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To the memory of Professor B.G. Pachpatte

FRACTIONAL ORDER RIEMANN-LIOUVILLE INTEGRAL INCLUSIONS WITH TWO INDEPENDENT VARIABLES AND MULTIPLE DELAY

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Abstract. In the present paper we investigate the existence of solutions for a system of integral inclusions of fractional order with multiple delay. Our results are obtained upon suitable fixed point theorems, namely the Bohnenblust-Karlin fixed point theorem for the convex case and the Covitz-Nadler for the nonconvex case.

Keywords: integral inclusion, left-sided mixed Riemann-Liouville integral, time delay, solution, fixed point.

Mathematics Subject Classification: 26A33.

1. INTRODUCTION

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as differential calculus and goes back to times when G.W. Leibniz and I. Newton invented differential calculus. We can find numerous applications in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [18, 25]. There has been significant development in ordinary and partial fractional differential equations and inclusions in recent years; see the monographs of Abbas *et al.* [6], Kilbas *et al.* [21], the papers of Abbas *et al.* [1–5, 7], Benchohra *et al.* [9–11], Ibrahim and H.A. Jalab [20], Kilbas and Marzan [22], Pachpatte [26–29], Vityuk [30], Vityuk and Golushkov [31] and the references therein.

Integral equations and inclusions occur in a natural way in the description of many physical phenomena; see the books by Corduneanu [14, 15].

Motivated by the above literature, this paper is concerned with the existence of solutions for the following system of fractional integral inclusions

$$u(x,y) - \sum_{i=1}^{m} b_i(x,y)u(x-\xi_i, y-\mu_i) \in I_\theta^r F(x,y,u(x,y)) \quad \text{if} \quad (x,y) \in J := [0,a] \times [0,b], \tag{1.1}$$

$$u(x,y) = \Phi(x,y) \text{ if } (x,y) \in \tilde{J} := [-\xi, a] \times [-\mu, b] \backslash (0,a] \times (0,b], \tag{1.2}$$

where $a, b > 0$, $\theta = (0,0)$, $\xi_i, \mu_i \geq 0$, $i = 1, \ldots, m$, $\xi = \max_{i=1,\ldots,m}\{\xi_i\}$, $\mu = \max_{i=1,\ldots,m}\{\mu_i\}$, $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a set-valued function with nonempty values in $\mathbb{R}^n$, $\mathcal{P}(\mathbb{R}^n)$ is the family of all nonempty subsets of $\mathbb{R}^n$, $I_\theta^r F(x,y,u(x,y))$ is the definite integral for the set-valued functions F of order $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, $b_i : J \to \mathbb{R}$; $i = 1, \ldots m$, and $\Phi : \tilde{J} \to \mathbb{R}^n$ are given continuous functions such that

$$\Phi(x,0) = \sum_{i=1}^{m} b_i(x,0)\Phi(x-\xi_i, -\mu_i), \quad x \in [0,a],$$

and

$$\Phi(0,y) = \sum_{i=1}^{m} b_i(0,y)\Phi(-\xi_i, y-\mu_i), \quad y \in [0,b].$$

We establish the existence results for the problem (1.1)–(1.2) when the right-hand side is convex as well as when it is non-convex valued. Our approach is based on appropriate fixed point theorems, namely the Bohnenblust-Karlin fixed point theorem for the convex case and the Covitz-Nadler for the nonconvex case. This approach is now standard, however its utilization is new in the framework of the considered integral inclusions.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J)$ we denote the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(x,y)\in J} \|w(x,y)\|,$$

where $\|.\|$ denotes a norm on $\mathbb{R}^n$. Also, $C := C([-\xi, a] \times [-\mu, b])$ is a Banach space endowed with the norm

$$\|w\|_C = \sup_{(x,y)\in[-\xi,a]\times[-\mu,b]} \|w(x,y)\|.$$

Let $L^1(J)$ be the space of Lebesgue-integrable functions $w : J \to \mathbb{R}^n$ with the norm

$$\|w\|_{L^1} = \int_0^a \int_0^b \|w(x,y)\| dydx,$$

and $L^\infty(J)$ be the Banach space of measurable functions $u : J \to \mathbb{R}^n$ which are essentially bounded, equipped with the norm

$$\|u\|_{L^\infty} = \inf\{c > 0 : \|u(x,y)\| \le c \text{ a.e. } (x,y) \in J\}.$$

Definition 2.1 ([31])**.** Let $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, $\theta = (0,0)$ and $u \in L^1(J)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I_\theta^r u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t)\,dt\,ds,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt$, $\xi > 0$.

