

CONCAVITY OF SOLUTIONS OF A $2n$ -TH ORDER PROBLEM WITH SYMMETRY

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Abstract. In this article we apply an extension of a Leggett-Williams type fixed point theorem to a two-point boundary value problem for a $2n$ -th order ordinary differential equation. The fixed point theorem employs concave and convex functionals defined on a cone in a Banach space. Inequalities that extend the notion of concavity to $2n$ -th order differential inequalities are derived and employed to provide the necessary estimates. Symmetry is employed in the construction of the appropriate Banach space.

Keywords: Fixed-point theorems, concave and convex functionals, differential inequalities, symmetry.

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1. INTRODUCTION

Richard Avery and co-authors [4–6, 8] have extended the Leggett-Williams fixed point theorem [16] in various ways; a recent extension [5] employs topological methods rather than index theory and as a result the recent extension does not require the functional boundaries to be invariant with respect to a functional wedge. It is shown [6] that this extension applies in a natural way to second order right focal boundary value problems. The concept of concavity provides estimates that are useful in multiple technical arguments with respect to the concave and convex functionals; the increasing nature of functions gives rise to natural constructions of convex or concave functions.

Recently, Al Twayt and Eloe [3] applied these types of theorems to a two point conjugate type boundary value problem for a second order ordinary differential equation. Concavity was employed as in [6] and symmetry of functions was employed to construct appropriate concave or convex functionals. Avery, Eloe and Henderson [7] successfully extended these results for second order problems to fourth order problems.

In this article we shall apply the fixed point theorem to a two-point conjugate type boundary value problem for a general $2n$ -th order ordinary differential equation.

Symmetry will be employed as in [3] and [7]. In [7] a new inequality representing concavity was obtained for functions satisfying a fourth order differential inequality (and more importantly, a new inequality will be obtained for an associated Green's function). So, in this article, we successfully extend the concept of concavity to functions satisfying $2n$ -th order differential inequalities. As a corollary, we shall exhibit sufficient conditions for the existence of solutions for a family of $2n$ -th order two-point conjugate boundary value problems.

We point out that there has been particular interest in the application of fixed point theory to two point boundary value problems for a fourth order equation as these boundary value problems serve as models for cantilever beam problems. Fixed point applications have been of interest for many years [2, 17] and interest has recently been renewed, [9, 15, 18, 20, 22], for example.

In Section 2 we shall introduce the appropriate definitions and state the fixed point theorem. In Section 3, we shall apply the fixed point theorem to a conjugate boundary value problem for the $2n$ -th order problem. To do so, we first obtain Lemma 3.1 which gives a new estimate for an associated Green's function and represents the primary contribution of this work.

2. PRELIMINARIES

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if it satisfies the following two conditions:

- (i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$,
- (ii) $x \in P, -x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \text{ if and only if } y - x \in P.$$

Definition 2.2. An operator is called *completely continuous* if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. A map α is said to be a *nonnegative continuous concave functional* on a cone P of a real Banach space E if $\alpha : P \rightarrow [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say the map β is a *nonnegative continuous convex functional* on a cone P of a real Banach space E if $\beta : P \rightarrow [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Let α and ψ be non-negative continuous concave functionals on P and δ and β be non-negative continuous convex functionals on P ; then, for non-negative real numbers a, b, c and d , we define the following sets:

$$A := A(\alpha, \beta, a, d) = \{x \in P : a \leq \alpha(x) \text{ and } \beta(x) \leq d\}, \tag{2.1}$$

$$B := B(\alpha, \delta, \beta, a, b, d) = \{x \in A : \delta(x) \leq b\}, \tag{2.2}$$

and

$$C := C(\alpha, \psi, \beta, a, c, d) = \{x \in A : c \leq \psi(x)\}. \tag{2.3}$$

We say that A is a *functional wedge with concave functional boundary* defined by the concave functional α and convex functional boundary defined by the convex functional β . We say that an operator $T : A \rightarrow P$ is *invariant with respect to the concave functional boundary*, if $a \leq \alpha(Tx)$ for all $x \in A$, and that T is *invariant with respect to the convex functional boundary*, if $\beta(Tx) \leq d$ for all $x \in A$. Note that A is a convex set. The following theorem, proved in [5], is an extension of the original Leggett-Williams fixed point theorem [16].

