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Dedicated to Professor Semyon Yakubovich on the occasion of his 50th birthday

ENERGY INTEGRAL OF THE STOKES FLOW IN A SINGULARLY PERTURBED EXTERIOR DOMAIN

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Abstract. We consider a pair of domains Ω^b and Ω^s in $\mathbb{R}^n$ and we assume that the closure of Ω^b does not intersect the closure of $\epsilon\Omega^s$ for $\epsilon \in (0,\epsilon_0)$. Then for a fixed $\epsilon \in (0,\epsilon_0)$ we consider a boundary value problem in $\mathbb{R}^n \setminus (\Omega^b \cup \epsilon\Omega^s)$ which describes the steady state Stokes flow of an incompressible viscous fluid past a body occupying the domain Ω^b and past a small impurity occupying the domain $\epsilon\Omega^s$. The unknown of the problem are the velocity field u and the pressure field p, and we impose the value of the velocity field u on the boundary both of the body and of the impurity. We assume that the boundary velocity on the impurity displays an arbitrarily strong singularity when ϵ tends to 0. The goal is to understand the behaviour of the strain energy of (u,p) for ϵ small and positive. The methods developed aim at representing the limiting behaviour in terms of analytic maps and possibly singular but completely known functions of ϵ, such as ϵ^{-1}, $\log \epsilon$.

Keywords: boundary value problem for the Stokes system, singularly perturbed exterior domain, real analytic continuation in Banach space.

Mathematics Subject Classification: 76D07, 35J57, 31B10, 45F15.

1. INTRODUCTION

In this paper we present an application of a functional analytic approach to the analysis of a boundary value problem for the Stokes system in a singularly perturbed exterior domain. We now introduce our boundary value problem. We fix once and for all

$$n \in \mathbb{N} \setminus \{0,1\}, \qquad \alpha \in (0,1).$$

Here $\mathbb{N}$ denotes the set of natural numbers including 0. Then we fix two sets Ω^b and Ω^s in the n-dimensional Euclidean space $\mathbb{R}^n$. The letter "b" stands for "body" and the letter "s" stands for "small impurity". We assume that Ω^b and Ω^s satisfy the following condition:

$$\begin{aligned}&\Omega^b \text{ and } \Omega^s \text{ are open bounded connected subsets of } \mathbb{R}^n \text{ of}\\&\text{class } C^{1,\alpha} \text{ with connected exterior, the origin 0 of } \mathbb{R}^n\\&\text{belongs to } \Omega^s \text{ but not to the closure } \mathrm{cl}\Omega^b \text{ of } \Omega^b.\end{aligned} \tag{1.1}$$

For the definition of the functions and sets of the usual Schauder class $C^{0,\alpha}$, $C^{1,\alpha}$ we refer for example to Gilbarg and Trudinger [8, §§4.1,6.2] (see also [3, §2]). We note that condition (1.1) implies that Ω^b and Ω^s have no holes and that there exists a real number ϵ_0 such that

$$\epsilon_0 \in (0,1) \text{ and } \mathrm{cl}\Omega^b \cap (\epsilon\mathrm{cl}\Omega^s) = \emptyset \text{ for all } \epsilon \in (0,\epsilon_0). \tag{1.2}$$

Then we denote by $\Omega^e(\epsilon)$ the exterior domain defined by

$$\Omega^e(\epsilon) \equiv \mathbb{R}^n \setminus \{\mathrm{cl}\Omega^b \cup (\epsilon\mathrm{cl}\Omega^s)\}, \qquad \forall \epsilon \in (0,\epsilon_0).$$

Here the letter "e" stands for "exterior domain". A simple topological argument shows that $\Omega^e(\epsilon)$ is connected, and that $\mathbb{R}^n \setminus \mathrm{cl}\Omega^e(\epsilon)$ has exactly the two connected components Ω^b and $\epsilon\Omega^s$, and that the boundary $\partial\Omega^e(\epsilon)$ of $\Omega^e(\epsilon)$ has exactly the two connected components $\partial\Omega^b$ and $\epsilon\partial\Omega^s$, for all $\epsilon \in (0,\epsilon_0)$.

Now,

$$\begin{aligned}&\text{let } \gamma \text{ be a function from } (0,\epsilon_0) \text{ to } (0,+\infty)\\&\text{such that } \gamma_0 \equiv \lim_{\epsilon\to 0}\gamma(\epsilon) \text{ exists in } [0,+\infty).\end{aligned} \tag{1.3}$$

Let $(f^b, f^s) \in C^{1,\alpha}(\partial\Omega^b,\mathbb{R}^n) \times C^{1,\alpha}(\partial\Omega^s,\mathbb{R}^n)$. Let $\epsilon \in (0,\epsilon_0)$. We consider the following boundary value problem for a pair (u,p) of $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega^e(\epsilon),\mathbb{R}^n) \times C^{0,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega^e(\epsilon),\mathbb{R})$,

$$\begin{cases}\Delta u - \nabla p = 0 & \text{in } \Omega^e(\epsilon),\\ \operatorname{div} u = 0 & \text{in } \Omega^e(\epsilon),\\ u = f^b & \text{on } \partial\Omega^b,\\ u(x) = \gamma(\epsilon)^{-1} f^s(x/\epsilon) & \text{for } x \in \epsilon\partial\Omega^s.\end{cases} \tag{1.4}$$