In particular,

$$(I_\theta^\theta u)(x,y) = u(x,y),\ (I_\theta^\sigma u)(x,y) = \int_0^x \int_0^y u(s,t)\,dt\,ds \quad \text{for almost all } (x,y) \in J,$$

where $\sigma = (1,1)$.

For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I_\theta^r u) \in C(J)$, and moreover

$$(I_\theta^r u)(x,0) = (I_\theta^r u)(0,y) = 0, \quad x \in [0,a],\ y \in [0,b].$$

Example 2.2. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, then

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2} \text{ for almost all } (x,y) \in J.$$

Let $(X, \|\cdot\|)$ be a Banach space. Denote $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_{bd}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$ and $\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

Definition 2.3. A multivalued map $T : X \to \mathcal{P}(X)$ is convex (closed) valued if $T(x)$ is convex (closed) for all $x \in X$. T is bounded on bounded sets if $T(B) = \cup_{x\in B} T(x)$ is bounded in X for all $B \in \mathcal{P}_{bd}(X)$ (i.e. $\sup_{x\in B}\sup_{y\in T(x)} \|y\| < \infty$). T is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $T(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $T(x_0)$, there exists an open neighborhood N_0 of x_0 such that $T(N_0) \subseteq N$. T is said to be completely continuous if $T(B)$ is relatively compact for every $B \in \mathcal{P}_{bd}(X)$. T has a fixed point if there is $x \in X$ such that $x \in T(x)$. The fixed point set of the multivalued operator T will be denoted by $FixT$. A multivalued map $G : X \to P_{cl}(\mathbb{R}^n)$ is said to be measurable if for every $v \in \mathbb{R}^n$, the function $x \longmapsto d(v, G(x)) = \inf\{\|v - z\| : z \in G(x)\}$ is measurable.

For more details on multivalued maps see the books of Aubin and Cellina [8], Górniewicz [17], Hu and Papageorgiou [19], and Kisielewicz [23].

Lemma 2.4 ([19]). *Let G be a completely continuous multivalued map with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $u_n \to u$, $w_n \to w$, $w_n \in G(u_n)$ imply $w \in G(u)$).*

Definition 2.5. A multivalued map $F : J\times\mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be Carathéodory if

(i) $(x, y) \longmapsto F(x, y, u)$ is measurable for each $u \in \mathbb{R}^n$,
(ii) $u \longmapsto F(x, y, u)$ is upper semicontinuous for almost all $(x, y) \in J$.

For each $u \in C(J)$, define the set of selections of F by

$$S_{F,u} = \{w \in L^1(J) : w(x, y) \in F(x, y, u(x, y)) \text{ a.e. } (x, y) \in J\}.$$

Lemma 2.6 ([24]). *Let X be a Banach space. Let $F : J \times X \longrightarrow \mathcal{P}_{cp,cv}(X)$ be a Carathéodory multivalued map and let Λ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$, then the operator*

$$\begin{aligned} \Lambda \circ S_F : C(J, X) &\longrightarrow \mathcal{P}_{cp,cv}(C(J, X)),\\ u &\longmapsto (\Lambda \circ S_F)(u) := \Lambda(S_{F,u}) \end{aligned}$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Proposition 2.7 ([13]). *Let X be a separable Banach space. Let $F_1, F_2 : J \to \mathcal{P}_{cp}(X)$ be measurable multivalued maps, then the multivalued map $t \to F_1(t) \cap F_2(t)$ is measurable.*