Theorem 2.4. *Suppose P is a cone in a real Banach space E , α and ψ are non-negative continuous concave functionals on P , δ and β are non-negative continuous convex functionals on P , and for non-negative real numbers a, b, c and d the sets A, B and C are as defined in (2.1), (2.2) and (2.3). Furthermore, suppose that A is a bounded subset of P , that $T : A \rightarrow P$ is completely continuous and that the following conditions hold:*

- (A1) $\{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset, \{x \in P : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset,$
- (A2) $\alpha(Tx) \geq a$ for all $x \in B,$
- (A3) $\alpha(Tx) \geq a$ for all $x \in A$ with $\delta(Tx) > b,$
- (A4) $\beta(Tx) \leq d$ for all $x \in C,$ and
- (A5) $\beta(Tx) \leq d$ for all $x \in A$ with $\psi(Tx) < c.$

Then T has a fixed point $x^* \in A$.

A fixed point of T will also be called a solution of T .

3. THE APPLICATION

Let $f : \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous map. Let n denote a positive integer. We consider the two point conjugate boundary value problem:

$$(-1)^n x^{(2n)}(t) = f(x(t)), \quad t \in [0, 1], \tag{3.1}$$

$$x^{(i)}(0) = 0, \quad x^{(i)}(1) = 0, \quad i = 0, 1, \dots, n. \tag{3.2}$$

The Green’s function for this problem has the form, see [10] or [1, Lemma 3.4],

$$G(t, s) = \frac{1}{(2n - 1)!} \begin{cases} t^n(1 - s)^n \sum_{j=0}^{n-1} \binom{n-1+j}{j} (s - t)^{n-1-j} s^j (1 - t)^j, & 0 \leq t < s \leq 1, \\ s^n(1 - t)^n \sum_{j=0}^{n-1} \binom{n-1+j}{j} (t - s)^{n-1-j} t^j (1 - s)^j, & 0 \leq s < t \leq 1. \end{cases}$$

Note that G satisfies the symmetry property

$$G(1 - t, 1 - s) = G(t, s), \quad (t, s) \in [0, 1] \times [0, 1].$$

We shall state and prove a lemma that characterizes a generalized notion of concavity and motivates the construction of the appropriate cone in which to apply Theorem 2.4. The inequalities obtained here are very closely related to those obtained in [11] or [21]; however, we believe these inequalities are new.

Lemma 3.1. *If $y, w \in [0, 1]$ with $y < w$ and $t \in (0, 1)$, then*

$$\frac{G(t, y)}{G(t, w)} \geq \frac{y^n}{w^n}. \tag{3.3}$$

Proof. First, set

$$g(t, s) = \sum_{j=0}^{n-1} \binom{n - 1 + j}{j} (t - s)^{n-1-j} t^j (1 - s)^j,$$

and write

$$G(t, s) = \frac{1}{(2n - 1)!} \begin{cases} t^n(1 - s)^n g(s, t), & 0 \leq t < s \leq 1, \\ s^n(1 - t)^n g(t, s), & 0 \leq s < t \leq 1. \end{cases}$$

Suppose $y, w \in [0, 1]$ with $y < w$ and $t \in (0, 1)$. First, consider the case $y < w < t$. Note that

$$g(t, y) > g(t, w) > 0, \quad 0 \leq y < w < t,$$

and so

$$\frac{G(t, y)}{G(t, w)} = \frac{y^n(1 - t)^n g(t, y)}{w^n(1 - t)^n g(t, w)} = \frac{y^n g(t, y)}{w^n g(t, w)} \geq \frac{y^n}{w^n}.$$

