Opuscula Mathematica • Vol. 32 • No. 4 • 2012

http://dx.doi.org/10.7494/OpMath.2012.32.4.719

OSCILLATION BEHAVIOR OF SECOND ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

Elmetwally M. Elabbasy, T.S. Hassan, and O. Moaaz

Abstract. Oscillation criteria are established for second order nonlinear neutral differential equations with deviating arguments of the form

$$r(t)\psi(x(t))\left|z'(t)\right|^{\alpha-1}z'(t)+\int_a^b q(t,\xi)f(x(g(t,\psi)))d\sigma(\xi)=0, t\geq t_0,$$

where $\alpha > 0$ and $z(t) = x(t) + p(t)x(t-\tau)$. Our results improve and extend some known results in the literature. Some illustrating examples are also provided to show the importance of our results.

Keywords: oscillation, second order, neutral differential equations, deviating arguments.

Mathematics Subject Classification: 34C10, 34C15.

1. INTRODUCTION

This paper is concerned with the oscillation problem of the second order nonlinear neutral differential equation with distributed argument:

$$\Big(r(t)\psi(x(t))\left|z'(t)\right|^{\alpha-1}z'(t)\Big)+\int_a^b q(t,\xi)f(x(g(t,\psi)))d\sigma(\xi)=0,\quad t\geq t_0,\tag{1.1}$$

where $\alpha > 0$ and $z(t) = x(t) + p(t)x(t-\tau)$. We assume that:

(A_1) $r, p \in C(I, R)$ and $-\mu \leq p(t) \leq 1$, $\mu \in (0,1)$, $r(t) > 0$, $\int_{t_0}^{\infty} r^{-1/\alpha}(t)dt$;

(A_2) $\psi \in C(R, R)$ and there exists a positive real number L such that $0<\psi(x)\leq L^{-1}$;

(A_3) $\psi \in C(R,R), f(x)sgnx > |x|^{\alpha}$ for $x \neq 0$ and $t \geq t_0$;
(A_4) $q \in C(I \times [a,b],[0,\infty)), q(t,\xi)$ is not zero on any half line $[t_{\mu},\infty) \times [a,b], t_{\mu} \geq t_0$;
(A_5) $g \in C(I \times [a,b],[0,\infty))$, for $t \geq t_0$ and $\xi \in [a,b]$, $g(t,\xi)$ is continuous, has positive partial derivative on $I \times [a,b]$ with respect to t, nondecreasing with respect to ξ and $\lim_{t\to\infty} \inf g(t,\xi) = \infty$;
(A_6) $\sigma \in C([a,b],R)$, σ is nondecreasing and the integral of Eq. (1.1) is in the sense of Riemann-Stieltijes.

We restrict our attention to those solutions $x(t)$ of Eq. (1.1) which exist on some half linear $[t_0,\infty)$ and satisfy $\sup\{|x(t)| : t \geq t_x\} \neq 0$ for any $T \geq t_0$. As usual, such a solution of Eq. (1.1) is called oscillatory if the set of its zeros is unbounded from above, otherwise, it is said to be nonoscillatory. Eq. (1.1) is called oscillatory if all its solutions are oscillatory.

We note that second order neutral delay differential equations have various applications in problems dealing with vibrating masses attached to an elastic bar and in some variational problems. For further applications and questions concerning existence and uniqueness of solutions of neutral delay differential equations, see [1, 2].

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and/or nonoscillation of solutions of second order linear and nonlinear neutral delay differential equations with distributed deviating arguments.

The oscillation problem for the nonlinear delay equation such as

$$(r(t)x'(t))' + q(t)f(x(\sigma(t))) = 0, \quad t > t_0,$$

and the neutral delay differential equation

$$(x(t) + p(t)x(t-\tau)'' + q(t)x(t-\sigma) = 0$$

have been studied by many authors with different methods. Some results can be found in [3–5] and the references therein.