(For the definition of $C^{1,\alpha}_{\mathrm{loc}}$ and $C^{0,\alpha}_{\mathrm{loc}}$ we refer, e.g., to [3, §2].) As is well known problem (1.4) admits a unique solution which satisfies the decay condition

$$\sup_{|x|>R}\Big\{|x|^{n-2}|u(x)|,\, |x|^{n-1}|Du(x)|,\, |x|^{n-1}|p(x)|\Big\} < +\infty \tag{1.5}$$

with $R \equiv \sup_{x\in\{\Omega^b\cup(\epsilon\Omega^s)\}}|x|$. The uniqueness of the solution of (1.4), (1.5) can be deduced by the results of Chang and Finn in [1, §4], we refer to Varnhorn [23, Lemma 1.1] for a proof. For a proof of the existence in case $n \geq 3$ we refer to Ladyzhenskaya [15, Chap. 3, Sec. 3], for a proof of the existence in case $n = 2$ we refer to Power [21, §§2,3].

We denote by (u_ϵ, p_ϵ) the unique solution in the space $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega^e(\epsilon), \mathbb{R}^n) \times C^{0,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega^e(\epsilon), \mathbb{R})$ of (1.4), (1.5) with $R \equiv \sup_{x \in \{\Omega^b \cup (\epsilon\Omega^s)\}} |x|$. Then we denote by $\mathcal{E}_\epsilon$ the strain energy integral of the Stokes flow (u_ϵ, p_ϵ). Namely, we set

$$\mathcal{E}_\epsilon \equiv \frac{1}{2} \int_{\Omega^e(\epsilon)} \mathrm{tr}\left[(Du_\epsilon + D^t u_\epsilon) \cdot (Du_\epsilon + D^t u_\epsilon)\right] dx, \qquad \forall \epsilon \in (0, \epsilon_0). \tag{1.6}$$

The aim of this paper is to understand the behaviour of $\mathcal{E}_\epsilon$ when ϵ shrinks to 0. If $\gamma_0 = 0$ then the velocity field u_ϵ on the boundary of $\epsilon\Omega^s$ displays a singular behaviour as $\epsilon \to 0$ due to the presence of the factor $\gamma(\epsilon)^{-1}$ in the right hand side of the fourth equation in (1.4). As we shall see in our main Theorem 4.1, the rate of convergence of γ as $\epsilon \to 0$ and its limit value γ_0 determine the limiting behaviour of $\mathcal{E}_\epsilon$. In particular, if

$$\gamma_1 \equiv \lim_{\epsilon \to 0} \frac{\epsilon^{n-2}}{\gamma(\epsilon)^2} \quad \text{exists in } [0, +\infty), \tag{1.7}$$

then Theorem 4.1 implies that

$$\lim_{\epsilon \to 0} \mathcal{E}_\epsilon = \mathcal{E}^b + \gamma_1 \mathcal{E}^s, \tag{1.8}$$

where $\mathcal{E}^b$ denotes the energy integral of the unique solution (u^b, p^b) in the space $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega^b, \mathbb{R}^n) \times C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega^b, \mathbb{R})$ of the “unperturbed boundary value problem”

$$\begin{cases} \Delta u^b - \nabla p^b = 0 & \text{in } \mathbb{R}^n \setminus \mathrm{cl}\Omega^b, \\ \mathrm{div}\, u^b = 0 & \text{in } \mathbb{R}^n \setminus \mathrm{cl}\Omega^b, \\ u^b = f^b & \text{on } \partial\Omega^b, \end{cases} \tag{1.9}$$

which satisfies the decay condition in (1.5) for $R \equiv \sup_{x \in \Omega^b} |x|$, and $\mathcal{E}^s$ denotes the energy integral of the unique solution $(u^s, p^s) \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega^s, \mathbb{R}^n) \times C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega^s, \mathbb{R})$ of the “limiting boundary value problem”

$$\begin{cases} \Delta u^s - \nabla p^s = 0 & \text{in } \mathbb{R}^n \setminus \mathrm{cl}\Omega^s, \\ \mathrm{div}\, u^s = 0 & \text{in } \mathbb{R}^n \setminus \mathrm{cl}\Omega^s, \\ u^s = f^s & \text{on } \partial\Omega^s, \end{cases} \tag{1.10}$$

which satisfies the decay condition in (1.5) for $R \equiv \sup_{x \in \Omega^s} |x|$. Namely, we set

$$\mathcal{E}^b \equiv \frac{1}{2} \int_{\mathbb{R}^n \setminus \Omega^b} \mathrm{tr}\left[(Du^b + D^t u^b) \cdot (Du^b + D^t u^b)\right] dx, \tag{1.11}$$

$$\mathcal{E}^s \equiv \frac{1}{2} \int_{\mathbb{R}^n \setminus \Omega^s} \mathrm{tr}\left[(Du^s + D^t u^s) \cdot (Du^s + D^t u^s)\right] dx. \tag{1.12}$$

It is worth noting that condition (1.7) can be satisfied also with $\gamma_0 = 0$. In particular, the strain energy $\mathcal{E}_\epsilon$ may converge to the finite value $\mathcal{E}^b + \gamma_1 \mathcal{E}^s$ as $\epsilon \to 0$ also when the velocity on the boundary of the impurity displays a singular behaviour for $\epsilon \to 0$. Moreover, if $n = 2$, then the limit in (1.8) differs from $\mathcal{E}^b$ also for $\gamma_0 > 0$ (cf. condition (1.7)). If instead $\gamma_0 = 0$ and

$$\lim_{\epsilon \to 0} \frac{\epsilon^{n-2}}{\gamma(\epsilon)^2} \quad \text{does not exist in } \mathbb{R}, \tag{1.13}$$

then $\mathcal{E}_\epsilon$ may display a singular behaviour as $\epsilon \to 0$. In this case we can show that

$$\lim_{\epsilon \to 0} \frac{\gamma(\epsilon)^2}{\epsilon^{n-2}} (\mathcal{E}_\epsilon - \mathcal{E}^b) = \mathcal{E}^s. \tag{1.14}$$

