

EXISTENCE RESULTS FOR DIRICHLET PROBLEMS WITH DEGENERATED p -LAPLACIAN

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Abstract. In this article, we prove the existence of entropy solutions for the Dirichlet problem

$$(P) \quad \begin{cases} -\operatorname{div}[\omega(x)|\nabla u|^{p-2}\nabla u] = f(x) - \operatorname{div}(G(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{in } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of \mathbb{R}^N ($N \geq 2$), $f \in L^1(\Omega)$ and $G/\omega \in [L^{p'}(\Omega, \omega)]^N$.

Keywords: degenerate elliptic equations, entropy solutions, weighted Sobolev spaces.

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1. INTRODUCTION

The main purpose of this paper (see Theorem 4.2) is to establish the existence of entropy solutions for the Dirichlet problem

$$(P) \quad \begin{cases} -\operatorname{div}[\omega(x)|\nabla u|^{p-2}\nabla u] = f(x) - \operatorname{div}(G(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{in } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $f \in L^1(\Omega)$, $G/\omega \in [L^{p'}(\Omega, \omega)]^N$, ω is a weight function (i.e., a locally integrable function on \mathbb{R}^N such that $0 < \omega(x) < \infty$ a.e. $x \in \mathbb{R}^N$) and $1 < p < \infty$, $p \neq 2$.

The notion of an entropy solution was introduced in [1], where the authors studied the nondegenerate elliptic equation $-\operatorname{div}(a(x, Du)) = f(x)$, with $f \in L^1(\Omega)$. In [3] the author studied the degenerate elliptic equation $Lu = f$, where L is a degenerate elliptic operator in divergence form (i.e., $Lu = -\sum_{i,j=1}^n D_j(a_{ij}(x)D_i u)$) and $f \in L^1(\Omega)$. Note that, in the proof of our main result, many ideas have been adapted from [1] and [3].

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [4–7, 9] and [12]).

A class of weights, which is particularly well understood, is the class of A_p weights that was introduced by B. Muckenhoupt in the early 1970’s (see [9]).

We propose to solve the problem (P) by approximation with variational solutions: we take $f_n \in C_0^\infty(\Omega)$ such that $f_n \rightarrow f$ in $L^1(\Omega)$, $G_n/\omega \in [L^{p'}(\Omega, \omega)]^N$ such that $G_n/\omega \rightarrow G/\omega$ in $[L^{p'}(\Omega, \omega)]^N$, we find a solution $u_n \in W_0^{1,p}(\Omega, \omega)$ for the problem with right-hand side f_n and G_n and we will try to pass to the limit as $n \rightarrow \infty$.

The paper is organized as follows. In Section 2 we present the definitions and basic results. In Section 3 we prove the existence and uniqueness of solutions when $f/\omega \in L^{p'}(\Omega, \omega)$, $G/\omega \in [L^{p'}(\Omega, \omega)]^N$ and in Section 4 we state and prove our main result about existence of entropy solutions for problem (P) (when $f \in L^1(\Omega)$ and $G/\omega \in [L^{p'}(\Omega, \omega)]^N$).

2. DEFINITIONS AND BASIC RESULTS

By weight we mean a locally integrable function ω on \mathbb{R}^N such that $0 < \omega(x) < \infty$ for a.e. $x \in \mathbb{R}^N$. Every weight ω gives rise to a measure on the measurable subsets of \mathbb{R}^N through integration. This measure will be denoted by μ . Thus, $\mu(E) = \int_E \omega(x) dx$ for measurable sets $E \subset \mathbb{R}^N$.

Definition 2.1. Let $1 \leq p < \infty$. A weight ω is said to be an A_p -weight, if there is a positive constant $C = C(p, \omega)$ such that, for every ball $B \subset \mathbb{R}^N$

$$\left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)}(x) dx \right)^{p-1} \leq C \quad \text{if } p > 1,$$

$$\left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\operatorname{ess\,sup}_{x \in B} \frac{1}{\omega(x)} \right) \leq C \quad \text{if } p = 1,$$

where $|\cdot|$ denotes the N -dimensional Lebesgue measure in \mathbb{R}^N .

If $1 < q \leq p$, then $A_q \subset A_p$ (see [6, 7] or [12] for more information about A_p -weights). As an example of an A_p -weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, is in A_p if and only if $-N < \alpha < N(p - 1)$ (see [11, Chapter IX, Corollary 4.4]). If $\varphi \in \text{BMO}(\mathbb{R}^N)$, then $\omega(x) = e^{\alpha \varphi(x)} \in A_2$ for some $\alpha > 0$ (see [10]).

Remark 2.2. If $\omega \in A_p$, $1 < p < \infty$, then

$$\left(\frac{|E|}{|B|} \right)^p \leq C \frac{\mu(E)}{\mu(B)}$$

for all measurable subsets E of B (see 15.5 strong doubling property in [7]). Therefore, if $\mu(E) = 0$, then $|E| = 0$. Thus, if $\{u_n\}$ is a sequence of functions defined in B and $u_n \rightarrow u$ μ -a.e., then $u_n \rightarrow u$ a.e.

Definition 2.3. Let ω be a weight. We shall denote by $L^p(\Omega, \omega)$ ($1 \leq p < \infty$) the Banach space of all measurable functions f defined in Ω for which

$$\|f\|_{L^p(\Omega, \omega)} = \left(\int_{\Omega} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty.$$

We denote $[L^{p'}(\Omega, \omega)]^N = L^{p'}(\Omega, \omega) \times \dots \times L^{p'}(\Omega, \omega)$.