Theorem 2.8 ([13]). *Let X be a separable metric space, $(T, \mathcal{L})$ a measurable space, F a multivalued map from T to complete non empty subsets of X. If for each open set U in X, $F^-(U) = \{t : F(t) \cap U \neq \emptyset\}$ belongs to $\mathcal{L}$, then F admits a measurable selection.*

Let $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a set-valued function with nonempty values in $\mathbb{R}^n$. $I_\theta^r F(x, y, u(x, y))$ are the definite integral for the set-valued functions F of order $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$ which is defined as

$$\begin{aligned} &I_\theta^r F(x, y, u(x, y)) =\\ &= \left\{ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)dtds : f(x,y) \in S_{F,u} \right\}. \end{aligned}$$

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{ \sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b) \right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(\mathcal{P}_{bd,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [23]).

Definition 2.9. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(u), N(v)) \leq \gamma d(u, v) \quad \text{for each} \quad u, v \in X,$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 2.10 ([12], Bohnenblust-Karlin). *Let X be a Banach space and $K \in \mathcal{P}_{cl,cv}(X)$ and suppose that the operator $G : K \to \mathcal{P}_{cl,cv}(K)$ is upper semicontinuous and the set $G(K)$ is relatively compact in X. Then G has a fixed point in K.*

Lemma 2.11 ([16], Covitz-Nadler). *Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then N has fixed points.*

3. EXISTENCE OF SOLUTIONS

Let us start by defining what we mean by a solution of the problem (1.1)–(1.2).

Definition 3.1. A function $u \in C$ is said to be a solution of (1.1)–(1.2) if there exists an essentially bounded function $f \in S_{F,u}$ such that u satisfies the equation

$$u(x, y) = \sum_{i=1}^{m} b_i(x, y)u(x - \xi_i, y - \mu_i) + (I_\theta^r f)(x, y)$$

on J and condition (1.2) on $\tilde{J}$.

Set

$$B = \max_{i=1,\dots,m} \{ \sup_{(x,y)\in J} |b_i(x, y)| \}.$$

Theorem 3.2 (Convex Case). *Assume that the following conditions hold:*

(H_1) *The multifunction F is Carathéodory,*
(H_2) *There exists a non-negative function $h \in L^\infty(J)$ such that*

$$\|F(x, y, u)\|_{\mathcal{P}} \leq h(x, y), \text{ a.e. } (x, y) \in J \text{ for all } u \in \mathbb{R}^n.$$

If $mB < 1$, then the problem (1.1)–(1.2) *has at least one solution u on* $[-\xi, a] \times [-\mu, b]$.

Proof. Transform the problem (1.1)–(1.2) into a fixed point problem. Consider the multivalued operator $N : C \to \mathcal{P}(C)$ defined by

$$N(u)(x, y) = \left\{ g \in C : g(x, y) = \begin{cases} \Phi(x, y), & (x, y) \in \tilde{J} \\ \displaystyle\sum_{i=1}^{m} b_i(x, y)u(x - \xi_i, y - \mu_i) + & \\ +(I_\theta^r f)(x, y), \ f \in S_{F,u}, & (x, y) \in J \end{cases} \right\}. \tag{3.1}$$

Since the selection function f is essentially bounded, the operator N is well defined, that is, maps C into $\mathcal{P}(C)$. Clearly, the fixed points of N are solutions to (1.1)–(1.2). Let $h^* = \|h\|_{L^\infty}$,

$$R = \max\left\{\|\Phi\|, \frac{a^{r_1}b^{r_2}h^*}{(1-mB)\Gamma(1+r_1)\Gamma(1+r_2)}\right\},$$

and

$$B_R = \{u \in C : \|u\|_C \leq R\}.$$

Clearly, B_R is a closed, bounded and convex subset of C. We shall show that N satisfies the assumptions of Lemma 2.10. The proof will be given by several steps.

Step 1. $N(B_R) \subset B_R$.

