RECURSIVELY ARBITRARILY VERTEX-DECOMPOSABLE GRAPHS

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Abstract. A graph $G = (V, E)$ is arbitrarily vertex decomposable if for any sequence τ of positive integers adding up to $|V|$, there is a sequence of vertex-disjoint subsets of V whose orders are given by τ, and which induce connected graphs. The main aim of this paper is to study the recursive version of this problem. We present a solution for trees, suns, and partially for a class of 2-connected graphs called balloons.

Keywords: arbitrary vertex decomposable (AVD) graph, recursively AVD graphs.

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1. INTRODUCTION

We deal with finite, simple and undirected graphs. Let $G = (V, E)$ be a graph of order n. A sequence $\tau = (n_1, ..., n_k)$ of positive integers is called *admissible for G* if it sums up to n. If $\tau = (n_1, ..., n_k)$ is an admissible sequence for G and there exists a partition $(V_1, ..., V_k)$ of the vertex set V such that V_i induces a connected subgraph of order n_i for each $i \in [1, k]$, then τ is called *realizable in G*, and the sequence $(V_1, ..., V_k)$ is said to be *a realization of τ in G*. A graph G is *arbitrarily vertex decomposable* (AVD, for short) if for each admissible sequence τ for G there exists a realization of τ in G. Clearly, every AVD graph has to be connected.

The notion of AVD graphs was first introduced by Barth *et al.* in [1] and motivated by the following problem in computer science. Consider a network connecting different computing resources; such a network is modeled by a graph. Suppose there are k different users, where the i-th one needs n_i resources in our graph. The subgraph induced by the set of resources attributed to each user should be connected and each resource should be attributed to one user. So we are seeking a realization of the sequence $\tau = (n_1, \ldots, n_k)$ in this graph. Thus, such a network should be an AVD graph.

This problem can also be considered as a natural analogy of the similar notion in which vertices are replaced by edges. Some references concerning arbitrarily edge decomposable graphs can be found in [8].

The problem of deciding whether a given graph is arbitrarily vertex decomposable has been considered in several papers. The investigation of AVD trees is motivated by the fact that a connected graph is AVD if its spanning tree is AVD. It turned out, however, that the structure of AVD trees is not obvious in general.

On the other hand, it is clear that each path, and therefore each traceable graph, is AVD. So, the problems concerning AVD graphs can be considered as a generalization of hamiltonian problems. Some results in this direction were obtained by use of some characterizations of AVD graphs in the family of graphs with a large dominating cycle, called suns. This justifies investigating AVD graphs in this family.

2. TERMINOLOGY AND PRELIMINARY RESULTS

In this paper, we deal only with simple graphs, that means graphs without loops or multiple edges. We denote by n the number of vertices, also called *order* of the graph and by m the number of edges. If $G = (V, E)$ and $A \subseteq V$, $G[A]$ will denote the subgraph of G induced by A. For more definitions on graphs, please refer to [6].

2.1. ARBITRARILY VERTEX-DECOMPOSABLE GRAPHS

Let $n, \tau_1, \ldots, \tau_k$ be positive integers such that $\tau_1 + \ldots + \tau_k = n$. $\tau = (\tau_1, \ldots, \tau_k)$ is called a *decomposition* of n.

Let $G = (V, E)$ be a graph of order n, and $\tau = (\tau_1, \ldots, \tau_k)$ a decomposition of n. G is τ-*Vertex-Decomposable* iff there exists a partition of V: $V_1, \ldots, V_k$ such that for each $i, 1 \leq i \leq k$,

— $|V_i| = \tau_i$,
— $G[V_i]$ is connected.

A graph $G = (V, E)$ of order n is *arbitrarily vertex-decomposable* (for short *AVD*) iff for each decomposition τ of n, G is τ-vertex-decomposable.

2.2. RECURSIVELY ARBITRARILY VERTEX-DECOMPOSABLE GRAPHS

We present here a new definition of graph decomposition into connected components. This definition introduces the main topic of this paper. The goal is to characterize graphs for which it is possible to perform arbitrarily decomposition recursively on each subgraph.

Definition 2.1. A graph $G = (V, E)$ of order n is *recursively arbitrarily vertex-decomposable* (for short *R-AVD)* iff:

— $G = K_1$
    or

— G is connected and for each decomposition $\tau = (\tau_1, \ldots, \tau_k)$ of n, $k \geq 2$, there exists a partition of V: $V_1, \ldots, V_k$ such that for all $i, 1 \leq i \leq k$,
  — $|V_i| = \tau_i$,
  — $G[V_i]$ is R-AVD.

A stronger version of the recursive decomposition of graphs into connected components will also be studied in the last section of this paper, based on the following definition.

Definition 2.2. A graph $G = (V, E)$ of order n is *strongly recursively arbitrarily vertex-decomposable* (for short *SR-AVD*) iff:

— G is AVD,
— for each sequence τ of length $k \geq 2$ admissible for G and for each realization $V_1, \ldots, V_k$ of τ the graph $G[V_i]$, $i = 1, \ldots, k$, is SR-AVD.

2.3. FAMILIES OF GRAPHS

The following definitions presents some families of graphs and their notations, used in further sections.

Definition 2.3.