Remark 2.4. If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega, \omega) \subset L^1_{\text{loc}}(\Omega)$ (see [12, Remark 1.2.4]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 2.5. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $1 < p < \infty$, k a nonnegative integer and $\omega \in A_p$. We shall denote by $W^{k,p}(\Omega, \omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega, \omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega, \omega)$, $1 \leq |\alpha| \leq k$. The norm in the space $W^{k,p}(\Omega, \omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega, \omega)} = \left(\int_{\Omega} |u(x)|^p \omega(x) dx + \sum_{1 \leq |\alpha| \leq k} \int_{\Omega} |D^\alpha u(x)|^p \omega(x) dx \right)^{1/p}. \tag{2.1}$$

We also define the space $W_0^{k,p}(\Omega, \omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{k,p}(\Omega, \omega)} = \left(\sum_{1 \leq |\alpha| \leq k} \int_{\Omega} |D^\alpha u(x)|^p \omega(x) dx \right)^{1/p}.$$

The dual space of $W_0^{1,p}(\Omega, \omega)$ is the space $[W_0^{1,p}(\Omega, \omega)]^* = W^{-1,p'}(\Omega, \omega)$,

$$W^{-1,p'}(\Omega, \omega) = \left\{ T = f - \text{div}(G) : G = (g_1, \dots, g_N), \frac{f}{\omega}, \frac{g_j}{\omega} \in L^{p'}(\Omega, \omega) \right\}.$$

It is evident that a weight function ω which satisfies $0 < C_1 \leq \omega(x) \leq C_2$, for a.e. $x \in \Omega$, gives nothing new (the space $W^{k,p}(\Omega, \omega)$ is then identical with the classical Sobolev space $W^{k,p}(\Omega)$). Consequently, we shall be interested in all above such weight functions ω which either vanish somewhere in $\Omega \cup \partial\Omega$ or increase to infinity (or both).

We need the following basic result.

Theorem 2.6 (The weighted Sobolev inequality). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let ω be an A_p -weight, $1 < p < \infty$. Then there exists positive constants C_Ω and δ such that for all $f \in C_0^\infty(\Omega)$ and $1 \leq \eta \leq N/(N-1) + \delta$*

$$\|f\|_{L^{\eta p}(\Omega, \omega)} \leq C_\Omega \| |\nabla f| \|_{L^p(\Omega, \omega)}. \tag{2.2}$$

Proof. See [5, Theorem 1.3]. □

Definition 2.7. We say that $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$ if $T_k(u) \in W_0^{1,p}(\Omega, \omega)$ for all $k > 0$, where the function $T_k : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k \text{ sign}(s), & \text{if } |s| > k. \end{cases}$$

Remark 2.8. (i) Note that for given $h > 0$ and $k > 0$ we have

$$T_h(u - T_k(u)) = \begin{cases} 0 & \text{if } |u| \leq k, \\ (|u| - k) \operatorname{sign}(u) & \text{if } k < |u| \leq k + h, \\ h \operatorname{sign}(u), & \text{if } |u| > k + h. \end{cases}$$

Moreover, if $\alpha \in \mathbb{R}$, $\alpha \neq 0$, we have $T_k(\alpha u) = \alpha T_{k/|\alpha|}(u)$.

(ii) If $u \in W_{\text{loc}}^{1,1}(\Omega, \omega)$, then we have

$$\nabla T_k(u) = \chi_{\{|u| < k\}} \nabla u,$$

where χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{R}^N$.

Definition 2.9. Let $f \in L^1(\Omega)$, $G/\omega \in [L^{p'}(\Omega, \omega)]^N$ and $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$. We say that u is an entropy solution to problem (P) if

$$\int_{\Omega} \omega(x) |\nabla u|^{p-2} \langle \nabla u, \nabla T_k(u - \varphi) \rangle dx = \int_{\Omega} f T_k(u - \varphi) dx + \int_{\Omega} \langle G, \nabla T_k(u - \varphi) \rangle dx \quad (2.3)$$

for all $k > 0$ and all $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in \mathbb{R}^N .

We recall that the gradient of u which appears in (2.3) is defined as in Remark 2.8 of [3], that is to say that $\nabla u = \nabla T_k(u)$ on the set where $|u| < k$.

Remark 2.10. Note that if $u_1, u_2 \in W_0^{1,p}(\Omega, \omega)$, then $\varphi = T_k(u_1 + u_2) \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$ and we have

$$\nabla \varphi = \nabla T_k(u_1 + u_2) = \nabla(u_1 + u_2) \chi_{\{|u_1 + u_2| \leq k\}}.$$

Definition 2.11. Let $0 < p < \infty$ and let ω be a weight function. We define the weighted Marcinkiewicz space $\mathcal{M}^p(\Omega, \omega)$ as the set of measurable functions $f : \Omega \rightarrow \mathbb{R}$ such that the function

$$\Gamma_k(f) = \mu(\{x \in \Omega : |f(x)| > k\}), \quad k > 0,$$

satisfies an estimate of the form $\Gamma_f(k) \leq Ck^{-p}$, $0 < C < \infty$.

Remark 2.12. If $1 \leq q < p$ and $\Omega \subset \mathbb{R}^N$ is a bounded set, we have that

$$L^p(\Omega, \omega) \subset \mathcal{M}^p(\Omega, \omega) \text{ and } \mathcal{M}^p(\Omega, \omega) \subset L^q(\Omega, \omega).$$

(the proof follows the lines of Theorem 2.18.8 in [8]).