Now, consider the case $y < t < w$. For $w \in (t, 1)$ (t is fixed), define the function

$$z(w) = \frac{(1 - w)^n g(w, t)}{w^n}. \tag{3.4}$$

The function z is a decreasing function in w which is seen by taking the derivative with respect to w in (3.4) to obtain

$$z'(w) = \frac{(1 - w)^{n-1}}{w^{n+1}} \left(w(1 - w) \frac{dg}{dw}(w, t) - ng(w, t) \right)$$

or

$$\begin{aligned} & \frac{w^{n+1}}{(1-w)^{n-1}} z'(w) = \\ & = w(1-w) \left(\sum_{j=0}^{n-2} \binom{n-1+j}{j} (n-1-j)(w-t)^{n-2-j} w^j (1-t)^j + \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \binom{n-1+j}{j} j (w-t)^{n-1-j} w^{j-1} (1-t)^j \right) - \\ & \quad - \left(n(w-t)^{n-1} + \sum_{j=0}^{n-2} (n-(j+1)) \binom{n+j}{j+1} (w-t)^{n-j-2} w^{j+1} (1-t)^{j+1} + \right. \\ & \quad \left. + \sum_{j=1}^{n-1} j \binom{n-1+j}{j} (w-t)^{n-j-1} w^j (1-t)^j \right) = \\ & = -n(w-t)^{n-1} + \\ & \quad + \sum_{j=0}^{n-2} (n-j-1) \binom{n-1+j}{j} w^{j+1} (w-t)^{n-2-j} (1-t)^j \left((1-w) - (1-t) \frac{n+j}{j+1} \right) - \\ & \quad - \sum_{j=1}^{n-1} j \binom{n-1+j}{j} (w-t)^{n-1-j} w^{j+1} (1-t)^j. \end{aligned}$$

Each term in the summation is negative and so, z is decreasing in w .

Now,

$$(1-t)^n g(t, t) = \frac{t^n (1-t)^n g(t, t)}{t^n} \geq \frac{t^n (1-w)^n g(w, t)}{w^n}.$$

Moreover,

$$(1-t)^n g(t, y) \geq (1-t)^n g(t, t) \geq t^n \frac{(1-w)^n}{w^n} g(w, t).$$

Thus,

$$(1-t)^n y^n g(t, y) \geq \frac{y^n}{w^n} t^n (1-w)^n g(w, t)$$

or

$$\frac{G(t, y)}{G(t, w)} \geq \frac{y^n}{w^n}.$$

Finally, consider the case $t < y < w$. Then $z(y) > z(w)$ or

$$\frac{t^n (1-y)^n g(y, t)}{y^n} \geq \frac{t^n (1-w)^n g(w, t)}{w^n};$$

that is,

$$\frac{G(t, y)}{G(t, w)} \geq \frac{y^n}{w^n}. \quad \square$$

Remark 3.2. Let $x \in C^{2n}[0, 1]$, $(-1)^n x^{(2n)}(t) \geq 0$, $0 < t < 1$, and x satisfies (3.2). Then for any $y, w \in [0, 1]$ with $y < w$ we have

$$\frac{x(y)}{x(w)} \geq \frac{y^n}{w^n}, \tag{3.5}$$

since

$$\begin{aligned} x(y) &= (-1)^n \int_0^1 G(y, s) x^{(2n)}(s) ds \geq \\ &\geq (-1)^n \int_0^1 \left(\frac{y^n}{w^n}\right) G(w, s) x^{(2n)}(s) ds = \left(\frac{y^n}{w^n}\right) x(w). \end{aligned}$$

Let $E = C[0, 1]$, equipped with the usual supremum norm denote the Banach space. Define the cone $P \subset E = C[0, 1]$ by

$$P := \{x \in E: x(1 - t) = x(t), 0 < t < 1, \ x(t) \geq 0, 0 < t < 1, \ x(t) \text{ nondecreasing on } t \in [0, 1/2], \text{ and if } 0 \leq y \leq w \leq 1, \text{ then } w^n x(y) \geq y^n x(w)\}.$$

