# A NOTE ON THE INDEPENDENT ROMAN DOMINATION IN UNICYCLIC GRAPHS

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**Abstract.** A Roman dominating function (RDF) on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. An RDF $f$ in a graph $G$ is independent if no two vertices assigned positive values are adjacent. The Roman domination number $\gamma_R(G)$ (respectively, the independent Roman domination number $i_R(G)$) is the minimum weight of an RDF (respectively, independent RDF) on $G$. We say that $\gamma_R(G)$ strongly equals $i_R(G)$, denoted by $\gamma_R(G) \equiv i_R(G)$, if every RDF on $G$ of minimum weight is independent. In this note we characterize all unicyclic graphs $G$ with $\gamma_R(G) \equiv i_R(G)$.

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## 1. INTRODUCTION

We consider finite, undirected, and simple graphs $G$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *degree* of $v$, denoted by $d_G(v)$, is the cardinality of its open neighborhood. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. If $v$ is a support vertex, then $v$ is called *strong* if $v$ is adjacent to at least two leaves.

For a graph $G$, let $f : V(G) \to \{0, 1, 2\}$ be a function, and let $(V_0; V_1; V_2)$ be the ordered partition of $V = V(G)$ induced by $f$, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$. There is a $1 - 1$ correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partitions $(V_0; V_1; V_2)$ of $V(G)$. So we will write $f = (V_0; V_1; V_2)$.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* **(RDF)** on $G$ if every vertex $u$ of $G$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ of $G$ for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. An RDF $f$ in a graph $G$ is independent if no two vertices assigned positive values

are adjacent. The *Roman domination number* $\gamma_R(G)$ (respectively, the *independent Roman domination number* $i_R(G)$) is the minimum weight of an RDF (respectively, independent RDF) on $G$. A function $f = (V_0; V_1; V_2)$ is called a $\gamma_R(G)$-function or $\gamma_R$-function for $G$ if it is a Roman dominating function on $G$ and $f(V(G)) = \gamma_R(G)$. An $i_R(G)$-function or $i_R$-function for $G$ is defined similarly. Let $f$ be a $\gamma_R(G)$-function, and $f(x) = 0$ for some vertex $x$. Then we say that $x$ is a *private neighbor* of a vertex $y$ with $f(y) = 2$ if $f$ is not an RDF for $G - xy$. Roman domination has been introduced by Cockayne et al. [3] and has been studied for example in [7]. The study of independent Roman domination has been initiated in [1].

We say that $\gamma_R(G)$ and $i_R(G)$ are *strongly equal* for $G$, denoted by $\gamma_R(G) \equiv i_R(G)$, if every $\gamma_R(G)$-function is an $i_R(G)$-function. In [2] a constructive characterization of all trees $T$ with $\gamma_R(T) \equiv i_R(T)$ is provided. Note that strong equality between two parameters was considered first by Haynes and Slater [6]. Later Haynes, Henning and Slater gave in [4] and [5] constructive characterizations of trees with strong equality between some domination parameters.

In this note we characterize all unicyclic graphs $G$ with $\gamma_R(G) \equiv i_R(G)$.


## 2. MAIN RESULT

We first describe the procedure given in [2] to built trees $T$ with $\gamma_R(T) \equiv i_R(T)$. Let $\mathcal{T}$ be the family of trees $T$ that can be obtained from $k$ $(k \geq 1)$ disjoint stars of centers $x_1, x_2, ..., x_k$, where each star has order at least three, attached by edges from their center vertices either to a single vertex or to the same leaf of a path $P_2$. Such a vertex is called a special vertex of $T$. Let $\mathcal{F}$ be the collection of trees $T$ that can be obtained from a sequence $T_1, T_2, ..., T_k$ $(k \geq 1)$ of trees, where $T_1$ is a star $K_{1,t}$ with $t \geq 2$, $T = T_k$, and, if $k \geq 2$, $T_{i+1}$ can be obtained recursively from $T_i$ by one of the following operations:

— **Operation $\mathcal{O}_1$** : Assume $y$ is a leaf of $T_i$ with $f_i(y) = 0$ and whose support vertex $z$ is either strong or satisfies $\gamma_R(T_i - z) > \gamma_R(T_i)$. Then $T_{i+1}$ is obtained from $T_i$ by adding a new vertex $x$ and adding the edge $xy$.
— **Operation $\mathcal{O}_2$** : Assume $y$ is a vertex of $T_i$. Then $T_{i+1}$ is obtained from $T_i$ by adding a tree $T \in \mathcal{T}$ of special vertex $x$ and adding the edge $xy$ with the condition that if $x$ is a support vertex, then $y$ satisfies $\gamma_R(T_i - y) \geq \gamma_R(T_i)$.
— **Operation $\mathcal{O}_3$** : Assume $y$ is a vertex of $T_i$ assigned $0$ or $1$ for every $\gamma_R(T_i)$-function. Then $T_{i+1}$ is obtained from $T_i$ by adding a path $P_3 = u$-$v$-$w$ and adding the edge $wy$.

**Theorem 2.1** (Chellali and Jafari Rad [2]). *Let $T$ be a tree. Then $\gamma_R(T) \equiv i_R(T)$ if and only if $T = K_1$ or $T \in \mathcal{F}$.*

Let $\mathcal{H}$ be the class of all graphs $G$ such that $G$ is obtained from a tree $T \in \mathcal{F}$ by joining two non-adjacent vertices $v_1, v_2$ such that:

(1) For every $\gamma_R(T)$-function $f$, $0 \in \{f(v_1), f(v_2)\}$,

(2) For $1 \leq i \neq j \leq 2$, there is no non-independent RDF $f$ for $T - v_i$ with weight $\gamma_R(T)$ such that $f(v_j) = 2$.