Lemma 2.13. Let $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$ and $\omega \in A_p$, $1 < p < \infty$, be such that

$$\frac{1}{k} \int_{\{|u| < k\}} |\nabla u|^p \omega dx \leq M, \quad (2.4)$$

for every $k > 0$. Then:

- (i) $u \in \mathcal{M}^{p_1}(\Omega, \omega)$, where $p_1 = \eta(p - 1)$ (where η is the constant in Theorem 2.6). More precisely, there exists $C > 0$ such that $\Gamma_k(u) \leq CM^\eta k^{-p_1}$.
- (ii) $|\nabla u| \in \mathcal{M}^{p_2}(\Omega, \omega)$, where $p_2 = pp_1/(p_1 + 1)$ and $p_1 = \eta(p - 1)$. More precisely, there exists $C > 0$ such that $\Gamma_k(|\nabla u|) \leq CM^{(p_1+\eta)/(p_1+1)} k^{-p_2}$.

Proof. See Lemma 3.3 and Lemma 3.4 in [3]. □

3. WEAK SOLUTIONS

In this section we prove the existence and uniqueness of weak solutions $u \in W_0^{1,p}(\Omega, \omega)$ to the Dirichlet problem

$$(P1) \quad \begin{cases} -\operatorname{div}[\omega(x)|\nabla u|^{p-2}\nabla u] = f(x) - \operatorname{div}(G(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{in } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of \mathbb{R}^N ($N \geq 2$), $f/\omega \in L^{p'}(\Omega, \omega)$ and $G/\omega \in [L^{p'}(\Omega, \omega)]^N$.

Definition 3.1. We say that $u \in W_0^{1,p}(\Omega, \omega)$ is a weak solution for problem (P1) if

$$\int_{\Omega} \omega(x)|\nabla u|^{p-2}\langle \nabla u, \nabla \varphi \rangle dx = \int_{\Omega} f \varphi dx + \int_{\Omega} \langle G, \nabla \varphi \rangle dx, \tag{3.1}$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega)$, with $f/\omega \in L^{p'}(\Omega, \omega)$ and $G/\omega \in [L^{p'}(\Omega, \omega)]^N$.

Theorem 3.2. Let $\omega \in A_p$, $1 < p < \infty$, $f/\omega \in L^{p'}(\Omega, \omega)$ and $G/\omega \in [L^{p'}(\Omega, \omega)]^N$. Then the problem (P1) has a unique solution $u \in W_0^{1,p}(\Omega, \omega)$.

Proof. (I) Existence. By Theorem 2.6, we have that

$$\begin{aligned} \left| \int_{\Omega} f \varphi dx \right| &\leq \left(\int_{\Omega} \left| \frac{f}{\omega} \right|^{p'} \omega dx \right)^{1/p'} \left(\int_{\Omega} |\varphi|^p \omega dx \right)^{1/p} \leq \\ &\leq C_{\Omega} \|f/\omega\|_{L^{p'}(\Omega, \omega)} \|\nabla \varphi\|_{L^p(\Omega, \omega)}. \end{aligned} \tag{3.2}$$

Define the functional $J_p : W_0^{1,p}(\Omega, \omega) \rightarrow \mathbb{R}$ by

$$J_p(\varphi) = \frac{1}{p} \int_{\Omega} |\nabla \varphi|^p \omega dx - \int_{\Omega} f \varphi dx - \int_{\Omega} \langle G, \nabla \varphi \rangle dx.$$

Using (3.2) and Young’s inequality, we have that

$$\begin{aligned}
 J_p(\varphi) &\geq \frac{1}{p} \int_{\Omega} |\nabla\varphi|^p \omega \, dx - (C_{\Omega} \|f/\omega\|_{L^{p'}(\Omega,\omega)} + \|G/\omega\|_{L^{p'}(\Omega,\omega)}) \|\nabla\varphi\|_{L^p(\Omega,\omega)} \geq \\
 &\geq \frac{1}{p} \int_{\Omega} |\nabla\varphi|^p \omega \, dx - \frac{1}{p} \|\nabla\varphi\|_{L^p(\Omega,\omega)}^p - \\
 &\quad - \frac{1}{p'} [C_{\Omega} \|f/\omega\|_{L^{p'}(\Omega,\omega)} + \|G/\omega\|_{L^{p'}(\Omega,\omega)}]^{p'} = \\
 &= - \frac{1}{p'} [C_{\Omega} \|f/\omega\|_{L^{p'}(\Omega,\omega)} + \|G/\omega\|_{L^{p'}(\Omega,\omega)}]^{p'},
 \end{aligned}$$

that is, J_p is bounded from below.

Let $\{u_n\}$ be a minimizing sequence, that is, a sequence such that

$$J_p(u_n) \rightarrow \inf_{\varphi \in W_0^{1,p}(\Omega,\omega)} J_p(\varphi).$$

Then for n large enough, we obtain

$$0 \geq J_p(u_n) = \frac{1}{p} \int_{\Omega} |\nabla u_n|^p \omega \, dx - \int_{\Omega} f u_n \, dx - \int_{\Omega} \langle G, \nabla u_n \rangle \, dx,$$

and we get

$$\begin{aligned}
 \|\nabla u_n\|_{L^p(\Omega,\omega)}^p &\leq p \left(\int_{\Omega} f u_n \, dx + \int_{\Omega} \langle G, \nabla u_n \rangle \, dx \right) \leq \\
 &\leq p (\|f/\omega\|_{L^{p'}(\Omega,\omega)} \|u_n\|_{L^p(\Omega,\omega)} + \|G/\omega\|_{L^{p'}(\Omega,\omega)} \|\nabla u_n\|_{L^p(\Omega,\omega)}) \leq \\
 &\leq p (C_{\Omega} \|f/\omega\|_{L^{p'}(\Omega,\omega)} + \|G/\omega\|_{L^{p'}(\Omega,\omega)}) \|\nabla u_n\|_{L^p(\Omega,\omega)}.
 \end{aligned}$$