Recently, in [6], by using the Riccati technique and the averaging functions method, Ruan established some general oscillation criteria for the second-order neutral delay differential equation

$$(r(t)(x(t) + p(t)x(t-\tau))')' + q(t)f(x(t-\sigma)) = 0.$$

Sahiner [7] obtained some general oscillation criteria for neutral delay differential equations

$$(r(t)\psi(x(t))z'(t))' + q(t)f(x(\sigma(t))) = 0$$

is oscillatory. In Wang [8] and Zhiting Xu [9], by using the Riccati technique and averaging functions method, have established some general oscillation criteria for second-order neutral delay differential equation with distributed deviating argument

$$(r(t)z'(t))' + \int_a^b q(t,\xi)x(g(t,\xi))d\sigma(\xi) = 0$$

and

$$(r(t)\psi(x(t))z'(t))' + \int_a^b q(t,\xi)f(x(g(t,\xi)))d\sigma(\xi) = 0.$$

Recently, Aydin [10] and Jiu-Gang Dong [11] obtained some oscillation criteria for the study of second order nonlinear neutral differential equations

$$(r(t)|x'(t)|^{\alpha-1}x'(t))' + q(t)f(x(\sigma(t))) = 0,$$
$$(r(t)|z'(t)|^{\alpha-1}z'(t))' + f(t, x(\sigma(t))) = 0.$$

In this paper, some oscillation criteria are established for Eq. (1.1). For the case when $0 \le p(t) \le 1$, our results which complement and extend the results in [10, 12]. On the other hand, we will establish some oscillation criteria for Eq. (1.1) in the case $-\mu \le p(t) \le 0$, $\mu \in (0,1)$.

2. OSCILLATION CRITERIA FOR $0 \le p(t) \le 1$

In this section, we establish some oscillation criteria for Eq. (1.1) in the case when $0 \le p(t) \le 1$. For simplicity, we define the following notation.

$$\Theta_1(t) = \int_a^b q(t,\xi)(1-p(g(t,\xi)))^\alpha d\sigma(\xi), \quad R_1(t) = \frac{\alpha L^{1/\alpha} g'(t,a)}{r^{1/\alpha}(g(t,a))}.$$

Lemma 2.1. *Let $x(t)$ be a positive solution of Eq.* (1.1)*. Then there exists $T_0 \ge t_0$ such that*

$$z(t) \ge 0, z'(t) \ge 0 \text{ and } (r(t)\psi(x(t))|z'(t)|^{\alpha-1}z'(t))' \le 0 \text{ for } t \ge T_0. \tag{2.1}$$

Moreover, for $t \ge T_0$

$$(r(t)\psi(x(t))|z'(t)|^{\alpha-1}z'(t))' + z^\alpha(g(t,a)) \int_a^b q(t,\xi)(1-p(g(t,\xi)))^\alpha d\sigma(\xi) \le 0. \tag{2.2}$$

Proof. Let $x(t)$ be eventually a positive solution of Eq. (1.1). Note that in view of (A_5), there exists a $t_1 \ge t_0$ such that

$$x(t) > 0, x(t-\tau) > 0 \text{ and } x(g(t,\xi)) > 0 \quad \text{for} \quad t \ge t_1 \ge t_0, \xi \in [a,b].$$

Then $z(t) = x(t) + p(t)x(t-\tau) \ge 0$ and from Eq. (1.1), (A_3) and (A_4), we have

$$\begin{aligned}(r(t)\psi(x(t))|z'(t)|^{\alpha-1}z'(t))' &= -\int_a^b q(t,\xi)f(x(g(t,\xi)))d\sigma(\xi) \le \\ &\le -\int_a^b q(t,\xi)\,|x(g(t,\xi))|^\alpha\, d\sigma(\xi).\end{aligned} \tag{2.3}$$