However, our main interest is focused on the description of the behaviour of $\mathcal{E}_\epsilon$ when ϵ is near 0, and not only on the limiting behaviour. Actually, we pose the following question:

What can be said of the function which takes ϵ to $\mathcal{E}_\epsilon$ when ϵ is small and positive? (1.15)

Questions of this type have long been investigated with the methods of asymptotic analysis, which aims at giving complete asymptotic expansions of the solutions in terms of the parameter ϵ. It is perhaps difficult to provide a complete list of the contributions. Here, we mention the books of Kozlov, Maz'ya and Movchan [14] and Maz'ya, Nazarov and Plamenewskii [20] for the analysis of general elliptic boundary value problems in singularly perturbed domains by means of the so called compound asymptotic expansion method. For application of the matched asymptotic expansion method to the analysis of singularly perturbed problems in fluid mechanics we refer to the book of Van Dyke [22]. For the asymptotic analysis of a Navier-Stokes flow with low Reynold number we mention the work of Kevorkian and Cole [11], Hsiao and MacCamy [10], Hsiao [9], Fischer, Hsiao and Wendland [7]. For application of the matched asymptotic expansion method to the analysis of a Stokes flow past a porous media with low or high permeability we mention Kohr, Sekhar and Wendland [12,13]. We note that, by the techniques of the asymptotic analysis, one can expect to obtain results which are expressed by means of a regular functions of ϵ plus a remainder which is smaller than a positive known function of ϵ. Instead, the approach adopted in this paper aims to express the dependence upon ϵ in terms of real analytic maps and in terms of possibly singular but completely known functions of ϵ such as ϵ^{-1}, $\log \epsilon$, $\gamma(\epsilon)^{-1}$. In particular, our main Theorem 4.1 shows that for ϵ small and positive $\mathcal{E}_\epsilon$ can be expressed by means of a real analytic map of four variables defined in a neighborhood of $(0, 0, 1 - \delta_{2,n}, \gamma_0)$ and evaluated at $(\epsilon, \epsilon(\log \epsilon)^{\delta_{2,n}}, (\log \epsilon)^{-\delta_{2,n}}, \gamma(\epsilon))$ and by means of the possibly singular function $\epsilon^{n-2}/\gamma(\epsilon)^2$ of $\epsilon \in (0, \epsilon_0)$. This point of view has been adopted by Lanza de Cristoforis and his collaborators in several problems for elliptic equations (see, e.g., [16–19]), and for the elliptic system of equation of linearized elastostatic (see, e.g., [4–6]), and for the Stokes system (see [2, 3].) In this paper we present an application of the results in [2, 3] to answer to the question in (1.15).

The paper is organized as follows. Section 2 is a section of preliminaries, where we introduce some notation and we show the validity of a suitable Green formula for an exterior Stokes flow (u,p) which satisfies the decay condition in (1.5). In Section 3 we collect some results of [2] and [3]. In particular, in Theorem 3.2 we consider the map which takes ϵ to $(u_{\epsilon|\mathrm{cl}\Omega_M}, p_{\epsilon|\mathrm{cl}\Omega_M})$ for ϵ small and positive. Here Ω_M is an open bounded subset of $\mathbb{R}^n$ contained in $\mathbb{R}^n \setminus \Omega^b$ such that $0 \notin \mathrm{cl}\Omega_M$. The letter "$M$" stands for "macroscopic". Instead in Theorem 3.3 we investigate the "microscopic" behaviour of (u_ϵ, p_ϵ) in the proximity of $\epsilon\Omega^s$. To do so, we consider the function which takes ϵ to $(u_\epsilon(\epsilon\,\cdot\,)_{|\mathrm{cl}\Omega_m}, p_\epsilon(\epsilon\,\cdot\,)_{|\mathrm{cl}\Omega_m})$ for ϵ small and positive. Here Ω_m is an open bounded set contained in $\mathbb{R}^n \setminus \Omega^s$ and the letter "m" stands for 'microscopic". In the Section 4, we prove our main Theorem 4.1 which answers the question in (1.15). Moreover, Theorem 4.1 implies the validity of the limits in (1.8) and (1.14) under conditions (1.7) and (1.13), respectively (cf. equalities (4.2) and (4.3)).