Hence $\|\nabla u_n\|_{L^p(\Omega,\omega)} \leq [p(C_{\Omega} \|f/\omega\|_{L^{p'}(\Omega,\omega)} + \|G/\omega\|_{L^{p'}(\Omega,\omega)})]^{1/(p-1)}$. Therefore $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega,\omega)$. Since $W_0^{1,p}(\Omega,\omega)$ is reflexive, there exists $u \in W_0^{1,p}(\Omega,\omega)$ such that $u_n \rightharpoonup$ in $W_0^{1,p}(\Omega,\omega)$. Since $W_0^{1,p}(\Omega,\omega) \ni \varphi \mapsto \int_{\Omega} f \varphi \, dx + \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx$, and $\varphi \mapsto \|\nabla \varphi\|_{L^p(\Omega,\omega)}$ are continuous then J_p is continuous. Moreover since $1 < p < \infty$ we have that J_p is convex and thus lower semi-continuous for the weak convergence. It follows that

$$J_p(u) \leq \liminf_n J_p(u_n) = \inf_{\varphi \in W_0^{1,p}(\Omega,\omega)} J_p(\varphi),$$

and thus u is a minimizer of J_p on $W_0^{1,p}(\Omega,\omega)$. For any $\varphi \in W_0^{1,p}(\Omega,\omega)$ the function

$$\lambda \mapsto \frac{1}{p} \int_{\Omega} |\nabla(u + \lambda\varphi)|^p \omega \, dx - \int_{\Omega} (u + \lambda\varphi) f \, dx - \int_{\Omega} \langle G, \nabla(u + \lambda\varphi) \rangle \, dx$$

has a minimum at $\lambda = 0$. Hence

$$\left. \frac{d}{d\lambda} \left(J_p(u + \lambda\varphi) \right) \right|_{\lambda=0} = 0, \quad \forall \varphi \in W_0^{1,p}(\Omega,\omega).$$

We have

$$\frac{d}{d\lambda} \left(|\nabla(u + \lambda\varphi)|^p \omega \right) = p \{ |\nabla(u + \lambda\varphi)|^{p-2} (\langle \nabla u, \nabla \varphi \rangle + \lambda |\nabla \varphi|^2) \} \omega,$$

and we obtain

$$\begin{aligned} 0 &= \frac{d}{d\lambda} \left(J_p(u + \lambda\varphi) \right) \Big|_{\lambda=0} = \\ &= \left[\frac{1}{p} \left(p \int_{\Omega} |\nabla(u + \lambda\varphi)|^{p-2} (\langle \nabla u, \nabla \varphi \rangle + \lambda |\nabla \varphi|^2) \omega \, dx \right) - \right. \\ &\quad \left. - \int_{\Omega} \varphi f \, dx - \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx \right] \Big|_{\lambda=0} = \\ &= \int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \omega \, dx - \int_{\Omega} f \varphi \, dx - \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx. \end{aligned}$$

Therefore $\int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \omega \, dx = \int_{\Omega} f \varphi \, dx + \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx$, that is, $u \in W_0^{1,p}(\Omega, \omega)$ is a solution of problem (P1).

(II) Uniqueness. If $u_1, u_2 \in W_0^{1,p}(\Omega, \omega)$ are two weak solutions of problem (P1), we have

$$\int_{\Omega} |\nabla u_i|^{p-2} \langle \nabla u_i, \nabla \varphi \rangle \omega \, dx = \int_{\Omega} f \varphi \, dx + \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx, \quad i = 1, 2,$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega)$. Hence

$$\int_{\Omega} \left(|\nabla u_1|^{p-2} \langle \nabla u_1, \nabla \varphi \rangle - |\nabla u_2|^{p-2} \langle \nabla u_2, \nabla \varphi \rangle \right) \omega \, dx = 0.$$

Taking $\varphi = u_1 - u_2$, and using that for every $x, y \in \mathbb{R}^N$ there exist two positive constants α_p and β_p such that

$$\alpha_p (|x| + |y|)^{p-2} |x - y| \leq \langle |x|^{p-2} x - |y|^{p-2} y, x - y \rangle \leq \beta_p (|x| + |y|)^{p-2} |x - y|,$$

we obtain

$$\begin{aligned} 0 &= \int_{\Omega} \left(|\nabla u_1|^{p-2} \langle \nabla u_1, \nabla u_1 - \nabla u_2 \rangle - |\nabla u_2|^{p-2} \langle \nabla u_2, \nabla u_1 - \nabla u_2 \rangle \right) \omega \, dx = \\ &= \int_{\Omega} \left(|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2, \nabla u_1 - \nabla u_2 \right) \omega \, dx \geq \\ &\geq \alpha_p \int_{\Omega} \left(|\nabla u_1| + |\nabla u_2| \right)^{p-2} |\nabla u_1 - \nabla u_2|^2 \omega \, dx. \end{aligned}$$

Therefore $\nabla u_1 = \nabla u_2$ μ -a.e. and since $u_1, u_2 \in W_0^{1,p}(\Omega, \omega)$, then $u_1 = u_2$ a.e. (by Remark 2.2). \square

4. MAIN RESULT

In this section, we prove the main result of this paper. We need the following results.