— Let a be a positive integer. P_a denotes the path of order a, C_a the cycle of order a (cf. Figures 1a and 1b).
— A k-pode $T_k(t_1, \ldots, t_k)$ is a tree of order $1 + \sum_{i=1}^{k} t_i$ composed by k paths of respective orders $t_1, \ldots, t_k$, connected to a unique node, called the *root* of the k-pode (cf. Figure 1c).
— Let a and b be two positive integers. A caterpillar Cat (a, b) is a tree of order $a+b$, composed by three paths of order a, b and 2, sharing exactly one node, called the *root* of the caterpillar. Cat (a, b) is isomorphic to $T_3(a-1, b-1, 1)$ (cf. Figure 1d).
— A sun with r rays is a graph of order $n \geq 2r$ with r hanging vertices $u_1, \ldots, u_r$ whose deletion yields a cycle C_{n-r}, and each vertex v_i adjacent to u_i is of degree three. If the sequence of vertices v_i is situated on the cycle C_{n-r} in such a way that there are exactly $a_i \geq 0$ vertices, each of degree two, between v_i and v_{i+1}, $i = 1, \ldots, r$ (the indices taken modulo r), then this sun is denoted by Sun $(a_1, \ldots, a_r)$, and is unique up to isomorphism (cf. Figure 1e).
  Note that the order of Sun $(a_1, \ldots, a_r)$ equals $n = 2r + a_1 + \ldots + a_r$.
— Let $b_1, \ldots, b_k$ be positive integers. A k-*balloon* $B(b_1, \ldots, b_k)$ is a graph of order $2 + \sum_{i=1}^{k} b_i$ composed by two vertices (called *roots*) linked by k paths (called *branches*) of *widths* (the number of internal vertices) $b_1, \ldots, b_k$ (cf. Figure 1f).

Definition 2.4. A graph of order n even (resp. odd) has a *perfect matching* (resp. *quasi-perfect matching*) iff it contains a set of $\lfloor \frac{n}{2} \rfloor$ disjoint edges. A graph is *traceable* iff it contains a hamiltonian path.
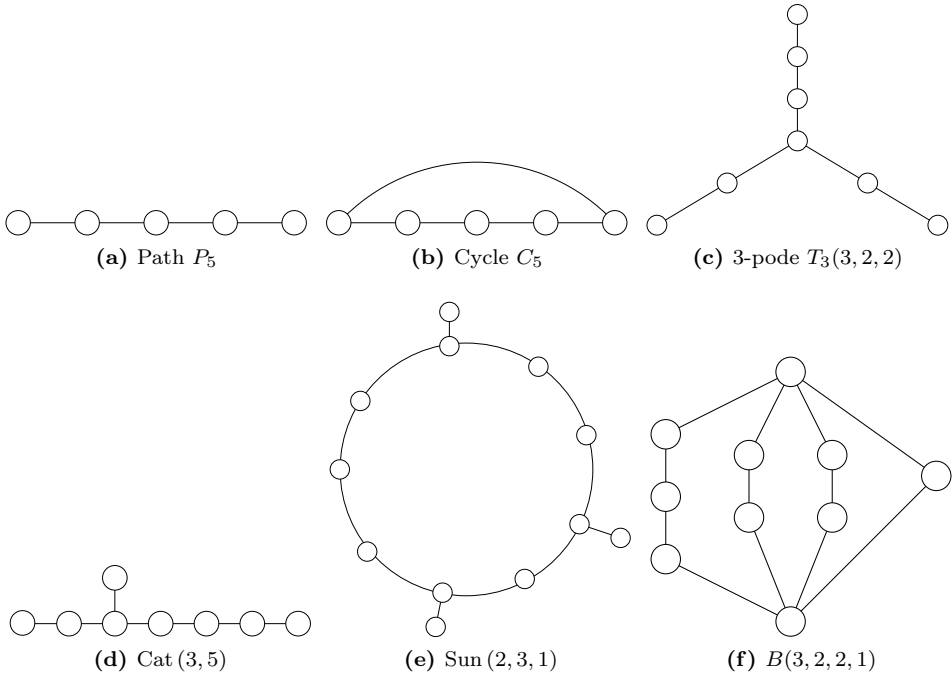
(a) Path P_5 **(b)** Cycle C_5 **(c)** 3-pode $T_3(3, 2, 2)$

(d) Cat $(3, 5)$ **(e)** Sun $(2, 3, 1)$ **(f)** $B(3, 2, 2, 1)$

Fig. 1. Examples of graphs

2.4. PRELIMINARY RESULTS

Remark 2.5. A graph G having an AVD spanning subgraph is AVD. But there are AVD-graphs without any AVD-spanning tree.

Proof. The first part of Remark 2.5 is trivial. We now prove that the graph G of Figure 2[1] is AVD, but has no AVD spanning tree.

To prove that the graph is AVD, we show that for any value of $\lambda \leq n-1$, we may find a connected induced subgraph $G[V_\lambda]$ of G of order λ such that the remaining graph $G[V \setminus V_\lambda]$ is traceable. The values of V_λ are given in Table 1.

The graph G is of order $n = 24$ and any of its spanning trees is of the form $G - e_j$, $j \in \{1, \ldots, 19\}$. If $G - e_j$ is AVD, then the sequence $(i, \ldots, i)$ is realizable in $G - e_j$, $i = 2, 3, 4, 6$. For $i \in \{2, 3, 4, 6\}$ let R_i be the set of all $j \in \{1, \ldots, 19\}$ such that $(i, \ldots, i)$ is realizable in $G - e_j$. Then $R_2 = \{1, 2, 3, 5, 7, 9, 11, 13, 15, 17, 18, 19\}$, $R_3 = \{1, 4, 7, 10, 13, 16, 19\}$, $R_4 = \{1, 5, 9, 13, 17\}$ and $R_6 = \{3, 4, 9, 10, 15, 16, 18, 19\}$. Since $R_2 \cap R_3 \cap R_4 \cap R_6 = \emptyset$, the graph G has no AVD spanning tree. $\square$

