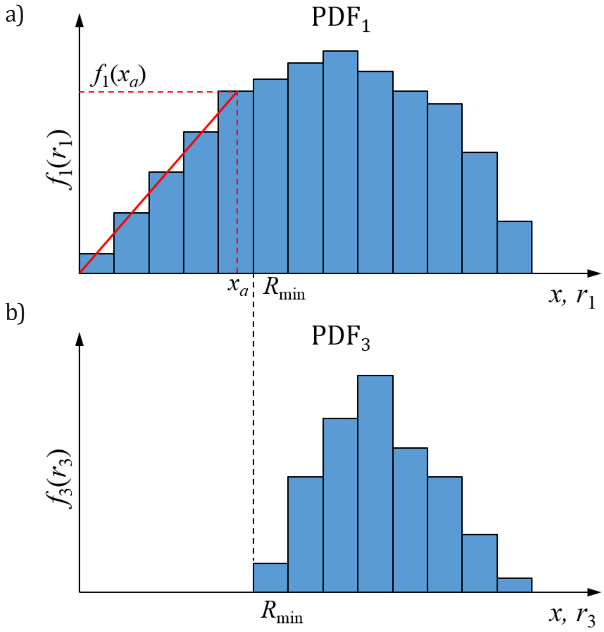

Fig. 5. Scheme presenting how to graphically determine the value of x_a needed to estimate the unknown parameter: a) a histogram of a chord size distribution; b) a histogram of a nodule size distribution

This fact allows us to state that if the beginning of a histogram of the PDF_1 is proportional, this range does not contain particles with this size because the $F_3(x)$ is zero. The proportional dependency can be determined graphically as it is presented on the histogram in Figure 5a. In this case, having the values of x_a and $f_1(x_a)$ we can use a simpler formula to estimate the unknown $\overline{S}$ by substituting these values to Equation (41) in order to get:

$$\overline{S} = \frac{8\pi}{f_1(x_a)}x_a \tag{42}$$

Further analysis of Equation (34) permits us to extend the property described in the previous paragraph. This means that if any range of the $f_1(x)$ histogram has a proportional dependency, the function $F_3(x)$ must be constant in this range to fulfil this proportionality. In turn, according to the obvious relation in Equation (6) the function $f_3(x)$ takes the value of 0 everywhere where $F_3(x)$ is constant. This allows us to claim that each proportional scope of the PDF_1 has no nodules with radii which belong to this range.