Lemma 4.1. *Let $\omega \in A_p$, $1 < p < \infty$ and a sequence $\{u_n\}$, $u_n \in W_0^{1,p}(\Omega, \omega)$ satisfies:*

- (1) $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, \omega)$ and μ -a.e. in Ω .
- (2) $\int_{\Omega} \langle |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u, \nabla(u_n - u) \rangle \omega \, dx \rightarrow 0$ with $n \rightarrow \infty$.

Then $u_n \rightarrow u$ in $W_0^{1,p}(\Omega, \omega)$.

Proof. The proof of this lemma follows the lines of Lemma 5 in [2]. □

Theorem 4.2. *Let $\omega \in A_p$, $1 < p < \infty$, $f \in L^1(\Omega)$ and $G/\omega \in [L^{p'}(\Omega, \omega)]^N$. There exists an entropy solution u of problem (P). Moreover, $u \in \mathcal{M}^{p_1}(\Omega, \omega)$ and $|\nabla u| \in \mathcal{M}^{p_2}(\Omega, \omega)$, with $p_1 = \eta(p - 1)$ and $p_2 = p_1 p / (p_1 + 1)$ (where η is the constant in Theorem 2.6).*

Proof. Considering a sequence $\{f_n\}$, $f_n \in C_0^\infty(\Omega)$, where

$$f_n \rightarrow f \text{ in } L^1(\Omega) \text{ and } \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)},$$

and a sequence $\{G_n\}$, with $G_n/\omega \in [L^{p'}(\Omega, \omega)]^N$ such that $\frac{G_n}{\omega} \rightarrow \frac{G}{\omega}$ in $[L^{p'}(\Omega, \omega)]^N$ and $\| |G_n|/\omega \|_{L^{p'}(\Omega, \omega)} \leq \| |G|/\omega \|_{L^{p'}(\Omega, \omega)}$. For each n , by Theorem 3.2, there exists a solution $u_n \in W_0^{1,p}(\Omega, \omega)$ of the Dirichlet problem

$$(P_n) \quad \begin{cases} -\operatorname{div}[\omega(x)|\nabla u_n|^{p-2} \nabla u_n] = f_n(x) - \operatorname{div}(G_n(x)) & \text{in } \Omega, \\ u_n(x) = 0 & \text{in } \partial\Omega, \end{cases}$$

that is,

$$\int_{\Omega} \omega |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla \varphi \rangle \, dx = \int_{\Omega} f_n \varphi \, dx + \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx \tag{4.1}$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega)$. For $\varphi = T_k(u_n)$ we obtain in (4.1) that

$$\int_{\Omega} \omega |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla T_k(u_n) \rangle \, dx = \int_{\Omega} f_n T_k(u_n) \, dx + \int_{\Omega} \langle G_n, \nabla T_k(u_n) \rangle \, dx. \tag{4.2}$$

We have

$$\left| \int_{\Omega} f_n T_k(u_n) \, dx \right| \leq \int_{\Omega} |f_n| |T_k(u_n)| \, dx \leq k \|f_n\|_{L^1(\Omega)} \leq k \|f\|_{L^1(\Omega)}, \tag{4.3}$$

and since $\nabla T_k(u_n) = \chi_{\{|u_n| < k\}} \nabla u_n$, we obtain

$$\begin{aligned} \int_{\Omega} \omega |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla T_k(u_n) \rangle \, dx &= \int_{\Omega} \omega |\nabla T_k(u_n)|^{p-2} \langle \nabla T_k(u_n), \nabla T_k(u_n) \rangle \, dx = \\ &= \int_{\Omega} |\nabla T_k(u_n)|^p \omega \, dx. \end{aligned} \tag{4.4}$$

We also have, using Young’s inequality, that there exists a constant $C_1 > 0$ (depending only on p) such that

$$\begin{aligned}
 \left| \int_{\Omega} \langle G_n, \nabla T_k(u_n) \rangle dx \right| &\leq \int_{\Omega} \left| \frac{G_n}{\omega} \right| |\nabla T_k(u_n)| \omega dx \leq \\
 &\leq \left(\int_{\Omega} |G_n/\omega|^{p'} \omega dx \right)^{1/p'} \left(\int_{\Omega} |\nabla T_k(u_n)|^p \omega dx \right)^{1/p} \leq \\
 &\leq \frac{1}{2} \int_{\Omega} |\nabla T_k(u_n)|^p \omega dx + C_1 \int_{\Omega} |G_n/\omega|^{p'} \omega dx \leq \\
 &\leq \frac{1}{2} \int_{\Omega} |\nabla T_k(u_n)|^p \omega dx + C_1 \int_{\Omega} |G/\omega|^{p'} \omega dx.
 \end{aligned}
 \tag{4.5}$$