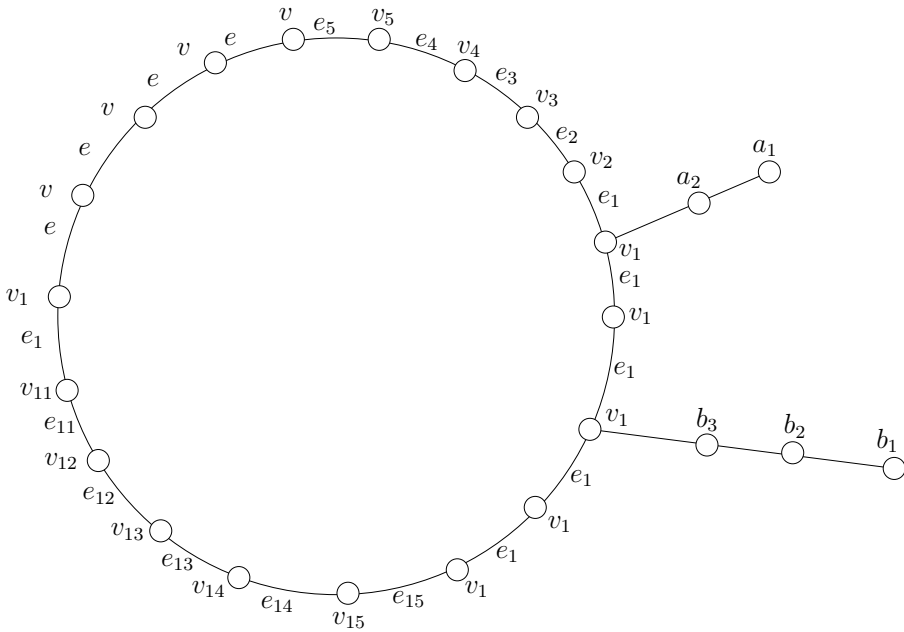[1] This counter-example has been proposed by Hervé Fournier. Another counter-example may be find in [12].

Fig. 2. An AVD graph without any AVD spanning tree

Table 1. Values of V_λ

λ	V_λ
1	$\{v_{19}\}$
2	$\{a_1, a_2\}$
3	$\{b_1, b_2, b_3\}$
4	$\{a_1, a_2, v_1, v_{19}\}$
≥ 5	$\{a_1, a_2, v_{19}, v_1, v_2, \ldots, v_{\lambda-3}\}$

Remark 2.6. A graph $G = (V, E)$ of order n is R-AVD iff for each integer $1 \leq \lambda \leq \lfloor \frac{n}{2} \rfloor$, there exists a subset V_λ of V such that:

— $|V_\lambda| = \lambda$,
— $G[V_\lambda]$ is R-AVD,
— $G[V \setminus V_\lambda]$ is R-AVD.

The notion of an on-line arbitrarily vertex decomposable graph has been introduced by Horňák *et al.* in [7]. This version of the problem is even more natural when applied to the problem in computer networks mentioned in Section 1.

Let $G = (V, E)$ be a graph. Imagine now the following decomposition procedure consisting of k stages, where k is a random variable attaining (integer) values from interval $[1, n]$. In the ith stage, where $i \in [1, k]$, a positive integer τ_i arrives and we have to choose a subset V_i of V of order τ_i that is disjoint from all subsets of V chosen in previous stages (without a possibility of changing the choice in the future). More precisely, for every partial sequence $(\tau_1, \ldots \tau_i)$ whose sum is less than n, there is a sequence $(V_1, \ldots, V_i)$ of disjoint subsets of V such that for $1 \le j \le i$, $|V_j| = \tau_j$, with the following property: for all sequences $(\tau'_1, \ldots, \tau'_k)$ with $k \ge i$ and summing to n, such that $\tau'_r = \tau_r$ for $1 \le r \le i$, there is a decomposition of V into disjoint subsets $V'_1, \ldots, V'_k$ with $|V'_j| = \tau'_j$ and $G[V'_j]$ connected, for all j, and $V'_j = V_j$ for $1 \le j \le i$.

Definition 2.7 ([7])**.** If the decomposition procedure can be accomplished for any (random) sequence of positive integers $(\tau_1, \ldots, \tau_k)$ adding up to n, the graph G is said to be *on-line AVD*, (for short *OL-AVD*).

Lemma 2.8 ([7])**.** *A graph connected $G = (V, E)$ of order n is OL-AVD iff for each integer $1 \le \lambda \le n - 1$, there exists a subset V_λ of V such that:*

— *$|V_\lambda| = \lambda$,*
— *$G[V_\lambda]$ is connected,*
— *$G[V \backslash V_\lambda]$ is OL-AVD.*

Remark 2.9. A straightforward consequence of Lemma 2.8 and Remark 2.6 is that every R-AVD graph is OL-AVD.

The opposite is not true. For example, the caterpillar $\mathrm{Cat}\,(8, 11)$ is OL-AVD [7], but not R-AVD (cf. Section 3.1).

We denote by:

— $PF(n)$ the set of graphs of order n with a perfect matching or a quasi-perfect matching;
— $AVD(n)$ the set of AVD graphs of order n;
— $OL{-}AVD(n)$ the set of OL-AVD graphs of order n;
— $R{-}AVD(n)$ the set of R-AVD graphs of order n;
— $Traceable(n)$ the set of traceable graphs of order (n);
— $SR{-}AVD(n)$ the set of SR-AVD graphs of order n.

Theorem 2.10. $PF(n) \supsetneq AVD(n) \supsetneq OL{-}AVD(n) \supsetneq R{-}AVD(n) \supsetneq Traceable(n) \supsetneq SR{-}AVD(n)$.

Proof.

— $PF(n) \supsetneq AVD(n)$
  If G is AVD, then it is $(2, \ldots, 2)$ or $(2, \ldots, 2, 1)$-decomposable, following the parity of its order.
  The caterpillar $\mathrm{Cat}\,(3, 3)$ has a perfect matching, but is not (3,3)-decomposable.
— $AVD(n) \supsetneq OL{-}AVD(n)$
  The inclusion is trivial.
  The 3-pode $T_3(2, 3, 5)$ is AVD ([1]), but not OL-AVD ([7]).