Thus, $(r(t)\psi(x(t))|z'(t)|^{\alpha-1}z'(t))$ is a nonincreasing function. Now we have two possible cases for $z'(t) \geq 0$ either $z'(t) \geq 0$ eventually or $z'(t) \geq 0$ eventually. Suppose that $z'(t) \geq 0$ for $t \geq t_2 \geq t_1$. Then, from (2.3), we have

$$r(t)\psi(x(t))|z'(t)|^{\alpha-1}z'(t) \leq r(t_2)\psi(x(t_2))|z'(t_2)|^{\alpha-1}z'(t_2),$$

which implies that

$$z'(t) \leq -\frac{\delta L^{1/\alpha}}{r^{1/\alpha}(t)}|z'(t_2)|, \tag{2.4}$$

where $\psi(x) \leq L^{-1}$ and $\delta = (r(t_2)\psi(x(t_2)))^{1/\alpha} \geq 0$.

By integrating (2.4) from t_2 to t,

$$z(t) \leq z(t_2) - \delta L^{1/\alpha}|z'(t_2)| \int_{t_2}^{t} r(t)^{-1/\alpha}dt.$$

Therefore, from (A_1), we have $\lim_{t\to\infty} z(t) = -\infty$, this contradicts $z(t) \geq 0$ for $t \geq t_1$. then $z'(t) > 0$ for $t \geq t_1$. Using this fact together with $x(t) \leq z(t)$, we see that

$$x(g(t,\xi) - \tau) \leq z(g(t,\xi) - \tau) \leq z(g(t,\xi)).$$

Then,

$$x(g(t,\xi)) \geq (1 - p(g(t,\xi)))z(g(t,\xi)). \tag{2.5}$$

Thus, from (2.3), we get

$$(r(t)\psi(x(t))|z'(t)|^{\alpha-1}z'(t))' + \int_{a}^{b} q(t,\xi)(1 - p(g(t,\xi)))^{\alpha} z^{\alpha}(g(t,\xi))d\sigma(\xi) \leq 0. \tag{2.6}$$

Further, from (A_5) and $z'(t) > 0$ for $t \geq t_1$, we get $z(g(t,\xi)) \geq z(g(t,a))$ for $t \geq t_1$, $\xi \in [a,b]$. Thus, (2.6) implies that (2.2) hold. This completes the proof of Lemma 2.1. □

Theorem 2.2. *Suppose that Eq.* (1.1) *is nonoscillatory. Then there exists a positive function* $\omega(t)$ *on* $[T,\infty)$ *such that*

$$\int_{t}^{\infty} \Theta_1(u)du < \infty, \qquad \int_{t}^{\infty} R(u)\omega^{\frac{\alpha+1}{\alpha}}(u)du < \infty, \tag{2.7}$$

$$\int_{t}^{\infty} \Theta_1(u)du + \int_{t}^{\infty} R(u)\omega^{\frac{\alpha+1}{\alpha}}(u)du \leq \omega(t) \tag{2.8}$$

for $t \geq T \geq t_0$, *and*

$$\lim_{t\to\infty} \inf \omega(t) \left(\int_{t_0}^{g(t,\xi)} r^{-1/\alpha}(u)du \right)^{\alpha} \leq \frac{1}{L}. \tag{2.9}$$

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Without loss of generality, we assume that $x(t) > 0$. Define

$$\omega(t) = \frac{r(t)\psi(x(t))(z'(t))^\alpha}{z^\alpha(g(t,a))}. \tag{2.10}$$

Therefore, $\omega(t) > 0$. By differentiating (2.10) and using (2.2), it follows that

$$\begin{aligned}\omega'(t) \le -\int_a^b q(t,\xi)(1-p(g(t,\xi)))^\alpha d\sigma(\xi) - \alpha\frac{r(t)\psi(x(t))(z'(t))^\alpha}{z^{2\alpha}(g(t,a))}\times \\ \times z^{\alpha-1}(g(t,a))z'(g(t,a))g'(t,a).\end{aligned} \tag{2.11}$$