Define $T : E \rightarrow E$ by

$$Tx(t) = \int_0^1 G(t, s) f(x(s)) ds.$$

Lemma 3.3. Assume $f : \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous map. Then

$$T : P \rightarrow P.$$

Proof. To see that $Tx(1 - t) = Tx(t)$, we have

$$\begin{aligned} Tx(1 - t) &= \int_0^1 G(1 - t, s) f(x(s)) ds = - \int_1^0 G(1 - t, 1 - \sigma) f(x(1 - \sigma)) d\sigma = \\ &= \int_0^1 G(1 - t, 1 - \sigma) f(x(\sigma)) d\sigma = \int_0^1 G(t, \sigma) f(x(\sigma)) d\sigma = Tx(t). \end{aligned}$$

Clearly, $Tx(t) \geq 0$ on $[0, 1]$ since $G(t, s) \geq 0$ on $[0, 1] \times [0, n]$ and $f : \mathbb{R} \rightarrow \mathbb{R}^+$.

To see that Tx is nondecreasing on $[0, 1/2]$, assume for the moment that $f : \mathbb{R} \rightarrow (0, +\infty)$. For $i = 0, \dots, n - 1$, $(Tx)^{(i)}$ has roots at $t = 0$ and at $t = 1$. Since $Tx(1 - t) = Tx(t)$, $(Tx)'(1/2) = 0$. $(Tx)^{(2n)} = f(x)$, and so $(Tx)^{(2n)}$ has no roots in $[0, 1]$. One can then use repeated applications of Rolle's theorem (see [11] or [21]) and show that $(Tx)'$ only has roots at $t = 0, 1/2$, or 1 . In particular, $(Tx)'$ is strictly positive on $(0, 1/2)$.

If $f : \mathbb{R} \rightarrow [0, +\infty)$, consider $x_\epsilon = \int_0^1 G(t, s)(f(x(s)) + \epsilon)ds$. Then x'_ϵ is strictly positive on $(0, 1/2)$ and $x'_\epsilon \rightarrow (Tx)'$ as $\epsilon \rightarrow 0^+$. In conclusion, Tx is nondecreasing on $[0, 1/2]$.

Finally, $(-1)^n(Tx)^{(n)}(t) = f(x(t)) \geq 0, 0 < t < 1$ and Tx satisfies (3.2). So by Lemma 3.1, $w^nTx(y) \geq y^nTx(w)$ and Tx satisfies the concavity condition. \square

For fixed $\nu, \tau, \mu \in [0, \frac{1}{2}]$ and $x \in P$, define the concave functionals α and ψ on P by

$$\alpha(x) := \min_{t \in [\tau, \frac{1}{2}]} x(t) = x(\tau), \quad \psi(x) := \min_{t \in [\mu, \frac{1}{2}]} x(t) = x(\mu),$$

and the convex functionals δ and β on P by

$$\delta(x) := \max_{t \in [0, \nu]} x(t) = x(\nu), \quad \beta(x) := \max_{t \in [0, \frac{1}{2}]} x(t) = x\left(\frac{1}{2}\right).$$

Theorem 3.4. Assume $\tau, \nu, \mu \in (0, \frac{1}{2}]$ are fixed with $\tau \leq \mu < \nu, d$ and L are positive real numbers with $0 < L \leq 2^n \mu^n d$. Set $K = \frac{1}{(2n-1)!2^{2n-1}} \sum_{j=0}^{n-1} \binom{n-1+j}{j}$. Assume $f : [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that:

- (a) $f(w) \geq \frac{(2n-1)((n-1)!)^2 2^{3n-1} d}{\tau^{n-1}(\nu-\tau)} \equiv M$ for $w \in [2^n \tau^n d, 2^n \nu^n d]$,
- (b) $f(w)$ is decreasing for $w \in [0, L]$ with $f(L) \geq f(w)$ for $w \in [L, d]$, and
- (c) $\int_0^\mu s^n f\left(\frac{Ls^n}{\mu^n}\right) ds \leq \frac{d}{2K} - \frac{f(L)}{n+1} \left(\frac{1}{2^{n+1}} - \mu^{n+1}\right)$.