Hence, using (4.3), (4.4) and (4.5), we obtain

$$\int_{\Omega} |\nabla T_k(u_n)|^p \omega dx \leq 2k \|f\|_{L^1(\Omega)} + 2C_1 \|G/\omega\|_{L^{p'}(\Omega, \omega)}^{p'} \leq C_2 k,
 \tag{4.6}$$

where $C_2 = 2 \|f\|_{L^1(\Omega)} + 2C_1 \|G/\omega\|_{L^{p'}(\Omega, \omega)}^{p'}$. By Lemma 2.13, the sequence $\{u_n\}$ is bounded in $\mathcal{M}^{p_1}(\Omega, \omega)$ (with $p_1 = \eta(p - 1)$), and $\{|\nabla u_n|\}$ is bounded in $\mathcal{M}^{p_2}(\Omega, \omega)$ (with $p_2 = p_1 p / (p_1 + 1)$). Moreover, $\{u_n\}$ is a Cauchy sequence in the μ -measure. Consequently, there exists a function u and a subsequence, that we will still denote by $\{u_n\}$, such that

$$u_n \rightarrow u \quad \mu - \text{a.e. in } \Omega,
 \tag{4.7}$$

and $u_n \rightarrow u$ a.e. in Ω (by Remark 2.2). Using (4.6) and (4.7), we have

$$\begin{aligned}
 T_k(u_n) &\rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega, \omega), \\
 T_k(u_n) &\rightarrow T_k(u) \text{ strongly in } L^p(\Omega, \omega) \text{ and } \mu - \text{a.e. in } \Omega,
 \end{aligned}
 \tag{4.8}$$

for all $k > 0$. Hence $T_k(u) \in W_0^{1,p}(\Omega, \omega)$.

Furthermore, by the weak lower semicontinuity of the norm $W_0^{1,p}(\Omega, \omega)$, we have that (4.6) still holds for u , that is,

$$\int_{\Omega} |\nabla T_k(u)|^p \omega dx \leq k C_2.$$

Applying Lemma 2.13, we deduce that $u \in \mathcal{M}^{p_1}(\Omega, \omega)$ (with $p_1 = \eta(p - 1)$) and $|\nabla u| \in \mathcal{M}^{p_2}(\Omega, \omega)$ (with $p_2 = p_1 p / (p_1 + 1)$).

We need to show that $T_k(u_n) \rightarrow T_k(u)$ strongly in $W_0^{1,p}(\Omega, \omega)$ for all $k > 0$.

Let $h > k$ and applying (4.1) with function $\varphi_n = T_{2k}(u_n - T_h(u_n)) + T_k(u_n) - T_k(u)$, we get

$$\int_{\Omega} \omega |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla \varphi_n \rangle dx = \int_{\Omega} f_n \varphi_n dx + \int_{\Omega} \langle G, \nabla \varphi_n \rangle dx.
 \tag{4.9}$$

If we set $M = 4k + h$, we have $\nabla\varphi_n = 0$ for $|u_n| > M$. We can write

$$\int_{\Omega} \omega |\nabla T_M(u_n)|^{p-2} \langle \nabla T_M(u_n), \nabla\varphi_n \rangle dx = \int_{\Omega} f_n \varphi_n dx + \int_{\Omega} \langle G, \nabla\varphi_n \rangle dx. \tag{4.10}$$

In the left-hand side of (4.10), we have

$$\begin{aligned} & \int_{\Omega} \omega |\nabla T_M(u_n)|^{p-2} \langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx = \\ &= \int_{\{|u_n| \leq k\}} \omega |\nabla T_M(u_n)|^{p-2} \langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx + \\ &+ \int_{\{|u_n| > k\}} \omega |\nabla T_M(u_n)|^{p-2} \langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx \end{aligned} \tag{4.11}$$

(a) If $|u_n| \leq k$.

Since $h > k$, if $|u_n| \leq k < h$, then $T_h(u_n) = T_k(u_n) = u_n$. Hence, $u_n - T_h(u_n) + T_k(u_n) - T_k(u) = u_n - T_k(u)$. We also have that $|u_n - u| \leq 2k$. Then, since $\nabla T_M(u_n) = \nabla T_k(u_n)$ (because $|u_n| \leq k < M$),

$$\begin{aligned} & \int_{\{|u_n| \leq k\}} \omega |\nabla T_M(u_n)|^{p-2} \langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx = \\ &= \int_{\{|u_n| \leq k\}} \omega |\nabla T_k(u_n)|^{p-2} \langle \nabla T_k(u_n), \nabla(T_k(u_n) - T_k(u)) \rangle dx = \\ &= \int_{\Omega} \omega |\nabla T_k(u_n)|^{p-2} \langle \nabla T_k(u_n), \nabla(T_k(u_n) - T_k(u)) \rangle dx. \end{aligned}$$

(b) If $|u_n| > k$.

Since $u_n, T_k(u_n)$ and $T_k(u)$ are in $W_0^{1,p}(\Omega, \omega)$, if $|u_n - T_h(u_n) + T_k(u_n) - T_k(u)| \leq 2k$, we obtain

$$\begin{aligned} \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) &= \nabla(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) = \\ &= \nabla u_n - \nabla T_h(u_n) + \nabla T_k(u_n) - \nabla T_k(u) = \\ &= \nabla u_n - \nabla T_h(u_n) - \nabla T_k(u) \end{aligned}$$

(because $\nabla T_k(u_n) = 0$ if $|u_n| > k$). There are two possible cases:

(i) If $k < |u_n| < h$, we have $\nabla T_h(u_n) = \nabla u_n$. Then

$$\nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) = -\nabla T_k(u).$$

(ii) If $h < |u_n| \leq M$, we have $\nabla T_h(u_n) = 0$. Then

$$\nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) = \nabla u_n - \nabla T_k(u) = \nabla T_M(u_n) - \nabla T_k(u).$$