Since $g(t,\xi) \le t$, $z'(t) > 0$ and $(r(t)\psi(x(t))(z'(t))^\alpha)' \le 0$ for $t \ge T_0$, we have

$$r(t)\psi(x(t))(z'(t))^\alpha \le r(g(t,a))\psi(x(g(t,a)))(z'(g(t,a)))^\alpha.$$

Thus, from (A_2) and (2.11), we get

$$\begin{aligned}\omega'(t) \le -\int_a^b q(t,\xi)(1-p(g(t,\xi)))^\alpha d\sigma(\xi) - \left(\frac{r(t)\psi(x(t))(z'(t))^\alpha}{z^\alpha(g(t,a))}\right)^{\frac{\alpha+1}{\alpha}} \frac{\alpha L^{1/\alpha}g'(t,a)}{r^{1/\alpha}(g(t,a))} = \\ = -\Theta_1(t) - R(t)\omega^{\frac{\alpha+1}{\alpha}}(t) < 0.\end{aligned} \tag{2.12}$$

Integrating (2.12) from t to t^*, we obtain

$$\omega(t^*) - \omega(t) + \int_t^{t^*} \Theta_1(u)du + \int_t^{t^*} R(u)\omega^{\frac{\alpha+1}{\alpha}}(u)du \le 0. \tag{2.13}$$

We claim that $\int_t^\infty \Theta_1(u)du < \infty$ for $t \ge t_2$. Otherwise, from (2.13) it follows that

$$\omega(t^*) \le \omega(t) - \int_t^{t^*} R(u)\omega^{\frac{\alpha+1}{\alpha}}(u)du.$$

Therefore, from (2.7), we get $\lim_{t^*\to\infty}\omega(t^*) = -\infty$, this contradicts $\omega(t) \ge 0$ for $t \ge t_2$. Similarly, we can show

$$\int_t^\infty R(u)\omega^{\frac{\alpha+1}{\alpha}}(u)du < \infty \quad \text{for} \quad t \ge t_2.$$

From the above inequality and (2.12), we have $\lim_{t^*\to\infty}\omega(t^*) = 0$. Letting $t^* \to \infty$ in (2.13), we get (2.8). To prove (2.9) note that

$$\frac{1}{\omega(t)} = \frac{1}{r(t)\psi(x(t))}\left(\frac{z(g(t,a))}{z'(t)}\right)^\alpha \ge$$

$$\geq \frac{1}{r(t)\psi(x(t))}\left(\frac{z(t_2)+\int\limits_{t_2}^{g(t,a)} r^{-1/\alpha}(u)\psi^{-1/\alpha}(x(u))r^{1/\alpha}(u)\psi^{1/\alpha}(x(u))z'(u)du}{z'(t)}\right)^{\alpha} \geq$$

$$\geq \frac{1}{r(t)\psi(x(t))}\left(\frac{r^{1/\alpha}(t)\psi^{1/\alpha}(x(t))z'(t)\int\limits_{t_2}^{g(t,a)} r^{-1/\alpha}(u)\psi^{-1/\alpha}(x(u))du}{z'(t)}\right)^{\alpha} \geq$$

$$\geq \left(L^{1/\alpha}\int\limits_{t_2}^{g(t,a)} r^{-1/\alpha}(u)du\right)^{\alpha}.$$

Thus, we have

$$\omega(t)\left(\int\limits_{t_0}^{g(t,a)} r^{-1/\alpha}(u)du\right)^{\alpha} \leq \frac{1}{L}\left(\frac{\int_{t_0}^{g(t,a)} r^{-1/\alpha}(u)du}{\int_{t_2}^{g(t,a)} r^{-1/\alpha}(u)du}\right)^{\alpha},$$

which implies (2.9). This completes the proof of Theorem 2.2. □

Corollary 2.3. *Assume that* $\int_{t_0}^{\infty}\Theta_1(u)du=\infty$. *Then Eq.* (1.1) *is oscillatory.*