Then the operator T has at least one positive solution $x^* \in A(\alpha, \beta, 2^n \tau^n d, d)$.

Proof. Let $a = 2^n \tau^n d, b = 2^n \nu^n d = \frac{\nu^n}{\tau^n} a$, and $c = 2^n \mu^n d$. Let $x \in A(\alpha, \beta, a, d)$. An immediate corollary of Lemma 3.3 is

$$T : A(\alpha, \beta, a, d) \rightarrow P.$$

By the Arzelà-Ascoli Theorem it is a standard exercise to show that T is a completely continuous operator using the properties of G and f ; by the definition of β, A is a bounded subset of the cone P . Also, if $x \in P$ and $\beta(x) > d$, then by the properties of the cone P (in particular, the concavity of x),

$$\alpha(x) = x(\tau) \geq 2^n \tau^n x\left(\frac{1}{2}\right) = 2^n \tau^n \beta(x) > 2^n \tau^n d = a.$$

Thus,

$$\{x \in P : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset.$$

For any $r \in \left(\frac{(2n)!2^n d}{(1-\mu)^n}, \frac{(2n)!2^n d}{(1-\nu)^n}\right)$ define x_r by

$$x_r(t) \equiv \int_0^1 rG(t, s)ds = \frac{rt^n(1-t)^n}{(2n)!}.$$

We claim $x_r \in A$. We have

$$\alpha(x_r) = x_r(\tau) = \frac{r\tau^n(1-\tau)^n}{(2n)!} > \frac{(2n)!2^n d \tau^n(1-\tau)^n}{(1-\mu)^n (2n)!} \geq 2^n \tau^n d = a,$$

$$\beta(x_r) = x_r\left(\frac{1}{2}\right) = \frac{r\left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n}{(2n)!} < \frac{(2n)!2^n d \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n}{(1-\nu)^n (2n)!} = \frac{\left(\frac{1}{2}\right)^n}{(1-\nu)^n} d \leq d.$$

Thus, the claim is true. Moreover, x_r has the properties that

$$\psi(x_r) = x_r(\mu) = \frac{r\mu^n(1-\mu)^n}{(2n)!} > \left(\frac{(2n)!2^n d}{(1-\mu)^n}\right) \left(\frac{\mu^n(1-\mu)^n}{(2n)!}\right) = 2^n \mu^n d = c$$

and

$$\delta(x_r) = x_r(\nu) = \frac{r\nu^n(1-\nu)^n}{(2n)!} < \left(\frac{(2n)!2^n d}{(1-\nu)^n}\right) \left(\frac{\nu^n(1-\nu)^n}{(2n)!}\right) = 2^n \nu^n d = b.$$

In particular,

$$\{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset.$$

We have shown that condition (A1) of Theorem 2.4 is satisfied.

We now verify that condition (A2) of Theorem 2.4, $\alpha(Tx) \geq a$ for all $x \in B$, is satisfied. Let $x \in B$. Apply condition (a) of Theorem 3.4, and

$$\alpha(Tx) = \int_0^1 G(\tau, s) f(x(s)) ds \geq M \int_\tau^\nu G(\tau, s) ds.$$

Now, for $\tau \leq s \leq \nu \leq \frac{1}{2}$,

$$\begin{aligned} G(\tau, s) &\geq \frac{1}{(2n-1)!} \tau^n (1-s)^n \binom{2(n-1)}{n-1} s^{n-1} (1-\tau)^{n-1} \geq \\ &\geq \frac{\tau^{2n-1}}{(2n-1)((n-1)!)^2 2^{2n-1}} (1-s)^n \geq \frac{\tau^{2n-1}}{(2n-1)((n-1)!)^2 2^{2n-1}}. \end{aligned}$$

Thus,

$$\alpha(Tx) = \int_0^1 G(\tau, s) f(x(s)) ds \geq M \frac{\tau^{2n-1}}{(2n-1)((n-1)!)^2 2^{2n-1}} \int_\tau^\nu ds = 2^n \tau^n d = a.$$