— $OL-AVD(n) \supsetneq R-AVD(n)$
see Remark 2.9.
— $R-AVD(n) \supsetneq Traceable(n)$
The inclusion is trivial, using Remark 2.6.
The caterpillar $\mathrm{Cat}\,(2,3)$ is R-AVD (cf. Section 3.1) and not traceable.
— $Traceable(n) \supsetneq SR-AVD(n)$
The proof of the inclusion is given in Section 4, Theorems 4.2 and 4.3.
The graph from Figure 3 is traceable, but not SR-AVD. Consider the $(4,1)$ decomposition, where the set of size 4 is $S = \{v_1, v_2, v_3, v_5\}$. The induced subgraph $G[S]$ is isomorphic to $K_{1,3}$ and therefore is not $(2,2)$-decomposable. $\qquad \square$
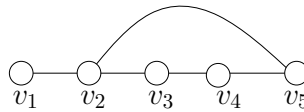

Fig. 3. A traceable graph, which is not SR-AVD

2.5. KNOWN RESULTS ON AVD AND OL-AVD GRAPHS

Theorem 2.11 ([2]). *There is no AVD k-pode for $k \geq 5$. All AVD 4-podes have a branch of length 1.*

A direct consequence of Theorem 2.11 is that the maximum degree of an AVD tree is at most 4. Nevertheless, this problem remains difficult on the class of trees, as shown by the next theorem.

Definition 2.12. Decision problem TP3:
INSTANCE: A tree T with maximum degree 3 and order n and τ a decomposition of n.
QUESTION: Is T τ-decomposable?

Theorem 2.13 ([2]). *TP3 is NP-complete.*

The two next results give a complete characterization of OL-AVD trees and suns.

Theorem 2.14 ([7]). *A tree T is OL-AVD if and only if either T is a path or T is a caterpillar $\mathrm{Cat}\,(a,b)$ with a and b given in Table 2 or T is the 3-pode $T_3(2,4,6)$.*

Theorem 2.15 ([10]). *A sun with one ray is always OL-AVD.*
A sun with two rays $\mathrm{Sun}\,(a,b)$ is OL-AVD iff a and b take values given in Table 3a.
A sun with three rays $\mathrm{Sun}\,(a,b,c)$ is OL-AVD iff a, b and c take values given in Table 3b.
A sun with four rays is OL-AVD iff it is isomorphic to $\mathrm{Sun}\,(0,0,1,d)$, where $d \equiv 2, 4 \pmod 6$.
A sun with five or more rays is never OL-AVD.

Table 2. Values a, b ($b \geq a$), such that $\mathrm{Cat}\,(a, b)$ is OL-AVD

a	b	a	b
2, 4	$\equiv 1 \pmod 2$	7	8, 9, 11, 13, 15
3	$\equiv 1, 2 \pmod 3$	8	11, 19
5	6, 7, 9, 11, 14, 19	9, 10	11
6	$\equiv 1, 5 \pmod 6$	11	12

Table 3. Values for OL-AVD Suns

(a) Values a, b ($b \geq a$), such that $\mathrm{Sun}\,(a, b)$ is OL-AVD

a	b
0	arbitrary
1, 3	$\equiv 0 \pmod 2$
2	$\not\equiv 3 \pmod 6$, 3, 9, 21
4	$\equiv 2, 4 \pmod 6$, $[4, 19] \backslash \{15\}$
5	$\equiv 2, 4 \pmod 6$, 6, 18
6	6, 7, 8, 10, 11, 12, 14, 16
7	8, 10, 12, 14, 16
8	8, 9, 10, 11,12
9	10, 12

(b) Values a, b, c ($c \geq b \geq a$), such that $\mathrm{Sun}\,(a, b, c)$ is OL-AVD

a	b	c
0	0	$\equiv 1, 2 \pmod 3$
	1	$\equiv 0 \pmod 2$
	2	$\equiv 2, 4 \pmod 6$, 3, 6, 7, 11, 18, 19
	3	$\equiv 2, 4 \pmod 6$
	4	4, 5, 6, 8, 10, 11, 12, 14, 16
	5	6, 8, 16
	6, 7	8, 10
	8	8, 9
1	2	$\equiv 2, 4 \pmod 6$, 6, 18
2	3	4, 8, 16

3. RECURSIVELY AVD GRAPHS

In this section, we give a complete characterization of R-AVD trees and suns and a tight upper bound of the maximum degree of R-AVD balloons.

3.1. TREES

Theorem 3.1. *A tree T is R-AVD if and only if either T is a path or T is a caterpillar $\mathrm{Cat}\,(a, b)$ with a and b given in Table 4 or T is the 3-pode $T_3(2, 4, 6)$.*

Proof. Only trees which are OL-AVD may be R-AVD (cf. Remark 2.9). The case of paths is trivial. Thus, we have only to check whether or not caterpillars from Table 2 and $T_3(2, 4, 6)$ are R-AVD. For that, we use Remark 2.6. That means that for each $\lambda \leq \frac{n}{2}$, where n denotes the order of the graph, we have to find a subset of vertices V_λ of size λ such that both $G[V\lambda]$ and $G[V \backslash V_\lambda]$ are R-AVD.

Labellings are those from Figure 4.

For $\mathrm{Cat}\,(a, b)$ and $\lambda = 1$, $V_\lambda = \{x\}$. The remaining graph is a path.

Table 4. Values a, b ($b \geq a$), such that Cat (a, b) is R-AVD

a	b
2, 4	$\equiv 1 \pmod 2$
3	$\equiv 1, 2 \pmod 3$
5	6, 7, 9, 11, 14, 19
6	7
7	8, 9, 11, 13, 15

For Cat (a, b) and $2 \leq \lambda \leq \frac{a+b}{2}$, we have only two possibilities: either a_{a-1} or b_{b-1} belongs to V_λ. If Cat (a, b) is R-AVD, the two graphs $G[V_\lambda]$ and $G[V \setminus V_\lambda]$ are either a path or a smaller R-AVD caterpillar. Table 5 gives the solution when it exists. For values a and b such that Cat (a, b) is OL-AVD, but not R-AVD, Table 6 gives a value λ for which it is not possible to find a set V_λ such that both $G[V_\lambda]$ and $G[V \setminus V_\lambda]$ are R-AVD.