Corollary 2.4. *Assume that*

$$\lim_{t\to\infty}\inf \frac{1}{Q_1(t)}\int\limits_{t}^{\infty} Q_1^{\frac{\alpha+1}{\alpha}}(u)R(u)du > \frac{\alpha}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}, \tag{2.14}$$

where

$$Q_1(t)=\int\limits_{t}^{\infty}\Theta_1(u)du.$$

Then Eq. (1.1) *is oscillatory.*

Proof. Suppose to the contrary that Eq. (1.1) is nonoscillatory. According to Theorem 2.2, we have that (2.8) holds. By (2.14), there exists a constant $\beta > \alpha/(\alpha+1)^{\frac{\alpha+1}{\alpha}}$ such that

$$\lim_{t\to\infty}\inf \frac{1}{Q_1(t)}\int\limits_{t}^{\infty} Q_1^{\frac{\alpha+1}{\alpha}}(u)R(u)du > \beta.$$

On the other hand, using (2.8), we get

$$\begin{aligned}\frac{\omega(t)}{Q_1(t)} &\geq 1+\frac{1}{Q_1(t)}\int_t^{\infty} R(u)\omega^{\frac{\alpha+1}{\alpha}}(u)du = \\ &= 1+\frac{1}{Q_1(t)}\int_t^{\infty} R(u)Q_1^{\frac{\alpha+1}{\alpha}}(u)\left(\frac{\omega(u)}{Q_1(u)}\right)^{\frac{\alpha+1}{\alpha}} du \geq \\ &\geq 1+\frac{1}{Q_1(t)}\left(\frac{\omega(t)}{Q_1(t)}\right)^{\frac{\alpha+1}{\alpha}}\int_t^{\infty} R(u)Q_1^{\frac{\alpha+1}{\alpha}}(u)du.\end{aligned} \tag{2.15}$$

Let $\lambda = \inf_{t\geq T}\frac{\omega(t)}{Q_1(t)}$, then $\lambda \geq 1$. (2.15) implies that $\lambda \geq 1+\lambda^{\frac{\alpha+1}{\alpha}}\beta$. By using the inequality

$$Ax - Bx^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}A^{\gamma+1}B^{-\gamma}, \quad B>0, A\geq 0, x\geq 0.$$

Taking $\gamma=\alpha, A=1, B=\beta$ and $x=\lambda$, we get

$$\lambda - \lambda^{\frac{\alpha+1}{\alpha}}\beta \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\beta^{-\alpha} < 1.$$

This contradicts $\lambda \geq 1+\lambda^{\frac{\alpha+1}{\alpha}}\beta < 1$ and completes the proof. □

3. OSCILLATION CRITERIA FOR $-\mu \leq p(t) \leq 0$.

In this section, we present some oscillation criteria for Eq. (1.1) under the case $-\mu \leq p(t) \leq 0$ for $\mu \in (0,1)$. It will be convenient to make use of the following notation.

$$\Theta_2(t) = \int_a^b q(t,\xi)d\sigma(\xi).$$

Lemma 3.1. *Let $x(t)$ be a solution of Eq. (1.1) which is neither oscillatory nor tends to zero. Then there exist $t_1 \geq t_0$ such that*

$$z(t)\geq 0, z'(t)\geq 0 \text{ and } (r(t)\psi(x(t))|z'(t)|^{\alpha-1}z'(t))' \leq 0 \quad \text{for} \quad t\geq t_1. \tag{3.1}$$

Moreover, for $t \geq t_1$

$$(r(t)\psi(x(t))|z'(t)|^{\alpha-1}z'(t))' + z^{\alpha}(g(t,a))\int_a^b q(t,\xi)d\sigma(\xi) \leq 0. \tag{3.2}$$