Let $u \in B_R$. We must show that $N(u) \subset B_R$. For each $g \in N(u)$, there exists $f \in S_{F,u}$ such that, for each $(x,y) \in J$, we have

$$\begin{aligned}\|g(x,y)\| \leq & \sum_{i=1}^{m} |b_i(x,y)| \|u(x-\xi_i, y-\mu_i)\| + \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t)\|dtds \leq \\ & \leq mB\|u\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)dtds \leq \\ & \leq mBR + \frac{a^{r_1}b^{r_2}h^*}{\Gamma(1+r_1)\Gamma(1+r_2)} = R,\end{aligned}$$

and, for all $(x,y) \in \tilde{J}$, we have

$$\|g(x,y)\| \leq \|\Phi\| \leq R.$$

Thus, for all $(x,y) \in [-\xi, a] \times [-\mu, b]$ and $g(x,y) \in N(u)$, we get

$$\|g(x,y)\| \leq R.$$

Step 2. $N(B_R)$ is a relatively compact set.

We must show that N is a compact operator. Since B_R is a bounded closed and convex set and $N(B_R) \subset B_R$, it follows that $N(B_R)$ is a bounded closed and convex set. Moreover, for $0 \le x_1 \le x_2 \le a$ and $0 \le y_1 \le y_2 \le b$ and $u \in B_R$, then for each $g \in N(u)$, there exists $f \in S_{F,u}$ such that, for each $(x,y) \in J$, we have

$$
\begin{aligned}
&\|g(x_1,y_1) - g(x_2,y_2)\| \le \\
&\le \sum_{i=1}^{m} \|b_i(x_1,y_1)u(x_1-\xi_i, y_1-\mu_i) - b_i(x_2,y_2)u(x_2-\xi_i,y_2-\mu_i)\| + \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1}\int_0^{y_1} [(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} - \\
&\quad - (x_1-s)^{r_1-1}(y_1-t)^{r_2-1}]\|f(s,t)\|dtds + \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t)\|dtds + \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{0}^{x_1}\int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t)\|dtds + \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_{0}^{y_1} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t)\|dtds \le \\
&\le B\sum_{i=1}^{m} \|u(x_1-\xi_i,y_1-\mu_i) - u(x_2-\xi_i,y_2-\mu_i)\| + \\
&\quad + \frac{h^*}{\Gamma(r_1)\Gamma(r_2)}[2y_2^{r_2}(x_2-x_1)^{r_1} + 2x_2^{r_1}(y_2-y_1)^{r_2} + \\
&\quad + x_1^{r_1}y_1^{r_2} - x_2^{r_1}y_2^{r_2} - 2(x_2-x_1)^{r_1}(y_2-y_1)^{r_2}].
\end{aligned}
$$

As $x_1 \to x_2$ and $y_1 \to y_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $x_1 < x_2 < 0$, $y_1 < y_2 < 0$ and $x_1 \le 0 \le x_2$, $y_1 \le 0 \le y_2$ is obvious. An application of the Arzelà-Ascoli Theorem yields that N maps B_R into C a compact set in C, that is $N : B_R \to \mathcal{P}(C)$ is a compact operator. Thus $N(B_R)$ is relatively compact.

Step 3. N is upper semi-continuous on B_R.

Let $u_n \to u_*$, $h_n \in N(u_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(u_*)$. $h_n \in N(u_n)$ means that there exists $f_n \in S_{F,u_n}$ such that

$$
\begin{cases}
h_n(x,y) = \Phi(x,y), & (x,y) \in \tilde{J}, \\
h_n(x,y) = \sum_{i=1}^{m} b_i(x,y)u_n(x-\xi_i, y-\mu_i) + & \\
+\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}f_n(s,t)dtds, & (x,y) \in J.
\end{cases}
$$

We must show that there exists $f_* \in S_{F,u_*}$ such that

$$\begin{cases} h_*(x,y) = \Phi(x,y), & (x,y) \in \tilde{J}, \\ h_*(x,y) = \sum\limits_{i=1}^{m} b_i(x,y) u_*(x-\xi_i, y-\mu_i) + \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^x \int\limits_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f_*(s,t)dtds, & (x,y) \in J. \end{cases}$$