Table 5. Leaf of Cat (a, b) in V_λ following values of a, b and λ

a	b	λ	$\in V_\lambda$	a	b	λ	$\in V_\lambda$
any	any	1	x	5	14	2, 4, 6, 9	a_4
2	$\equiv 1 \pmod 2$	$\equiv 1 \pmod 2$	a_1			3, 5, 7, 8	b_{13}
		$\equiv 0 \pmod 2$	b_{b-1}		19	2, 3, 4, 6, 7, 9, 11, 12	a_4
3	$\equiv 1, 2 \pmod 3$	$\equiv 1, 2 \pmod 3$	a_2			5, 8, 10	b_{18}
		$\equiv 0 \pmod 3$	b_{b-1}	6	7	2, 3, 4, 5	a_5
4	$\equiv 1 \pmod 2$	$\equiv 1 \pmod 2$	a_3			6	b_6
		$\equiv 0 \pmod 2$	b_{b-1}		8	6	a_6
	6	4	a_4			2, 3, 4, 5, 7	b_7
		2, 3, 5	b_5		9	2, 5, 6	a_6
5	7	2, 3, 4	a_4			3, 4, 7, 8	b_8
		5, 6	b_6	7	11	2, 3, 4, 5, 6, 8	a_6
	9	4	a_4			7, 9	b_{10}
		2, 3, 5, 6, 7	b_8		13	3, 5, 6, 8	a_6
	11	3, 4, 6	a_4			2, 4, 7, 9, 10	b_7
		2, 5, 7, 8	b_{10}		15	3, 5, 6, 8	a_6
						2, 4, 7, 9 10	b_{14}

For $T_3(2, 4, 6)$, values of V_λ are given in Table 7, using labelling from Figure 4.
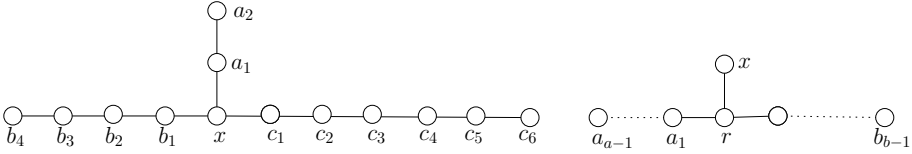

Fig. 4. 3-pode $T_3(2, 4, 6)$ and Cat (a, b)

                                                      □

Table 6. Values a and b with $b \geq a$, such that Cat (a, b) is OL-AVD, but not R-AVD

a	b	λ
6	$\equiv 1 \pmod 6$ and $\neq 7$	9
	$\equiv 5 \pmod 6$	8
8	11	2
	19	8

a	b	λ
9	11	3
10	11	4
11	12	2

Table 7. Values of $G[V_\lambda]$ and $G[V \setminus V_\lambda]$ for $T_3(2, 4, 6)$

λ	V_λ	$G[V_\lambda]$	$G[V \setminus V_\lambda]$	λ	V_λ	$G[V_\lambda]$	$G[V \setminus V_\lambda]$
1	$\{a_2\}$	P_1	Cat $(5, 7)$	4	$\{b_1, b_2, b_3, b_4\}$	P_4	P_9
2	$\{a_1, a_2\}$	P_2	P_{11}	5	$\{c_2, c_3, c_4, c_5, c_6\}$	P_5	Cat $(3, 5)$
3	$\{b_2, b_3, b_4\}$	P_3	Cat $(3, 7)$	6	$\{c_1, c_2, c_3, c_4, c_5, c_6\}$	P_6	P_7

3.2. SUNS

Theorem 3.2.

A sun with one ray is always R-AVD.

A sun with two rays Sun (a, b) is R-AVD if and only if a and b take values given in Table 8a.

A sun with three rays Sun (a, b, c) is R-AVD if and only if a, b and c take values given in Table 8b.

A sun with four rays is R-AVD if and only if it is isomorphic to Sun $(0, 0, 1, 2)$ or to Sun $(0, 0, 1, 4)$.

A sun with five or more rays in never R-AVD.

The proof of this theorem uses arguments that are similar to those used for Theorem 3.1. Because it is quite long, it has been given in a separate paper [4].

Table 8. Values for R-AVD Suns

(a) Values a, b ($b \geq a$), such that $\text{Sun}(a,b)$ is R-AVD	

a	b
0	arbitrary
1	$\equiv 0 \,(\text{mod}\,2)$
2	$\not\equiv 0 \,(\text{mod}\,3)$, 3, 6, 9, 12, 18, 21, 24, 36
3	$\equiv 0 \,(\text{mod}\,2)$
4	$4 \leq b \leq 19$ except for $b = 15$, $\equiv 2, 4 \,(\text{mod}\,6)$ with $20 \leq b \leq 46$
5	$\equiv 2, 4 \,(\text{mod}\,6)$ with $8 \leq b \leq 32$, 6, 18
6	6, 7, 8, 10, 11, 12, 14, 16

(b) Values a, b, c ($c \geq b \geq a$), such that $\text{Sun}(a,b,c)$ is R-AVD		

a	b	c
0	0	$\equiv 1, 2 \,(\text{mod}\,3)$
	1	$\equiv 0 \,(\text{mod}\,2)$
	2	2, 3, 4, 6, 7, 8, 10, 11, 14, 16, 18, 19
	3	4, 8, 10
	4	4, 5, 6, 8, 10, 11, 12, 14, 16
	5	6
1	2	2, 4, 6, 8, 10, 14, 16, 18
2	3	4