Proof. Let $x(t)$ be a solution of Eq. (1.1) which is neither oscillatory nor tends to zero. Without loss of generality, we assume that $x(t) > 0$. Note that in view of (A_5), there exists a $t_1 \geq t_0$ such that

$$x(t) > 0, x(t-\tau) > 0 \text{ and } x(g(t,\xi)) > 0 \text{ for } t \geq t_1 \geq t_0, \xi \in [a,b].$$

From Eq. (1.1), (A_3) and (A_4),

$$(r(t)\psi(x(t))|z'(t)|^{\alpha-1}z'(t))' = -\int_a^b q(t,\xi)f(x(g(t,\xi)))d\sigma(\xi) \leq$$
$$\leq -\int_a^b q(t,\xi)\left|x(g(t,\xi))\right|^{\alpha} d\sigma(\xi).$$

Thus, $(r(t)\psi(x(t))|z'(t)|^{\alpha-1}z'(t))$ is a nonincreasing function. As a result, $z'(t)$ and $z(t)$ are eventually of constant sign. Now we have two possible cases for $z(t)$ either $z(t) < 0$ eventually or $z(t) > 0$ eventually. Suppose that $z(t) < 0$ eventually, say, $z(t) < 0$ for $t > t_2 \geq t_1$. Consider two cases for $x(t)$: (a) $x(t)$ is unbounded, (b) $x(t)$ is bounded.

(a) Assume that $x(t)$ is unbounded. For $t > t_2$, we have

$$x(t) = z(t) - p(t)x(t-\tau) < -p(t)x(t-\tau) < x(t-\tau). \tag{3.3}$$

On the other hand, since $x(t)$ is unbounded we can choose a sequence $\{T_n\}_{n=1}^{\infty}$ satisfying that $\lim_{n\to\infty} T_n = \infty, \lim_{n\to\infty} x(T_n) = \infty$ and $\max_{T_1\leq t\leq T_n} x(t) = x(T_n)$. By picking N so large that $T_N > t_2$ and $(T_N - \tau) > T_1$. Therefore, $\max_{(T_N-\tau)\leq t\leq T_N} x(t) = x(T_N)$ which contradicts (3.3).

(b) Assume that $x(t)$ is bounded. We claim that $\lim_{t\to\infty} x(t) = 0$. To see this, note that

$$\begin{aligned} 0 &\geq \lim_{t\to\infty} \sup z(t) = \lim_{t\to\infty} \sup\left(x(t) + p(t)x(t-\tau)\right)) \geq \\ &\geq \lim_{t\to\infty} \sup x(t) + \lim_{t\to\infty} \inf p(t)x(t-\tau) \geq \\ &\geq \lim_{t\to\infty} \sup x(t) - \mu \lim_{t\to\infty} \sup x(t-\tau) \geq (1-\mu) \lim_{t\to\infty} \sup x(t), \end{aligned}$$

which has proved the claim and this contradicts the assumptions that $x(t)$ is neither oscillatory nor tends to zero. Then $z(t) > 0$ for $t \geq t_2$. Proceeding as in the proof of Theorem 2.2 until we prove $z'(t) > 0$. Now we have

$$x(t) = z(t) - p(t)x(t-\tau) \geq z(t).$$

Therefore, from Eq. (1.1), we get

$$(r(t)\psi(x(t))|z'(t)|^{\alpha-1}z'(t))' + \int_a^b q(t,\xi)\left|z(g(t,\xi))\right|^{\alpha} d\sigma(\xi) \leq 0. \tag{3.4}$$

Further, from (A_5) and $z'(t) > 0$ for $t \geq t_2$, we get $z(g(t,\xi)) \geq z(g(t,a))$ for $t \geq t_2, \xi \in [a,b]$. Thus, (3.4) implies that (3.2) hold. This completes the proof of Lemma 3.1. □