Now, we consider the linear continuous operator

$$\begin{aligned} \Lambda : L^\infty([-\xi, a] \times [-\mu, b]) &\longrightarrow C, \\ f &\longmapsto \Lambda(f)(x,y) \end{aligned}$$

such that

$$\begin{cases} \Lambda(f)(x,y) = \Phi(x,y), & (x,y) \in \tilde{J}, \\ \Lambda(f)(x,y) = \sum\limits_{i=1}^{m} b_i(x,y) u_*(x-\xi_i, y-\mu_i) + \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^x \int\limits_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)dtds, & (x,y) \in J. \end{cases}$$

From Lemma 2.6, it follows that $\Lambda \circ S_F$ is a closed graph operator. Clearly, for each $(x,y) \in J$, we have

$$\begin{aligned} &\left\| \left[h_n(x,y) - \sum_{i=1}^{m} b_i(x,y) u_n(x-\xi_i, y-\mu_i) \right] - \right. \\ &\left. - \left[h_*(x,y) - \sum_{i=1}^{m} b_i(x,y) u_*(x-\xi_i, y-\mu_i) \right] \right\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Moreover, from the definition of Λ, we have

$$h_n(x,y) - \sum_{i=1}^{m} b_i(x,y) u_n(x-\xi_i, y-\mu_i) \in \Lambda(S_{F,u_n}).$$

Since $u_n \to u_*$, it follows from Lemma 2.6 that, for some $f_* \in \Lambda(S_{F,u_*})$, we have

$$\begin{aligned} &h_*(x,y) - \sum_{i=1}^{m} b_i(x,y) u_*(x-\xi_i, y-\mu_i) = \\ &= \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^x \int\limits_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f_*(s,t)dtds. \end{aligned}$$

From Lemma 2.4, we can conclude that N is u.s.c.

Step 4. N has convex values.

Let $u \in C$ and $g_1, g_2 \in N(u)$, then there exists $f_1, f_2 \in S_{F,u}$ such that

$$g_k(x,y) = \sum_{i=1}^{m} b_i(x,y)u(x-\xi_i, y-\mu_i) + I_\theta^r f_k(x,y), \quad k = 1, 2.$$

Let $0 \le \zeta \le 1$, then for each $(x,y) \in J$, we have

$$\begin{aligned}[\zeta g_1 + (1-\zeta)g_2](x,y) &= \sum_{i=1}^{m} b_i(x,y)[\zeta u(x-\xi_i, y-\mu_i) + (1-\zeta)u(x-\xi_i, y-\mu_i)]+ \\ &\quad + I_\theta^r[\zeta f_1 + (1-\zeta)f_2](x,y) = \\ &= \sum_{i=1}^{m} b_i(x,y)u(x-\xi_i, y-\mu_i) + I_\theta^r[\zeta f_1 + (1-\zeta)f_2](x,y),\end{aligned}$$

and for each $(x,y) \in \tilde{J}$, we have $[\zeta g_1 + (1-\zeta)g_2](x,y) = \Phi(x,y)$.

The convexity of $S_{F,u}$ and $F(x,y,u)$ implies that $[\zeta g_1 + (1-\zeta)g_2] \in N(u)$. Hence $N(u)$ is convex for each $u \in C$. As a consequence of Lemma 2.10, we deduce that N has a fixed point which is a solution for the problem (1.1)–(1.2). □

Theorem 3.3 (Non-convex Case). *Assume that the following conditions hold:*

(H_3) *The multifunction $F : J \times \mathbb{R}^n \to \mathcal{P}_{cp}(\mathbb{R}^n)$ has the property that*

$$F(\cdot,\cdot,u) : J \to \mathcal{P}_{cp}(\mathbb{R}^n) \text{ is measurable for each } u \in \mathbb{R}^n,$$

(H_4) *There exists a non-negative function $m \in L^\infty(J)$ such that*

$$H_d(F(x,y,u), F(x,y,v)) \le m(x,y)\|u-v\| \text{ for every } u, v \in \mathbb{R}^n,$$

and

$$d(0, F(x,y,0)) \le m(x,y) \text{ a.e. } (x,y) \in J.$$

If

$$mB + \frac{m^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{3.2}$$

where $m^* = \|m\|_{L^\infty}$, then the problem (1.1)–(1.2) *has at least one solution on* $[-\xi, a] \times [-\mu, b]$.