2. SOME PRELIMINARIES

We denote by $S_{V,n} \equiv (S^{i,j}_{V,n})_{i,j=1,\dots,n}$, $S_{P,n} \equiv (S^i_{P,n})_{i=1,\dots,n}$ the functions from $\mathbb{R}^n \setminus \{0\}$ to $M_n(\mathbb{R})$ and to $\mathbb{R}^n$, respectively, defined by

$$S^{i,j}_{V,n}(x) \equiv \frac{1}{2s_n(\delta_{2,n} + (2-n))} \Big\{ \left(\delta_{2,n} \log|x| + (1-\delta_{2,n})|x|^{2-n}\right) \delta_{i,j} - \\ - (\delta_{2,n} + (2-n)) \frac{x_i x_j}{|x|^n} \Big\},$$

$$S^i_{P,n}(x) \equiv -\frac{1}{s_n} \frac{x_i}{|x|^n}$$

for all $i,j \in \{1,\dots,n\}$ and all $x \in \mathbb{R}^n \setminus \{0\}$. Here $M_n(\mathbb{R})$ denotes the space of real $n \times n$-matrices, and s_n denotes the $(n-1)$ dimensional measure of the unit sphere $\partial\mathbb{B}_n$ in $\mathbb{R}^n$, and $\delta_{i,j}$ is defined by $\delta_{i,j} \equiv 0$ if $i \neq j$ and $\delta_{i,j} \equiv 1$ if $i = j$, for all $i,j \in \mathbb{N}$. We also find convenient to set

$$S^j_{V,n} \equiv (S^{i,j}_{V,n})_{i=1,\dots,n},$$

which we think of as column vectors for all $j \in \{1,\dots,n\}$. For each scalar $\rho \in \mathbb{R}$ and each matrix $A \in M_n(\mathbb{R})$ we set

$$\mathcal{T}(\rho, A) \equiv -\rho\, I + (A + A^t),$$

where I denotes the unit matrix in $M_n(\mathbb{R})$ and A^t denotes the transpose matrix to A. Now, let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. We denote by $\partial\Omega$ the boundary of Ω and by ν_Ω the outward unit normal to $\partial\Omega$. We set

$$w_V[\mu](x) \equiv -\left\{\int_{\partial\Omega} \mu^t(y)\mathcal{T}(S^j_{P,n}(x-y), D_x S^j_{V,n}(x-y))\nu_\Omega(y)\, d\sigma_y\right\}_{j=1,\dots,n},$$

$$w_P[\mu](x) \equiv -2\,\mathrm{div}\left(\int_{\partial\Omega} \mu^t(y) S_{P,n}(x-y)\,\nu_\Omega(y)\; d\sigma_y\right), \qquad \forall x \in \mathbb{R}^n,$$

for all $\mu \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^n)$.

Then we have the following Lemma 2.1, where we show that the decay condition of a solution (u,p) of problem (1.4), (1.5) can be improved in the case $n = 2$. In the sequel $\mathbb{B}_n$ denotes the open unit ball in $\mathbb{R}^n$, namely we set $\mathbb{B}_n \equiv \{x \in \mathbb{R}^n : |x| < 1\}$.

Lemma 2.1. *Let Ω be an open bounded subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$. Let (u,p) be a pair of function of $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Omega, \mathbb{R}^2) \times C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Omega, \mathbb{R})$ such that*

$$\Delta u - \nabla p = 0 \text{ and } \mathrm{div}\, u = 0 \text{ in } \mathbb{R}^2 \setminus \mathrm{cl}\Omega.$$

Assume that (u,p) satisfies the decay condition in (1.5) for $R \equiv \sup_{x\in\Omega} |x|$. Then there exists a vector $\lambda \in \mathbb{R}^2$ such that

$$\sup_{|x|>R} \left\{|x|\,|u(x) - \lambda|,\ |x|^2\,|Du(x)|,\ |x|^2\,|p(x)|\right\} < \infty.$$

Proof. Let $R \equiv \sup_{x\in\Omega} |x|$. Let Z_2 denote the skew-symmetric matrix defined by

$$(Z_2)_{ij} \equiv \delta_{i,1}\delta_{j,2} - \delta_{i,2}\delta_{j,1}, \qquad \forall i,j \in \{1,2\}.$$

Then there exists $\mu \in C^{1,\alpha}(R\partial\mathbb{B}_2, \mathbb{R}^2)$ such that

$$u(x) = w_V[\mu](x) + \frac{1}{4\pi}\int_{R\partial\mathbb{B}_2} \mu\, d\sigma + \frac{Z_2 \cdot x}{|x|^2}\int_{R\partial\mathbb{B}_2} \mu^t(y)\, Z_2 \cdot y\, d\sigma_y,$$

$$p(x) = w_P[\mu](x,) \qquad \forall x \in \mathbb{R}^2 \setminus R\mathbb{B}_2$$

(cf. Power [21, §2], see also [3, Prop. 3.2]). We set $\lambda \equiv (4\pi)^{-1}\int_{R\partial\mathbb{B}_2} \mu\, d\sigma$. Then the validity of Lemma follows by a straightforward calculation. □

Now we have the following Lemma 2.2, where we introduce a suitable Green formula for an exterior Stokes flow (u,p).

Lemma 2.2. *Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ such that $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ is connected. Let (u,p) be a pair of function of $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega, \mathbb{R}^n) \times C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega, \mathbb{R})$ such that*

$$\Delta u - \nabla p = 0 \text{ and } \mathrm{div}\, u = 0 \text{ in } \mathbb{R}^n \setminus \mathrm{cl}\Omega.$$

Assume that (u,p) satisfies the decay condition in (1.5) for $R \equiv \sup_{x\in\Omega} |x|$. Then the following equality holds:

$$\frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega} \mathrm{tr}\left[(Du + D^t u)\cdot(Du + D^t u)\right] dx = -\int_{\partial\Omega} u^t\, \mathcal{T}(p, Du)\nu_\Omega\, d\sigma.$$