Theorem 3.2. *Assume that every solution of Eq.* (1.1) *is neither oscillatory nor tends to zero. Then there exists a positive function $\omega(t)$ on $[T,\infty)$ such that*

$$\int_t^\infty \Theta_2(u)du < \infty, \qquad \int_t^\infty R(u)\omega^{\frac{\alpha+1}{\alpha}}(u)du < \infty, \tag{3.5}$$

$$\int_t^\infty \Theta_2(u)du + \int_t^\infty R(u)\omega^{\frac{\alpha+1}{\alpha}}(u)du \le \omega(t), \tag{3.6}$$

for $t \ge T \ge t_0$, and

$$\lim_{t\to\infty} \inf \omega(t) \left(\int_{t_0}^{g(t,\xi)} r^{-1/\alpha}(u)du \right)^\alpha \le \frac{1}{L}. \tag{3.7}$$

Proof. Replacing Θ_2 by Θ_1 and following the similar steps as in the proof of Theorem 2.2, we can get all desired results. □

Corollary 3.3. *Assume that $\int_{t_0}^\infty \Theta_2(u)du = \infty$. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero.*

Corollary 3.4. *Assume that*

$$\lim_{t\to\infty} \inf \frac{1}{Q_2(t)} \int_t^\infty Q_2^{\frac{\alpha+1}{\alpha}}(u)R(u)du > \frac{\alpha}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}, \tag{3.8}$$

where

$$Q_2(t) = \int_t^\infty \Theta_2(u)du.$$

Then every solution of Eq. (1.1) *is either oscillatory or tends to zero.*

4. EXAMPLES

Example 4.1. Consider the nonlinear differential equation

$$\left(\frac{1}{2(1+x^2)}\left(x(t)+\frac{1}{t+2}x(t-1)\right)'\right)' + \int_0^1 \frac{\gamma(t+\xi+2)}{t^2(t+\xi+1)}x(t+\xi)d\xi = 0, \tag{4.1}$$

where $\alpha = 1$, $r(t) = 1$, $\psi(x) = \frac{1}{2(1+x^2)}$, $p(t) = \frac{1}{t+2}$, $q(t,\xi) = \frac{\gamma(t+\xi+2)}{t^2(t+\xi+1)}$, $\gamma > \frac{1}{4}$, $g(t,\xi) = t+\xi$, $f(x) = x$ and $\sigma(\xi) = \xi$. If we take $L = 1$, then

$$R(t) = 1, \Theta_1(t) = \frac{\gamma}{t^2}, Q_1(t) = \frac{\gamma}{t}$$

and

$$\liminf_{t\to\infty} \frac{1}{Q_1(t)} \int_t^{\infty} Q_1^{\frac{\alpha+1}{\alpha}}(u)R(u)du = \gamma.$$

Hence, by Corollary 2.4, Eq. (4.1) is oscillatory if $\gamma > 1/4$.

Example 4.2. Consider the nonlinear differential equation

$$\left(\frac{1}{e^t(1+x^2(t))}\left|\left(x(t)+(1-e^{-\frac{1}{4}t})x(t-1)\right)'\right|\left(x(t)+(1-e^{-\frac{1}{4}t})x(t-1)\right)'\right)' +$$

$$+\int_0^1 e^{-\frac{1}{2}(t+\xi)}x(t+\xi)d\xi = 0, \tag{4.2}$$

where $\alpha = 2$, $r(t) = e^{-t}$, $\psi(x) = \frac{1}{1+x^2(t)}$, $p(t) = 1 - e^{-\frac{1}{4}t}$, $q(t,\xi) = e^{-\frac{1}{2}(t+\xi)}$, $g(t,\xi) = t+\xi$, $f(x) = x$ and $\sigma(\xi) = \xi$. If we take $L = 1$, then

$$R(t) = 2e^{\frac{1}{2}t}, \Theta_1(t) = e^{-t}(1-e^{-1}), Q_1(t) = e^{-t}(1-e^{-1})$$

and

$$\liminf_{t\to\infty} \frac{1}{Q_1(t)} \int_t^{\infty} Q_1^{\frac{\alpha+1}{\alpha}}(u)R(u)du = 2(1-e^{-1})^{1/2} > 2/\sqrt{27}.$$

Therefore, Eq. (4.2) is oscillatory by Corollary 2.4.