Proof. For each $u \in C$ the set $S_{F,u}$ is nonempty since by (H_3), F has a nonempty measurable selection (Theorem 2.8). We shall show that N defined in Theorem 3.2 satisfies the assumptions of Lemma 2.11. The proof will be given in two steps.

Step 1. $N(u) \in \mathcal{P}_{cl}(C)$ for each $u \in C$.

Indeed, let $(g_n)_{n\geq 0} \in N(u)$ such that $g_n \to g$. Then, $g \in C$ and there exists $f_n \in S_{F,u}$ such that, for each $(x,y) \in J$,

$$\begin{cases} g_n(x,y) = \Phi(x,y), & (x,y) \in \tilde{J}, \\ g_n(x,y) = \sum\limits_{i=1}^{m} b_i(x,y)u(x-\xi_i, y-\mu_i)+ \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^x \int\limits_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f_n(s,t)dtds, & (x,y) \in J. \end{cases}$$

Using the fact that F has compact values and from (H_4), we may pass to a subsequence if necessary to get that $f_n(\cdot,\cdot)$ converges to f almost everywhere on J. Using (H_4) we have for a.e. $(x,y) \in J$

$$|f_n(x,y)| \leq m(x,y)\|u\|_\infty + m(x,y), \quad n \in \mathbb{N}.$$

The Lebesgue dominated convergence theorem implies that

$$\|f_n - f\|_{L^1} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $f \in S_{F,u}$. Then, for each $(x,y) \in [-\xi, a] \times [-\mu, b]$, $g_n(x,y) \to g(x,y)$, where

$$\begin{cases} g(x,y) = \Phi(x,y), & (x,y) \in \tilde{J}, \\ g(x,y) = \sum\limits_{i=1}^{m} b_i(x,y)u(x-\xi_i, y-\mu_i)+ \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^x \int\limits_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)dtds, & (x,y) \in J. \end{cases}$$

So, $g \in N(u)$.

Step 2. There exists $\gamma < 1$ such that $H_d(N(u), N(v)) \leq \gamma\|u-v\|_C$ for each $u, v \in C$.

Let $u, v \in C$ and $g \in N(u)$. Then, there exists $f \in S_{F,u}$ such that

$$\begin{cases} g(x,y) = \Phi(x,y), & (x,y) \in \tilde{J}, \\ g(x,y) = \sum\limits_{i=1}^{m} b_i(x,y)u(x-\xi_i, y-\mu_i)+ \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^x \int\limits_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)dtds, & (x,y) \in J. \end{cases}$$

From (H_4) it follows that

$$H_d(F(x,y,u(x,y)), F(x,y,v(x,y))) \leq m(x,y)\|u(x,y) - v(x,y)\|.$$

Hence, there exists $w \in S_{F,v}$ such that

$$\|f(x,y) - w(x,y)\| \leq m(x,y)\|u(x,y) - v(x,y)\|, \quad (x,y) \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R}^n)$ given by

$$U(x,y) = \{w(x,y) \,|\, w : J \to \mathbb{R}^n \text{ is Lebesgue integrable and} \\ \|f(x,y) - w(x,y)\| \le m(x,y)\|u(x,y) - v(x,y)\|\}.$$

Since the multivalued operator $U(x,y) \cap F(x,y,v(x,y))$ is measurable (Proposition 2.7), there exists a function $\overline{f}(x,y)$ which is a measurable selection for U. So, $\overline{f}(x,y) \in S_{F,v}$, and for each $(x,y) \in J$,

$$\|f(x,y) - \overline{f}(x,y)\| \le m(x,y)\|u(x,y) - v(x,y)\|.$$

Let us define

$$\begin{cases} \overline{g}(x,y) = \Phi(x,y), & (x,y) \in \tilde{J}, \\ \overline{g}(x,y) = \sum\limits_{i=1}^{m} b_i(x,y)v(x - \xi_i, y - \mu_i) + \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^x \int\limits_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\overline{f}(s,t)dtds, & (x,y) \in J. \end{cases}$$