3.3. BALLOONS

3.3.1. Maximum degree of RAVD Balloons

Remark 3.3. $B(b_1, \ldots, b_k)$ is AVD (resp. R-AVD) iff $B(b_1, \ldots, b_k, 0)$ is AVD (resp. R-AVD). Thus, we will always consider non-zero values for $b_i, 1 \leq i \leq k$.
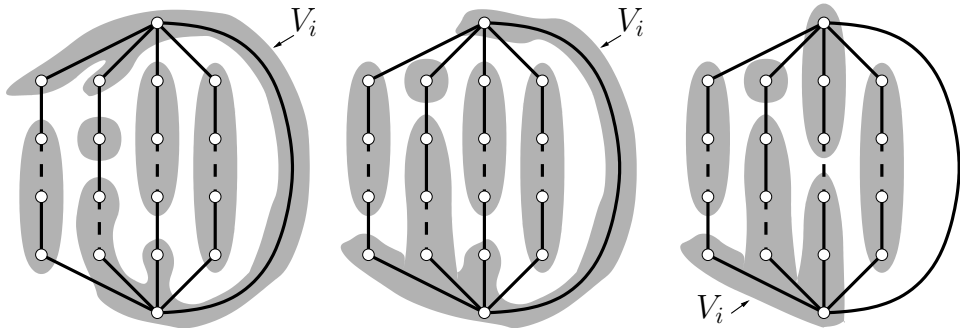

Fig. 5. Removing the empty branch

Proof. Consider a balloon $B(b_1, \ldots, b_k, 0)$ and a decomposition $\lambda = (\lambda_1, \ldots, \lambda_p)$ of $n = b_1 + \ldots + b_k + 2$ (cf. Figure 5). The branch of width 0 means that there exists an edge between the two roots. The existence of such a branch is essential (i.e., it guarantees connectedness of a part $G[V_i]$) only if the part $G[V_i]$ is a tree containing both roots of the balloon. Thus, the other parts V_j, $j \neq i$, are all situated on the branches. We may then move them in such a way that for one root, V_i contains exactly one of its neighbours, that means the other root. And then, we may move one

of the V_j, $j \neq i$, on this root. Now, the edge of the branch with width zero is not necessary and may be removed.   □

Theorem 3.4. *An R-AVD k-balloon has maximum degree at most 5. This bound is tight.*

The proof of Theorem 3.4 is based on Lemmas 3.5 to 3.7.

Lemma 3.5. *The two 5-balloons $B(3, 2, 2, 1, 1)$ and $B(4, 2, 1, 1, 1)$ are R-AVD.*
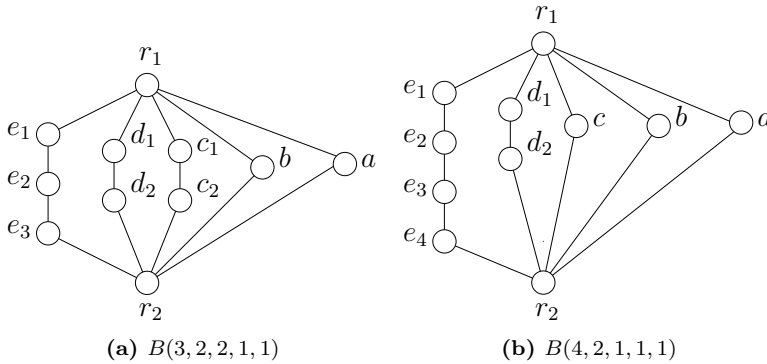

(a) $B(3, 2, 2, 1, 1)$ **(b)** $B(4, 2, 1, 1, 1)$

Fig. 6. RAVD 5-balloons

Proof. The proof is based on Remark 2.6. The values of V_λ are given in Table 9a for $B(3, 2, 2, 1, 1)$ and Table 9b for $B(4, 2, 1, 1, 1)$, using vertex labelings from Figure 6. Please remark that:

— $B(2, 2, 1, 1)$ contains Cat $(3, 5)$ as a spanning tree and then is R-AVD,
— $B(4, 2, 1, 1)$ contains Cat $(3, 7)$ as a spanning tree and then is R-AVD,
— $B(4, 1, 1, 1)$ contains Cat $(2, 7)$ as a spanning tree and then is R-AVD,
— $B(2, 1, 1, 1)$ contains Cat $(2, 5)$ as a spanning tree and then is R-AVD.   □

Table 9. Values of $G[V_\lambda]$ and $G[V \setminus V_\lambda]$ for $B(3, 2, 2, 1, 1)$ and $B(4, 2, 1, 1, 1)$

(a) Values of $G[V_\lambda]$ and $G[V \setminus V_\lambda]$ for $B(3, 2, 2, 1, 1)$

λ	V_λ	$G[V_\lambda]$	$G[V \setminus V_\lambda]$
1	$\{a\}$	P_1	Cat $(3, 7)$
2	$\{c_1, c_2\}$	P_2	Cat $(2, 7)$
3	$\{e_1, e_2, e_3\}$	P_3	$B(2, 2, 1, 1)$
4	$\{a, c_1, c_2, r_2\}$	P_4	Cat $(3, 4)$
5	$\{a, b, c_1, c_2, r_2\}$	Cat $(2, 3)$	P_6

(b) Values of $G[V_\lambda]$ and $G[V \setminus V_\lambda]$ for $B(4, 2, 1, 1, 1)$

λ	V_λ	$G[V_\lambda]$	$G[V \setminus V_\lambda]$
1	$\{a\}$	P_1	$B(4, 2, 1, 1)$
2	$\{d_1, d_2\}$	P_2	$B(4, 1, 1, 1)$
3	$\{a, b, r_2\}$	P_3	Cat $(3, 5)$
4	$\{e_1, e_2, e_3, e_4\}$	P_4	$B(2, 1, 1, 1)$
5	$\{a, b, d_1, d_2, r_2\}$	Cat $(2, 3)$	P_6

Lemma 3.6. *A k-balloon with $k \geq 7$ cannot be R-AVD.*

Proof. We have proved in subsection 3.1 that an R-AVD tree is a path or a 3-pode.