Example 4.3. Consider the nonlinear differential equation

$$(|z'(t)|z'(t))' + \int_0^1 \frac{\vartheta}{t^2}x(\sqrt{t+\xi})d\xi = 0, \tag{4.3}$$

where $z(t) = x(t)+p(t)x(t-\tau)$, $-\mu \le p(t) \le 1$, $\alpha = 2$, $r(t) \equiv 1$, $\psi(x) \equiv 1$, $q(t,\xi) = \frac{\vartheta}{t^2}$, $g(t,\xi) = \sqrt{t+\xi}$, $f(x) = x$ and $\sigma(\xi) = \xi$. If we take $L = 1$, then

$$R(t) = \frac{1}{\sqrt{t}}, \Theta_2(t) = \frac{\vartheta}{t^2}, Q_2(t) = \frac{\vartheta}{t}$$

and

$$\liminf_{t\to\infty} \frac{1}{Q_2(t)} \int_t^{\infty} Q_2^{\frac{\alpha+1}{\alpha}}(u)R(u)du = \liminf_{t\to\infty} \frac{t}{\vartheta} \int_t^{\infty} \left(\frac{\vartheta}{u}\right)^{3/2} \frac{1}{\sqrt{u}}du = \sqrt{\vartheta}.$$

Thus, if $\vartheta > 4/27$, by Corollary 3.4, we have that every solution of Eq. (4.3) is either oscillatory or tends to zero.

REFERENCES

[1] I. Gyori, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Oxford, 1991.

[2] L.H. Erbe, Q. Kong, B.G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, New York, 1995.

[3] X.Z. Liu, X.L. Fu, *Nonlinear differential inequalities with distributed deviating argument and applications*, Nonlinear World **4** (1994), 409–427.

[4] Y.H. Yu, X.L. Fu, *Oscillation of second order neutral equation with continuous distributed deviating argument*, Rad. Mat. **7** (1991), 167–176.

[5] J.K. Hale, *Theory of Functional Differential Equations*, Springer Verlag, New York, 1997.

[6] S. Ruan, *Oscillation of second-order neutral delay differential equations*, Can. Math. Bull. **36** (1993), 485–496.

[7] Y. Sahiner, *On oscillation of second-order neutral type delay differential equations*, Appl. Math. Comput. **150** (2004), 697–706.

[8] P.G. Wang, *Oscillation criteria for second-order neutral equations with distributed deviating arguments*, Comput. Math. Appl. **47** (2004), 1935–1946.

[9] Z.T. Xu, P.X. Weng, *Oscillation of second-order neutral equations with distributed deviating arguments*, J. Comput. Appl. Math. **202** (2007), 460–477.

[10] A. Tiryaki, *Oscillation criteria for a certain second-order nonlinear differential equations with deviating arguments*, Electron. J. Differential Equations **61** (2009), 1–11.

[11] Jiu-Gang Dong, *Oscillation behavior of second order nonlinear neutral differential equations with deviating arguments*, Appl. Math. Comput. **59** (2010), 3710–3717.

[12] G. Gui, Z.T. Xu, *Oscillation criteria for second-order neutral equations with distributed deviating arguments*, Electron. J. Differential Equations **10** (2007), 1–11.

[13] Samir H. Saker, *Oscillation Theory of Delay Differential and Difference Equations*, Mansoura University, 2010.

Elmetwally M. Elabbasy
emelabbasy@mans.edu.eg

Mansoura University
Department of Mathematics, Faculty of Science
Mansoura, Egypt

T.S. Hassan

Mansoura University
Department of Mathematics, Faculty of Science
Mansoura, Egypt

O. Moaaz

Mansoura University
Department of Mathematics, Faculty of Science
Mansoura, Egypt

Received: August 14, 2011.
Revised: February 8, 2012.
Accepted: February 20, 2012.