Now we are ready to state our main result.

**Theorem 2.2.** *Let $G$ be a unicyclic graph. Then $\gamma_R(G) \equiv i_R(G)$ if and only if $G \in \mathcal{H}$.*

*Proof.* Let $G$ be a unicyclic graph, where $C$ is its unique cycle. Assume that $\gamma_R(G) \equiv i_R(G)$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. By assumption $f$ is independent. Let $x \in V(C) \cap V_0$, and let $N(x) \cap V(C) = \{y, z\}$. Clearly $x$ cannot be a private neighbor for both $y$ and $z$. Hence we assume that $x$ is not a private neighbor of $y$ and let $T = G - xy$. Then $f$ is an IRDF for $T$, and so $\gamma_R(T) \leq i_R(T) \leq \gamma_R(G) = i_R(G)$. If $\gamma_R(T) < i_R(G)$, and $f_1$ is a $\gamma_R(T)$-function, then $f_1$ is an RDF for $G$ with weight less than $\gamma_R(G)$, a contradiction. Thus $\gamma_R(T) = i_R(T) = i_R(G) = \gamma_R(G)$. Next we show that any $\gamma_R(T)$-function is independent. Assume to the contrary that $f$ is a $\gamma_R(T)$-function and $f$ is not independent. Since $f$ is an RDF for $G$ and $\gamma_R(G) = \gamma_R(T)$, we obtain that $f$ is a $\gamma_R(G)$-function, contradicting the fact that $\gamma_R(G) \equiv i_R(G)$. Thus $f$ is independent and consequently, $\gamma_R(T) \equiv i_R(T)$. We deduce that $T \in \mathcal{F}$.

Next we prove (1). Suppose that there is a $\gamma_R(T)$-function $f$ such that $0 \notin \{f(x), f(y)\}$. If $\{f(x), f(y)\} = \{2, 1\}$ and $f(x) = 1$, then $g$ defined on $G$ by $g(x) = 0$ and $g(u) = f(u)$ if $u \neq x$ is an RDF for $G$ with weight less than $\gamma_R(G)$, a contradiction. Thus $\{f(x), f(y)\} \neq \{2, 1\}$ but then $f$ would be a non-independent $\gamma_R(G)$-function, a contradiction since $\gamma_R(G) \equiv i_R(G)$.

Finally, let us prove (2). Assume that there is a non-independent RDF $f$ for $T - x$ with weight $\gamma_R(T)$ such that $f(y) = 2$. Then $f$ is a $\gamma_R(G)$-function which is not independent, a contradiction.

Conversely, assume that $G \in \mathcal{H}$. Let $G$ be obtained from a tree $T \in \mathcal{F}$ by joining two vertices $x$ and $y$ such that (1) and (2) hold. First notice that $\gamma_R(G) \leq \gamma_R(T)$. Assume to the contrary that $\gamma_R(G) < \gamma_R(T)$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. If $\{f(x), f(y)\} \neq \{0, 2\}$, then $f$ is an RDF for $T$ with weight less than $\gamma_R(T)$, a contradiction. Thus $\{f(x), f(y)\} = \{0, 2\}$. Suppose that $f(y) = 0$. Then $N(y) \cap V_2 = \{x\}$. Now $g$ defined on $T$ by $g(y) = 1$ and $g(u) = f(u)$ if $u \neq y$, is an RDF for $T$. Then $w(g) = \gamma_R(T)$ for otherwise $g$ is an RDF for $T$ with weight less than $\gamma_R(T)$ which is impossible. Hence $g$ is a $\gamma_R(T)$-function and $0 \notin \{g(x), g(y)\}$, contradicting (1). Therefore $\gamma_R(G) = \gamma_R(T)$. Now let $h$ be an $i_R(T)$-function. Note that $h$ is a $\gamma_R(T)$-function since $\gamma_R(T) \equiv i_R(T)$. If $h$ is not an IRDF for $G$, then $0 \notin \{h(x), h(y)\}$, and $h$ is a $\gamma_R(T)$-function that does not satisfy (1), a contradiction. Thus $h$ is an IRDF for $G$, and so $i_R(G) \leq \gamma_R(T) = \gamma_R(G) \leq i_R(G)$, implying that $i_R(G) = \gamma_R(G) = \gamma_R(T) = i_R(T)$. So $h$ is an $i_R(G)$-function. We next show that each $\gamma_R(G)$-function is independent. Assume to the contrary that $f = (V_0, V_1, V_2)$ is a $\gamma_R(G)$-function and $f$ is not independent. If $0 \notin \{f(x), f(y)\}$, then $f$ is a $\gamma_R(T)$-function which is not independent, contradicting the fact that $T \in \mathcal{F}$. Thus $0 \in \{f(x), f(y)\}$, and we may assume that $f(y) = 0$. Furthermore, $N(y) \cap V_2 = \{x\}$. Then $f|_{T-y}$ is an IRDF for $T - y$ with weight $\gamma_R(T)$ and $f(x) = 2$, a contradiction with (2). We deduce that $\gamma_R(G) \equiv i_R(G)$. $\square$

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