Proof. Let $\rho > 0$ be such that $\mathrm{cl}\Omega \subseteq \rho\mathbb{B}_n$. By the Divergence Theorem we deduce that

$$\begin{aligned}
&\frac{1}{2}\int_{(\rho\mathbb{B}_n)\setminus\Omega} \mathrm{tr}\left[(Du(x)+D^t u(x))\cdot(Du(x)+D^t u(x))\right] dx = \\
&= -\int_{\partial\Omega} u^t(x)\,\mathcal{T}(p(x),Du(x))\nu_\Omega(x)\,d\sigma_x + \\
&\quad + \int_{\rho\partial\mathbb{B}_n} u^t(x)\,\mathcal{T}(p(x),Du(x))\nu_{\rho\mathbb{B}_n}(x)\,d\sigma_x.
\end{aligned} \tag{2.1}$$

By the decay conditions in (1.5) and by Lemma 2.1 we verify that

$$\lim_{\rho\to\infty}\int_{\rho\partial\mathbb{B}_n} u^t(x)\,\mathcal{T}(p(x),Du(x))\nu_{\rho\mathbb{B}_n}(x)\,d\sigma_x = 0.$$

Hence the validity of the lemma follows by taking the limit as $\rho \to \infty$ in equality (2.1) and by a standard argument based on the Monotone Convergence Theorem. □

3. REAL ANALYTIC REPRESENTATION THEOREMS FOR THE SOLUTION (u_ϵ, p_ϵ)

In Theorem 3.2 we describe the limiting behaviour of (u_ϵ, p_ϵ) when $\epsilon \to 0$. To do so, we need the following Lemma 3.1 where we introduce the auxiliary pair of functions (g^V_Ω, g^P_Ω). The validity of Lemma 3.1 can be verified by a standard argument based on the ellipticity properties of the Stokes system (see also [3, Lemma 5.2] for a proof).

Lemma 3.1. *Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ such that $0 \notin \mathrm{cl}\Omega$. Then there exists a unique pair of functions (g^V_Ω, g^P_Ω) in the space $L^1_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega, M_n(\mathbb{R}))\times L^1_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega, \mathbb{R}^n)$ such that*

$$\Delta g^V_\Omega - \nabla g^P_\Omega = \delta_0 I \text{ and } \mathrm{div} g^V_\Omega = 0 \text{ in the sense of distributions in } \mathbb{R}^n\setminus\mathrm{cl}\Omega$$

and such that the following conditions hold:

(i) *If $B \subseteq \mathbb{R}^n\setminus\mathrm{cl}\Omega$ is open and bounded and $0 \notin \mathrm{cl}B$, then $g^V_{\Omega|B}$ extends to a function of $C^{1,\alpha}(\mathrm{cl}B, M_n(\mathbb{R}))$ and $g^P_{\Omega|B}$ extends to a function of $C^{0,\alpha}(\mathrm{cl}B,\mathbb{R}^n)$,*
(ii) $g^V_{\Omega|\partial\Omega} = 0$,
(iii) *g^V_Ω and g^P_Ω satisfy the decay condition in (1.5) for $R \equiv \sup_{x\in\Omega}|x|$.*

We are now ready to state our Theorem 3.2, where we consider the behaviour of (u_ϵ, p_ϵ) when ϵ shrinks to 0. For a proof we refer to [3, Th. 5.1, 5.3]. To simplify our notation we introduce the following definition. Let ϵ_0 be as in (1.2). Let γ, γ_0 be as in (1.3). Then we denote by $\Psi_n[\cdot]$ the function from $(0,\epsilon_0)$ to $\mathbb{R}^4$ defined by

$$\Psi_n[\epsilon] \equiv \left(\epsilon, \epsilon(\log\epsilon)^{\delta_{2,n}}, (\log\epsilon)^{-\delta_{2,n}}, \gamma(\epsilon)\right), \qquad \forall \epsilon \in (0,\epsilon_0). \tag{3.1}$$

We note that $\lim_{\epsilon\to 0}\Psi_n[\epsilon]=(0,0,1-\delta_{2,n},\gamma_0)$.

Theorem 3.2. *Let Ω^b and Ω^s be as in* (1.1). *Let ϵ_0 be as in* (1.2). *Let γ, γ_0 be as in* (1.3). *Let $(f^b,f^s)\in C^{1,\alpha}(\partial\Omega^b,\mathbb{R}^n)\times C^{1,\alpha}(\partial\Omega^s,\mathbb{R}^n)$. Let $(g^V_{\Omega^b},g^P_{\Omega^b})$ be as in Lemma 3.1 with $\Omega=\Omega^b$. Let (u^b,p^b) denote the unique solution in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega^b,\mathbb{R}^n)\times C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega^b,\mathbb{R})$ of the boundary value problem in* (1.9) *which satisfies the decay condition in* (1.5) *for $R\equiv\sup_{x\in\Omega^b}|x|$. Let (u^s,p^s) denote the unique solution in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega^s,\mathbb{R}^n)\times C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega^s,\mathbb{R})$ of the boundary value problem in* (1.10) *which satisfy the decay condition in* (1.5) *for $R\equiv\sup_{x\in\Omega^s}|x|$. Let τ^s, $\lambda^s\in\mathbb{R}^n$ be defined by*

$$\tau^s\equiv\int_{\partial\Omega^s}\mathcal{T}(p^s,Du^s)\nu_{\Omega^s}\,d\sigma,\qquad \lambda^s\equiv\lim_{|x|\to\infty}u^s(x)$$