In both cases we obtain

$$\begin{aligned} & |\nabla T_M(u_n)|^{p-2} \langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle \geq \\ & \geq -|\nabla T_M(u_n)|^{p-2} \langle \nabla T_M(u_n), \nabla T_k(u) \rangle \geq \\ & \geq -|\nabla T_M(u_n)|^{p-2} |\nabla T_M(u_n)| |\nabla T_k(u)|. \end{aligned}$$

Therefore, we obtain in (4.11)

$$\begin{aligned} & \int_{\Omega} \omega |\nabla T_M(u_n)|^{p-2} \langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx = \\ & = \int_{\{|u_n| \leq k\}} \omega |\nabla T_M(u_n)|^{p-2} \langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx + \\ & + \int_{\{|u_n| > k\}} \omega |\nabla T_M(u_n)|^{p-2} \langle \nabla T_M(u_n), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx \geq \\ & \geq \int_{\Omega} \omega |\nabla T_M(u_n)|^{p-2} \langle \nabla T_k(u_n), \nabla (T_k(u_n) - T_k(u)) \rangle dx - \\ & - \int_{\{|u_n| > k\}} \omega |\nabla T_M(u_n)|^{p-2} |\nabla T_M(u_n)| |\nabla T_k(u)| dx. \end{aligned}$$

Hence, in (4.10) we obtain

$$\begin{aligned} & \int_{\Omega} \omega \langle |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) - |\nabla T_k(u)|^{p-2} \nabla T_k(u), \nabla (T_k(u_n) - T_k(u)) \rangle dx \leq \\ & \leq \int_{\{|u_n| > k\}} \omega |\nabla T_M(u_n)| |\nabla T_k(u)| dx + \\ & + \int_{\Omega} f_n T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) dx + \tag{4.12} \\ & + \int_{\Omega} \langle G_n, \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx - \\ & - \int_{\Omega} \omega |\nabla T_k(u)|^{p-2} \langle \nabla T_k(u), \nabla (T_k(u_n) - T_k(u)) \rangle dx. \end{aligned}$$

Considering the test function $\psi_n = T_{2k}(u_n - T_h(u_n))$ in (4.1), we have

$$\int_{\Omega} \omega |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla \psi_n \rangle dx = \int_{\Omega} f_n \psi_n dx + \int_{\Omega} \langle G_n, \nabla \psi_n \rangle dx,$$

and using that

$$\left| \int_{\Omega} f_n \psi_n \, dx \right| \leq \int_{\Omega} |f_n| |\psi_n| \, dx \leq (2k + 1) \|f_n\|_{L^1(\Omega)} \leq (2k + 1) \|f\|_{L^1(\Omega)},$$

and

$$\begin{aligned} \int_{\Omega} \omega |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla \psi_n \rangle \, dx &= \int_{\Omega} \omega |\nabla \psi_n|^{p-2} \langle \nabla \psi_n, \nabla \psi_n \rangle \, dx = \\ &= \int_{\Omega} |\nabla \psi_n|^p \omega \, dx = \int_{\Omega} |\nabla T_{2k}(u_n - T_h(u_n))|^p \omega \, dx, \end{aligned}$$

we obtain

$$\int_{\Omega} |\nabla T_{2k}(u_n - T_h(u_n))|^p \omega \, dx \leq (2k + 1) C_2.$$

Now using that $T_{2k}(u_n - T_h(u_n)) \rightharpoonup T_{2k}(u - T_h(u))$ weakly in $W_0^{1,p}(\Omega, \omega)$ (by (4.8) and Remark 2.8 (i)), we have

$$\int_{\Omega} |\nabla T_{2k}(u - T_h(u))|^p \omega \, dx \leq (2k + 1) C_2. \tag{4.13}$$

We have (by Remark 2.8 (i) and (ii) and (4.13))

$$\begin{aligned} \int_{\Omega} |G| |\nabla T_{2k}(u - T_h(u))| \, dx &= \int_{\{h < |u| < 2k+h\}} |G| |\nabla u| \, dx \leq \\ &\leq \left(\int_{\{|u| \geq h\}} |G/\omega|^{p'} \omega \, dx \right)^{1/p'} \left(\int_{\{h < |u| < 2k+h\}} |\nabla u|^p \omega \, dx \right)^{1/p} = \\ &= \left(\int_{\{|u| \geq h\}} |G/\omega|^{p'} \omega \, dx \right)^{1/p'} \left(\int_{\Omega} |\nabla T_{2k}(u - T_h(u))|^p \omega \, dx \right)^{1/p} \leq \\ &\leq C_3 \left(\int_{\{|u| \geq h\}} |G/\omega|^{p'} \omega \, dx \right)^{1/p'}, \end{aligned}$$

where C_3 depends on k but not on h . Therefore, we have

$$\lim_{h \rightarrow \infty} \int_{\Omega} \langle G, \nabla T_{2k}(u - T_h(u)) \rangle \, dx = 0.$$

We also have (by Theorem 2.6 and (4.13))

$$\int_{\Omega} |T_{2k}(u - T_h(u))|^p \omega \, dx \leq C_{\Omega} \int_{\Omega} |\nabla T_{2k}(u - T_h(u))|^p \omega \, dx \leq C_{\Omega} C_2 (2k + 1).$$

Moreover, by Lebesgue's theorem, we obtain

$$\lim_{h \rightarrow \infty} \int_{\Omega} f T_{2k}(u - T_h(u)) \, dx = 0.$$

We can fix a positive real number h_ε sufficiently large to have

$$\int_{\Omega} f T_{2k}(u - T_{h_\varepsilon}) \, dx + \int_{\Omega} \langle G, \nabla T_{2k}(u - T_{h_\varepsilon}(u)) \rangle \, dx \leq \varepsilon. \tag{4.14}$$