Consider a k-balloon $B(b_1, \ldots, b_k)$ of order $n = b_1 + \ldots + b_k + 2$ with $k \geq 7$, $b_1 \geq \ldots \geq b_k$, and the integer $\lambda = b_1 + 1$. The only way to obtain two R-AVD subgraphs with sizes λ and $n - \lambda$ is to separate the vertices into two trees. Then, at least one of these trees is a k'-pode with $k' \geq 4$.

Thus, a k-balloon with $k \geq 7$ cannot be R-AVD. $\qquad\square$

Lemma 3.7. *A 6-balloon cannot be R-AVD.*

Proof. We suppose there exists an R-AVD 6-balloon $B_{min} = B(b_1, \ldots, b_6)$ of order $n = b_1 + \ldots + b_6 + 2$ as small as possible.

Let λ be an integer, with $\lambda \leq \frac{n}{2}$. We want to split B_{min} into two R-AVD subgraphs $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ of respective orders λ and $n - \lambda$.

Suppose that $G[V_\lambda]$ is a path consisting of inner vertices of a branch of width $b_i > \lambda$. Then the 6-balloon $B(b_1, \ldots, b_{i-1}, b_i - \lambda, b_{i+1}, \ldots, b_6)$ must be R-AVD. Because this 6-balloon has a smaller order than B_{min}, it is impossible.

The two remaining possibilities is to have a branch of width $b_i = \lambda$ or three branches of respective widths $b_{i_1}, b_{i_2}, b_{i_3}$, with $b_{i_1} + b_{i_2} + b_{i_3} + 1 = \lambda$, and $T_3(b_{i_1}, b_{i_2}, b_{i_3})$ R-AVD.

Now, consider the values from 1 to 6 for λ:

For $\lambda \in \{1, 2, 3, 4, 6\}$, the only possibility is to have a branch of width λ.
For $\lambda = 5$, the two possibilities are $G[V_\lambda] = \mathrm{Cat}\,(2, 3)$ or there is a branch with width 5.

Then, the two candidates for B_{min} are $B1 = B(1, 1, 2, 3, 4, 6)$ and $B2 = B(1, 2, 3, 4, 5, 6)$:

— $B1 = B(1, 1, 2, 3, 4, 6)$
   Consider $\lambda = 5$. The only possibility for $G[V_\lambda]$ is $\mathrm{Cat}\,(2, 3)$, using the two branches of width 1 and those of width 2. Then the remaining graph is $T_3(3, 4, 6)$ which is not R-AVD. Thus, $B1$ is not R-AVD.
— $B2 = B(1, 2, 3, 4, 5, 6)$
   Consider $\lambda = 7$. The only possibility for $G[V_\lambda]$ is $\mathrm{Cat}\,(3, 4)$, using the branches of width 1, 2 and 3. The remaining graph is $T_3(4, 5, 6)$ which is not R-AVD. Thus, $B2$ is not R-AVD. $\qquad\square$

3.3.2. Other results

Lemma 3.8. *Let $b_1, \ldots, b_k$ be positive integers with $b_1 \geq \ldots \geq b_k \geq 1$. If $B(b_1, \ldots, b_k)$ is AVD, then*

$$\sum_{j=i+1}^{k} b_j \leq 2b_i \text{ for } i = 1, \ldots, k-1.$$

Proof. Consider the decomposition $\lambda = (b_i + 1, \ldots, b_i + 1, r) = (\lambda_1, \ldots, \lambda_l)$ of n and its $B(b_1, \ldots, b_k)$-realization $(V_1, \ldots, V_l)$. Note that each part $G[V_j]$ of order $b_i + 1$ contains at least one root of the balloon. Let s denote the sum of terms of the decomposition

λ corresponding to the minimal set of parts that cover inner vertices of branches of widths $b_i, \ldots, b_k$. Since $i \leq k-1$, at least one summand in s is equal to $b_i + 1$, and so at least one root is covered. On the other hand, there are two roots, hence at most two summands in s are equal to $b_i + 1$. Besides that, the summand r can be involved in s.

If s has just one summand equal to $b_i + 1$, then $\sum_{j=i}^{k} b_j + 1 \leq (b_i + 1) + r \leq 2(b_i + 1)$ and $\sum_{j=i+1}^{k} b_j \leq b_i < 2b_i$.

If, however, s has two summands equal to $b_i + 1$, then $\sum_{j=i}^{k} b_j + 2 \leq 2(b_i + 1) + r < 3(b_i + 1)$ and $\sum_{j=i+1}^{k} b_j \leq 2b_i$. $\qquad\square$

In the next lemma, we denote by $BP(b_1, b_2, \ldots, b_k)$ the graph formed by attaching a path of length b_1 to a $(k-1)$-balloon $B(b_2, \ldots, b_k)$. $BP(b_1, b_2, \ldots, b_k)$ may also be viewed as a spanning subgraph of the k-balloon $B(b_1, b_2, \ldots, b_k)$ after removing an edge joining the branch of width b_1 to a root.

Lemma 3.9. $BP(b_1, 2, 1, 1)$ with $b_1 \geq 0$ and $BP(b_1, 1, 1, 1)$ with $b_1 \equiv 0 \pmod{2}$ are R-AVD.

Proof. The proof is based on Lemma 2.6. The values of V_λ for $BP(b_1, 2, 1, 1)$ are given in Table 10, and for $BP(b_1, 1, 1, 1)$ in Table 11. These tables use the vertex labeling from Figure 7.