(*see also Lemma 2.1*). *Let $\Omega_M\subseteq\mathbb{R}^n\setminus\mathrm{cl}\Omega^b$ be open bounded and such that $0\notin\mathrm{cl}\Omega_M$. Then, there exist $\epsilon_{\Omega_M}\in(0,\epsilon_0)$, and an open neighborhood $\mathcal{U}_{\Omega_M}$ of $(0,1-\delta_{2,n},\gamma_0)$ in $\mathbb{R}^3$, and real analytic maps*

$$\begin{aligned}U_{\Omega_M}&:(-\epsilon_{\Omega_M},\epsilon_{\Omega_M})\times\mathcal{U}_{\Omega_M}\to C^{1,\alpha}(\mathrm{cl}\Omega_M,\mathbb{R}^n),\\ P_{\Omega_M}&:(-\epsilon_{\Omega_M},\epsilon_{\Omega_M})\times\mathcal{U}_{\Omega_M}\to C^{0,\alpha}(\mathrm{cl}\Omega_M,\mathbb{R})\end{aligned}$$

such that

$$\mathrm{cl}\Omega_M\cap(\epsilon\mathrm{cl}\Omega^s)=\emptyset \text{ and } \Psi_n[\epsilon]\in(-\epsilon_{\Omega_M},\epsilon_{\Omega_M})\times\mathcal{U}_{\Omega_M},\quad\forall\epsilon\in(0,\epsilon_{\Omega_M})\tag{3.2}$$

and

$$\begin{aligned}u_{\epsilon|\mathrm{cl}\Omega_M}&=u^b_{|\mathrm{cl}\Omega_M}+\frac{\epsilon^{n-2}}{\gamma(\epsilon)(\log\epsilon)^{\delta_{2,n}}}U_{\Omega_M}[\Psi_n[\epsilon]],\\ p_{\epsilon|\mathrm{cl}\Omega_M}&=p^b_{|\mathrm{cl}\Omega_M}+\frac{\epsilon^{n-2}}{\gamma(\epsilon)(\log\epsilon)^{\delta_{2,n}}}P_{\Omega_M}[\Psi_n[\epsilon]],\qquad\forall\epsilon\in(0,\epsilon_{\Omega_M}).\end{aligned}$$

Here (u_ϵ,p_ϵ) denotes the unique solution in $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega(\epsilon),\mathbb{R}^n)\times C^{0,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega(\epsilon),\mathbb{R})$ of the boundary value problem in (1.4) *which satisfies the decay condition in* (1.5) *for $R\equiv\sup_{x\in(\Omega^b\cup(\epsilon\Omega^s))}|x|$. Moreover,*

$$\begin{aligned}U_{\Omega_M}[0,0,1-\delta_{2,n},\gamma_0]&=-g^V_{\Omega^b|\mathrm{cl}\tilde\Omega}\cdot(\delta_{2,n}\lambda^s+(1-\delta_{2,n})\tau^s),\\ P_{\Omega_M}[0,0,1-\delta_{2,n},\gamma_0]&=-g^P_{\Omega^b|\mathrm{cl}\tilde\Omega}\cdot(\delta_{2,n}\lambda^s+(1-\delta_{2,n})\tau^s).\end{aligned}\tag{3.3}$$

In Theorem 3.3 we consider the "microscopic" behaviour of (u_ϵ,p_ϵ) near the impurity $\epsilon\Omega^s$ as $\epsilon\to 0$. For a proof we refer to [2, Th. 4.1].

Theorem 3.3. *Let Ω^b and Ω^s be as in* (1.1). *Let ϵ_0 be as in* (1.2). *Let γ, γ_0 be as in* (1.3). *Let $(f^b,f^s)\in C^{1,\alpha}(\partial\Omega^b,\mathbb{R}^n)\times C^{1,\alpha}(\partial\Omega^s,\mathbb{R}^n)$. Let (u^s,p^s) denote the unique solution in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega^s,\mathbb{R}^n)\times C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega^s,\mathbb{R})$ of the boundary value problem in* (1.10) *which satisfy the decay condition in* (1.5) *for $R\equiv\sup_{x\in\Omega^s}|x|$. Let $\Omega_m\subseteq\mathbb{R}^n\setminus\mathrm{cl}\Omega^s$*

be open and bounded. Then there exists $\epsilon_{\Omega_m} \in (0, \epsilon_0)$*, and a neighborhood* $\mathcal{U}_{\Omega_m}$ *of* $(0, 1 - \delta_{2,n}, \gamma_0)$ *in* $\mathbb{R}^3$*, and real analytic maps*

$$\begin{aligned} U_{\Omega_m} &: (-\epsilon_{\Omega_m}, \epsilon_{\Omega_m}) \times \mathcal{U}_{\Omega_m} \to C^{1,\alpha}(\mathrm{cl}\Omega_m, \mathbb{R}^n), \\ P_{\Omega_m} &: (-\epsilon_{\Omega_m}, \epsilon_{\Omega_m}) \times \mathcal{U}_{\Omega_m} \to C^{0,\alpha}(\mathrm{cl}\Omega_m, \mathbb{R}) \end{aligned}$$

such that

$$(\epsilon \mathrm{cl}\Omega_m) \cap \mathrm{cl}\Omega^b = \emptyset \text{ and } \Psi_n[\epsilon] \in (-\epsilon_{\Omega_m}, \epsilon_{\Omega_m}) \times \mathcal{U}_{\Omega_m}, \quad \forall \epsilon \in (0, \epsilon_{\Omega_m}) \tag{3.4}$$

and such that

$$\begin{aligned} u_\epsilon(\epsilon x) &= \frac{1}{\gamma(\epsilon)} U_{\Omega_m}[\Psi_n[\epsilon]](x), \\ p_\epsilon(\epsilon x) &= \frac{1}{\epsilon\gamma(\epsilon)} P_{\Omega_m}[\Psi_n[\epsilon]](x), \qquad \forall x \in \mathrm{cl}\Omega_m,\ \epsilon \in (0, \epsilon_{\Omega_m}). \end{aligned}$$