Considering $h = h_\varepsilon$ in (4.12) (and $M = M_\varepsilon = 4k + h_\varepsilon$), by (4.6), we have

$$\begin{aligned} \int_{\Omega} |\nabla T_M(u_n)|^{p-2} \nabla T_M(u_n)|^{p'} \omega \, dx &= \int_{\Omega} |\nabla T_M(u_n)|^{(p-2)p'} |\nabla T_M(u_n)|^{p'} \omega \, dx = \\ &= \int_{\Omega} |\nabla T_M(u_n)|^p \omega \, dx \leq M C_2, \end{aligned}$$

that is, $|\nabla T_M(u_n)|^{p-2} \nabla T_M(u_n)|$ is bounded in $L^{p'}(\Omega, \omega)$. Moreover,

$$\chi_{\{|u_n|>k\}} |\nabla T_k(u)| \rightarrow 0$$

in $L^p(\Omega, \omega)$ as $n \rightarrow \infty$. Therefore,

$$\lim_{n \rightarrow \infty} \int_{\{|u_n|>k\}} |\nabla T_M(u_n)|^{p-2} \nabla T_M(u_n)| |\nabla T_k(u)| \omega \, dx = 0. \tag{4.15}$$

Futhermore, we have that $T_{2k}(u_n - T_h(u_n)) + T_k(u_n) - T_k(u) \rightarrow T_{2k}(u - T_h(u))$, weakly in $W_0^{1,p}(\Omega, \omega)$, as $n \rightarrow \infty$.

Hence, by (4.8), (4.14) and (4.15), passing to the limit in (4.12), we have

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_{\Omega} \langle |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) - |\nabla T_k(u)|^{p-2} \nabla T_k(u), \nabla(T_k(u_n) - T_k(u)) \rangle \omega \, dx \leq \\ &\leq \int_{\Omega} f T_{2k}(u - T_{h_\varepsilon}) \, dx + \int_{\Omega} \langle G, \nabla T_{2k}(u - T_{h_\varepsilon}(u)) \rangle \, dx \leq \varepsilon \end{aligned}$$

for all $\varepsilon > 0$, that is,

$$\lim_{n \rightarrow \infty} \int_{\Omega} \langle |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) - |\nabla T_k(u)|^{p-2} \nabla T_k(u), \nabla(T_k(u_n) - T_k(u)) \rangle \omega \, dx = 0.$$

Applying Lemma 4.1 we get

$$T_k(u_n) \rightarrow T_k(u) \tag{4.16}$$

strongly in $W_0^{1,p}(\Omega, \omega)$ for every $k > 0$.

This convergence implies that for every fixed $k > 0$

$$|\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \rightarrow |\nabla T_k(u)|^{p-2} \nabla T_k(u) \tag{4.17}$$

in $(L^{p'}(\Omega, \omega))^N = L^{p'}(\Omega, \omega) \times \dots \times L^{p'}(\Omega, \omega)$.

Finally, we need to show that u is an entropy solution to the Dirichlet problem (P) . Let us take $\psi_n = T_k(u_n - \varphi)$ as test function in (4.1), with $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$. We obtain

$$\int_{\Omega} \omega |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla \psi_n \rangle dx = \int_{\Omega} f_n \psi_n dx + \int_{\Omega} \langle G_n, \nabla \psi_n \rangle dx. \tag{4.18}$$

If $M = k + \|\varphi\|_{L^\infty(\Omega)}$ and $n > M$, we have

$$\int_{\Omega} \omega |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla T_k(u_n - \varphi) \rangle dx = \int_{\Omega} \omega |\nabla T_M(u_n)|^{p-2} \langle \nabla T_M(u_n), \nabla T_k(u_n - \varphi) \rangle dx.$$

Hence, in (4.18) we obtain

$$\begin{aligned} & \int_{\Omega} \omega |\nabla T_M(u_n)|^{p-2} \langle \nabla T_M(u_n), \nabla T_k(u_n - \varphi) \rangle dx = \\ & = \int_{\Omega} f_n T_k(u_n - \varphi) dx + \int_{\Omega} \langle G, \nabla T_k(u - \varphi) \rangle dx. \end{aligned} \tag{4.19}$$

Therefore, by (4.8) and (4.17), passing to the limit as $n \rightarrow \infty$ in (4.19), we obtain

$$\int_{\Omega} \omega |\nabla u|^{p-2} \langle \nabla u, \nabla T_k(u - \varphi) \rangle dx = \int_{\Omega} f T_k(u - \varphi) dx$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$ and for each $k > 0$.

Therefore u is an entropy solution of problem (P) . □

Example 4.3. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, $\omega(x, y) = (x^2 + y^2)^{-1/6}$ ($\omega \in A_3$, $p = 3$), $f(x, y) = \frac{\sin(xy)}{(x^2+y^2)^{1/3}}$ ($f \in L^1(\Omega)$), $G(x, y) = ((x^2 + y^2) \sin(xy), (x^2 + y^2)^{-1/3} \cos(xy))$. By Theorem 4.2, the problem

$$(P) \quad \begin{cases} -\operatorname{div}[(x^2 + y^2)^{-1/6} |\nabla u| \nabla u] = \frac{\sin(xy)}{(x^2+y^2)^{1/3}} - \operatorname{div}(G(x, y)) & \text{in } \Omega, \\ u(x, y) = 0 & \text{in } \partial\Omega \end{cases}$$

has an entropy solution.

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