Then, for each $(x,y) \in [-\xi, a] \times [-\mu, b]$, we get

$$\begin{aligned} \|g(x,y) - \overline{g}(x,y)\| \le & \sum_{i=1}^{m} |b_i(x,y)| \|u(x - \xi_i, y - \mu_i) - v(x - \xi_i, y - \mu_i)\| + \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^x \int\limits_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\ & \times \|f(s,t) - \overline{f}(s,t)\| dtds \le \\ \le & \, mB\|u - v\|_C + \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^x \int\limits_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\ & \times m(s,t)\|u - v\|_C dtds \le \\ \le & \, mB\|u - v\|_C + m^* \frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \|u - v\|_C = \\ = & \left(mB + \frac{m^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right) \|u - v\|_C. \end{aligned}$$

Thus, for each $(x,y) \in [-\xi, a] \times [-\mu, b]$, we get

$$\|g - \overline{g}\|_C \le \left(mB + \frac{m^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right) \|u - v\|_C.$$

By an analogous relation, obtained by interchanging the roles of u and v, it follows that

$$H_d(N(u), N(v)) \le \left(mB + \frac{m^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right) \|u - v\|_C.$$

So by (3.2), N is a contraction and thus, by Lemma 2.11, N has a fixed point u which is a solution to (1.1)–(1.2) on $[-\xi, a] \times [-\mu, b]$. $\square$

4. AN EXAMPLE

As an application of our results we consider the following system of fractional integral inclusions of the form

$$\begin{aligned} &u(x,y) - \frac{x^3 y}{8} u(x-1, y-3) + \frac{x^4 y^2}{12} u\Big(x-2, y-\frac{1}{4}\Big) + \\ &+ \frac{1}{14} u\Big(x - \frac{3}{2}, y-2\Big) \in I_\theta^r F(x,y,u) \quad \text{if} \quad (x,y) \in J := [0,1] \times [0,1], \end{aligned} \tag{4.1}$$

$$u(x,y) = \Phi(x,y) \quad \text{if} \quad (x,y) \in \tilde{J} := [-2,1] \times [-3,1] \backslash (0,1] \times (0,1], \tag{4.2}$$

where $m = 3$, $r = \left(\frac{1}{2}, \frac{1}{5}\right)$,

$$F(x,y,u) = \left[\frac{e^{x+y}}{2+|u|}, \frac{e^{x+y}(1+|u|)}{2+|u|}\right] \text{ a.a. } (x,y) \in J \text{ and for all } u \in \mathbb{R},$$

and $\Phi : \tilde{J} \to \mathbb{R}$ is a continuous function satisfying

$$\Phi(x,0) = \frac{1}{14} \Phi\Big(x - \frac{3}{2}, -2\Big), \quad \Phi(0,y) = \frac{1}{14} \Phi\Big(-\frac{3}{2}, y-2\Big) \quad x,y \in [0,1]. \tag{4.3}$$

Notice that condition (4.3) is satisfied by $\Phi \equiv 0$.

Set

$$b_1(x,y) = \frac{x^3 y}{8}, \ b_2(x,y) = \frac{x^4 y^2}{12}, \ b_3(x,y) = \frac{1}{14}.$$

Then, $B = \frac{1}{8}$ and

$$\|F(x,y,u)\| \leq e^{x+y} \text{ for a.e. } (x,y) \in J \text{ and all } u \in \mathbb{R}.$$

It is clear that F is Carathéodory, hence condition (H_1) is satisfied. Also, condition (H_2) holds with $h^* = e^2$, and we have $mB = \frac{3}{8} < 1$. In view of Theorem 3.2, the problem (4.1)–(4.2) has a solution defined on $[-2,1] \times [-3,1]$.

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