Here (u_ϵ, p_ϵ) *denotes the unique solution in* $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega(\epsilon), \mathbb{R}^n) \times C^{0,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega(\epsilon), \mathbb{R})$ *of the boundary value problem in* (1.4) *which satisfies the decay condition in* (1.5) *for* $R \equiv \sup_{x \in (\Omega^b \cup (\epsilon\Omega^s))} |x|$*. Moreover,*

$$\begin{aligned} U_{\Omega_m}[0, 0, 1 - \delta_{2,n}, \gamma_0] &= u^s_{|\mathrm{cl}\Omega_m}, \\ P_{\Omega_m}[0, 0, 1 - \delta_{2,n}, \gamma_0] &= p^s_{|\mathrm{cl}\Omega_m}. \end{aligned} \tag{3.5}$$

4. REAL ANALYTIC REPRESENTATION THEOREM FOR THE ENERGY INTEGRAL $\mathcal{E}_\epsilon$

We are now ready to prove our main Theorem 4.1.

Theorem 4.1. *Let* Ω^b *and* Ω^s *be as in* (1.1)*. Let* ϵ_0 *be as in* (1.2)*. Let* γ*,* γ_0 *be as in* (1.3)*. Let* $(f^b, f^s) \in C^{1,\alpha}(\partial\Omega^b, \mathbb{R}^n) \times C^{1,\alpha}(\partial\Omega^s, \mathbb{R}^n)$*. Let* (u^b, p^b) *denote the unique solution in* $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega^b, \mathbb{R}^n) \times C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega^b, \mathbb{R})$ *of the boundary value problem in* (1.9) *which satisfies the decay condition in* (1.5) *for* $R \equiv \sup_{x \in \Omega^b} |x|$*. Let* $\mathcal{E}^b$ *be defined as in* (1.11)*. Let* (u^s, p^s) *denote the unique solution in* $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega^s, \mathbb{R}^n) \times C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega^s, \mathbb{R})$ *of the boundary value problem in* (1.10) *which satisfy the decay condition in* (1.5) *for* $R \equiv \sup_{x \in \Omega^s} |x|$*. Let* $\mathcal{E}^s$ *be defined as in* (1.12)*. Then there exist* $\epsilon_E \in (0, \epsilon_0)$*, and an open neighborhood* $\mathcal{U}_E$ *of* $(0, 1 - \delta_{2,n}, \gamma_0)$ *in* $\mathbb{R}^3$*, and a real analytic map*

$$E : (-\epsilon_E, \epsilon_E) \times \mathcal{U}_E \to \mathbb{R},$$

such that

$$\Psi_n[\epsilon] \in (-\epsilon_E, \epsilon_E) \times \mathcal{U}_E, \qquad \forall \epsilon \in (0, \epsilon_E) \tag{4.1}$$

and such that

$$\mathcal{E}_\epsilon = \mathcal{E}^b + \frac{\epsilon^{n-2}}{\gamma(\epsilon)^2} E[\Psi_n[\epsilon]], \qquad \forall \epsilon \in (0, \epsilon_E) \tag{4.2}$$

(*see also* (3.1)). *Moreover,*

$$E[0,0,1-\delta_{2,n},\gamma_0]=\mathcal{E}^s+(1-\delta_{2,n})\gamma_0\int_{\partial\Omega^b} f^b\,\mathcal{T}(g^P_{\Omega^b}\cdot\tau^s, Dg^V_{\Omega^b}\cdot\tau^s)\nu_{\Omega^b}\,d\sigma. \tag{4.3}$$

Here $\mathcal{E}_\epsilon$ *denotes the energy integral of the solution* $(u_\epsilon,p_\epsilon)\in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega(\epsilon),\mathbb{R}^n)\times C^{0,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega(\epsilon),\mathbb{R})$ *of the boundary value problem in* (1.4) *which satisfies the decay condition in* (1.5) *for* $R\equiv\sup_{x\in(\Omega^b\cup(\epsilon\Omega^s))}|x|$ (*see definition* (1.6)).

Proof. Let $\epsilon\in(0,\epsilon_0)$. Let (u_ϵ,p_ϵ) denote the solution in $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega(\epsilon),\mathbb{R}^n)\times C^{0,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega(\epsilon),\mathbb{R})$ of the boundary value problem in (1.4) which satisfies the decay condition in (1.5) for $R\equiv\sup_{x\in(\Omega^b\cup(\epsilon\Omega^s))}|x|$. By Lemma 2.2 and by definition (1.6), we deduce that

$$\begin{aligned}\mathcal{E}_\epsilon=&-\int_{\partial\Omega^b} u^t_\epsilon(x)\,\mathcal{T}(p_\epsilon(x),Du_\epsilon(x))\nu^b_\Omega(x)\,d\sigma_x-\\&-\int_{\epsilon\partial\Omega^s} u^t_\epsilon(x)\,\mathcal{T}\left(p_\epsilon(x),Du_\epsilon(x)\right)\nu_{\epsilon\Omega^s}(x)\,d\sigma_x=\\=&-\int_{\partial\Omega^b} u^t_\epsilon(x)\,\mathcal{T}(p_\epsilon(x),Du_\epsilon(x))\nu^b_\Omega(x)\,d\sigma_x-\\&-\epsilon^{n-1}\int_{\partial\Omega^s} u^t_\epsilon(\epsilon x)\,\mathcal{T}\left(p_\epsilon(\epsilon x),\frac{1}{\epsilon}D_x u_\epsilon(\epsilon x)\right)\nu_{\Omega^s}(x)\,d\sigma_x.\end{aligned} \tag{4.4}$$