Please remark that:

— for $BP(b_1, 2, 1, 1)$, $\lambda \geq 6 \Rightarrow b_1 \geq \lambda$,
— for $BP(b_1, 1, 1, 1)$, $\lambda \geq 5 \Rightarrow b_1 \geq \lambda$,
— $BP(b_1, 2, 1)$ and $BP(b_1, 1, 1)$ are traceable,
— $BP(b_1, 1, 1)$ is traceable,
— λ odd $\Rightarrow \lambda - 5$ even,
— λ and b_1 even $\Rightarrow b_1 - \lambda$ even. $\qquad\square$

Table 10. Values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$
for $BP(b_1, 2, 1, 1)$

λ	V_λ	$G[V_\lambda]$	$G[V \backslash V_\lambda]$
1	$\{a\}$	P_1	$BP(b_1, 2, 1)$
2	$\{c_1, c_2\}$	P_2	$BP(b_1, 1, 1)$
3	$\{a, b, r_2\}$	P_3	P_{b_1+3}
4	$\{a, c_1, c_2, r_2\}$	P_4	P_{b_1+2}
5	$\{a, b, c_1, c_2, r_2\}$	Cat $(2,3)$	P_{b_1+1}
6	$\{a, b, c_1, c_2, r_1, r_2\}$	$B(2,1,1)$	P_{b_1}
≥ 7	$\{v_{b_1}, v_{b_1-1}, \ldots, v_{b_1-\lambda+1}\}$	P_λ	$BP(b_1 - \lambda, 2, 1, 1)$

Table 11. Values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ for $BP(b_1, 1, 1, 1), b_1 \equiv 0 \ (mod \ 2)$

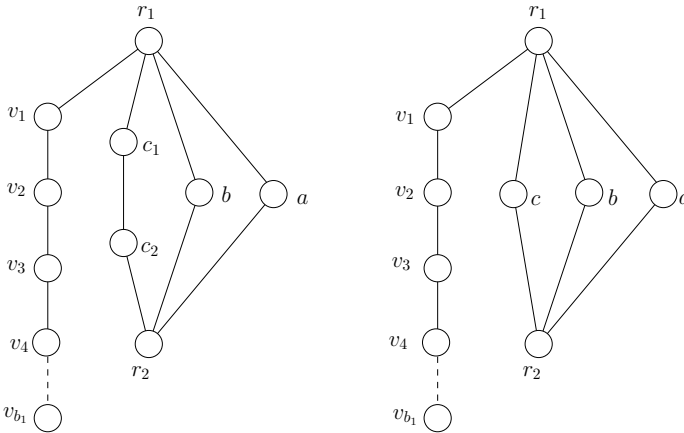| λ		V_λ	$G[V_\lambda]$	$G[V \backslash V_\lambda]$
1		$\{a\}$	P_1	$BP(b_1, 1, 1)$
2		$\{a, r_2\}$	P_2	$\mathrm{Cat}\,(2, b_1 + 1)$
3		$\{a, b, r_2\}$	P_3	$P_{b_1 + 2}$
4	$b_1 = 0$	$\{a, b, r_1, r_2\}$	C_4	P_1
	$b_1 = 2$	$\{v_2, v_1, r_1, c\}$	P_4	P_3
	$b_1 \geq 4$	$\{v_{b_1}, v_{b_1-1}, v_{b_1-2}, v_{b_1-3}\}$	P_4	$BP(b_1 - 4, 1, 1, 1)$
5		$\{a, b, c, r_1, r_2\}$	$B(1, 1, 1)$	P_{b_1}
≥ 5 and odd		$\{a, b, c, r_1, r_2, v_1, \ldots, v_{\lambda-5}\}$	$BP(\lambda - 5, 1, 1, 1)$	$P_{b_1 - \lambda + 5}$
≥ 5 and even		$\{v_{b_1}, \ldots, v_{b_1-\lambda+1}\}$	P_λ	$BP(b_1 - \lambda, 1, 1, 1)$

Fig. 7. $BP(b_1, 2, 1, 1)$ and $BP(b_1, 1, 1, 1)$

4. STRONGLY RECURSIVELY AVD GRAPHS

In this section, we give a characterization of SR-AVD graphs. This characterization, based on the exclusion of two induced subgraphs, leads a polynomial algorithm to check if a graph is SR-AVD or not.

Definition 4.1. A *claw* is a star isomorphic to $K_{1,3}$. A *net* is a graph obtained from a triangle by attaching to each vertex a new dangling edge (cf. Figure 8). A graph is said *claw- and net-free* if it has no induced subgraph isomorphic to either a claw or a net.
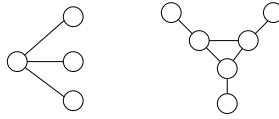
Fig. 8. A claw and a net

We use the following theorem on connected claw- and net-free graphs.

Theorem 4.2 ([5])**.** *Any connected claw- and net-free graph is traceable.*

Now, we may give a characterization of the SR-AVD graphs.

Theorem 4.3. *A connected graph G is SR-AVD if and only if it is claw- and net-free.*

Proof. First of all, we may remark that both claw and net are not SR-AVD (and even not AVD). The sequence $(2, 2)$ is not realizable in the claw and the sequence $(3, 3)$ is not realizable in the net. Thus, if a graph is SR-AVD, it cannot contains a claw or a net as induced subgraph.

The proof of the sufficient condition is based on induction. Let $P(n)$ be the following proposition: "Every connected claw- and net-free graph of order n is SR-AVD". $P(1)$ is trivially true.

Suppose that $P(i)$ is true for any $1 \leq i \leq n-1$ and let G be a connected claw- and net-free graph of order n. Thus G is traceable (and hence AVD) for any decomposition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of length $k \geq 2$ admissible for G we can find (using a hamiltonian path in G) a partition $(V_1, \ldots, V_k)$ of the vertex set of G in which $G[V_i]$ is connected and of order λ_i for $i = 1, \ldots, k$. Since clearly each $G[V_i]$ is claw-free, net-free and of order smaller than n, by the induction hypothesis it is SR-AVD. Thus, $P(n)$ is true for any $n \geq 1$. $\qquad\square$

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