Now let $\tilde\Omega_M$ be an open bounded neighborhood of $\mathrm{cl}\Omega^b$. Let $\tilde\Omega_m$ be an open bounded neighborhood of $\mathrm{cl}\Omega^s$. Let ϵ_{Ω_M}, $\mathcal{U}_{\Omega_M}$, U_{Ω_M}, P_{Ω_M} be as in Theorem 3.2 with $\Omega_M\equiv\tilde\Omega_M\setminus\mathrm{cl}\Omega^b$. Let ϵ_{Ω_m}, $\mathcal{U}_{\Omega_m}$, U_{Ω_m}, P_{Ω_m} be as in Theorem 3.3 with $\Omega_m\equiv\tilde\Omega_m\setminus\mathrm{cl}\Omega^s$. Let $\epsilon_E\equiv\inf\{\epsilon_{\Omega_M},\epsilon_{\Omega_m}\}$. Let $\mathcal{U}_E\equiv\mathcal{U}_{\Omega_M}\cap\mathcal{U}_{\Omega_m}$. Then the validity of (4.1) follows by (3.2) and (3.4). Moreover, if $\epsilon\in(0,\epsilon_E)$, then Theorems 3.2, 3.3 and the equality in (4.4) imply that

$$\begin{aligned}\mathcal{E}_\epsilon=&-\int_{\partial\Omega^b}(u^b(x))^t\,\mathcal{T}(p^b(x),Du^b(x))\nu^b_\Omega(x)\,d\sigma_x-\\&-\frac{\epsilon^{n-2}}{\gamma(\epsilon)(\log\epsilon)^{\delta_{2,n}}}\Bigg\{\int_{\partial\Omega^b}(U_{\Omega_M}[\Psi[\epsilon]](x))^t\,\mathcal{T}(p^b(x),Du^b(x))\nu_{\Omega^b}(x)d\sigma_x+\\&+\int_{\partial\Omega^b}(f^b(x))^t\,\mathcal{T}(P_{\Omega_M}[\Psi_n[\epsilon]](x),DU_{\Omega_M}[\Psi_n[\epsilon]](x))\nu_{\Omega^b}(x)d\sigma_x\Bigg\}-\\&-\frac{\epsilon^{n-1}}{\gamma(\epsilon)^2}\int_{\partial\Omega^s}(f^s(x))^t\mathcal{T}\left(P_{\Omega_m}[\Psi[\epsilon]](x),DU_{\Omega_m}[\Psi[\epsilon]](x)\right)\nu_{\Omega^s}(x)\,d\sigma_x.\end{aligned} \tag{4.5}$$

Thus it is natural to set

$$\begin{aligned} E[\mathbf{e}] \equiv -\epsilon_2\epsilon_3\Big\{ \int_{\partial\Omega^b} (U_{\Omega_M}[\mathbf{e}](x))^t\, \mathcal{T}(p^b(x), Du^b(x))\nu_{\Omega^b}(x)d\sigma_x + \\ + \int_{\partial\Omega^b} (f^b(x))^t\, \mathcal{T}(P_{\Omega_M}[\mathbf{e}](x), DU_{\Omega_M}[\mathbf{e}](x))\nu_{\Omega^b}(x)d\sigma_x \Big\} - \\ - \int_{\partial\Omega^s} (f^s(x))^t\, \mathcal{T}\left(P_{\Omega_m}[\mathbf{e}](x), DU_{\Omega_m}[\mathbf{e}](x)\right)\nu_{\Omega^s}(x)\, d\sigma_x \end{aligned} \tag{4.6}$$

for all $\mathbf{e} \equiv (\epsilon, \epsilon_1, \epsilon_2, \epsilon_3) \in (-\epsilon_E, \epsilon_E) \times \mathcal{U}_E$. By the real analyticity of U_{Ω_M}, P_{Ω_M}, U_{Ω_m} and P_{Ω_m} and by standard calculus in the Banach space we deduce that E is real analytic from $(-\epsilon_E, \epsilon_E) \times \mathcal{U}_E$ to $\mathbb{R}$. The validity of (4.2) follows by the definitions in (3.1) and (4.6), by the equality in (4.5), and by Lemma 2.2 (see also (1.11)). The validity of (4.3) follows by the definition in (3.1) and (4.6), and by the equalities in (3.3) and (3.5), and by the equality

$$U_{\Omega_M}[0, 0, (1-\delta_{2,n}), \gamma_0]_{|\partial\Omega^b} = -g^V_{\Omega^b|\partial\Omega^b} \cdot (\delta_{2,n}\lambda^s + (1-\delta_{2,n})\tau^s) = 0$$

(cf. Lemma 3.1 (ii)). □

We observe that the equalities in (4.2) and (4.3) immediately imply the validity of the limits in (1.8) and (1.14) under conditions (1.7) and (1.13), respectively (note that condition (1.13) implies that $\gamma_0 = 0$).

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