We now verify that condition (A3) of Theorem 2.4, $\alpha(Tx) \geq a$, for all $x \in A$ with $\delta(Tx) > b$, is satisfied. Let $x \in A$ with $\delta(Tx) > b$. Apply Lemma 3.1 to obtain

$$\alpha(Tx) = (Tx)(\tau) \geq \left(\frac{\tau}{\nu}\right)^n (Tx)(\nu) = \left(\frac{\tau}{\nu}\right)^n \delta(Tx) > \left(\frac{\tau}{\nu}\right)^n 2^n \nu^n d = a.$$

We now verify that condition (A4) of Theorem 2.4, $\beta(Tx) \leq d$, for all $x \in C$, is satisfied. Let $x \in C$. Since $c = 2^n \mu^n d$ and $0 < L \leq 2^n \mu^n d = c$, the concavity of x implies (see the remark following Lemma 3.1), for $s \in [0, \mu]$,

$$x(s) \geq \frac{s^n}{\mu^n} x(\mu) \geq \frac{cs^n}{\mu^n} \geq \frac{Ls^n}{\mu^n}.$$

Since x is symmetric about $\frac{1}{2}$ and $G(\frac{1}{2}, s)$ is symmetric about $s = \frac{1}{2}$, it follows that

$$\beta(Tx) = \int_0^1 G\left(\frac{1}{2}, s\right) f(x(s)) \, ds = 2 \int_0^{\frac{1}{2}} G\left(\frac{1}{2}, s\right) f(x(s)) \, ds$$

which we shall use to abbreviate our calculations. Note that for $s \leq \frac{1}{2}$,

$$G\left(\frac{1}{2}, s\right) \leq \frac{1}{(2n-1)!} \frac{s^n}{2^n} g\left(\frac{1}{2}, 0\right) = \frac{1}{(2n-1)!} \frac{s^n}{2^{(2n-1)}} \left(\sum_{j=0}^{n-1} \binom{n-1+j}{j} \right) = Ks^n.$$

Applying properties (b) and (c) of Theorem 3.4, we have

$$\begin{aligned} \beta(Tx) &= 2 \int_0^{\frac{1}{2}} G\left(\frac{1}{2}, s\right) f(x(s)) \, ds \leq 2K \int_0^{\frac{1}{2}} s^n f(x(s)) \, ds \leq \\ &\leq 2K \int_0^{\mu} s^n f\left(\frac{Ls^n}{\mu^n}\right) \, ds + 2Kf(L) \int_{\mu}^{\frac{1}{2}} s^n \, ds \leq \\ &\leq d - \frac{2Kf(L)}{(n+1)} \left(\frac{1}{2^{n+1}} - \mu^{n+1} \right) + \frac{2Kf(L)}{(n+1)} \left(\frac{1}{2^{n+1}} - \mu^{n+1} \right) = d. \end{aligned}$$

We close the proof by verifying that condition (A5), $\beta(Tx) \leq d$, for all $x \in A$ with $\psi(Tx) < c$ is satisfied. Let $x \in A$ with $\psi(Tx) < c$. Apply Lemma 3.1 to obtain

$$\beta(Tx) = (Tx)\left(\frac{1}{2}\right) \leq \left(\frac{1}{2^n \mu^n}\right) Tx(\mu) = \left(\frac{1}{2^n \mu^n}\right) \psi(Tx) \leq \left(\frac{c}{2^n \mu^n}\right) = d.$$

Therefore, the hypotheses of Theorem 2.4 have been satisfied; thus the operator T has at least one positive solution $x^* \in A(\alpha, \beta, a, d)$. □

REFERENCES

[1] R.P. Agarwal, *Boundary Value Problems for Higher Order Differential Equations*, World Scientific, Singapore, 1986.
 [2] R.P. Agarwal, *On fourth-order boundary value problems arising in beam analysis*, *Differential Integral Equations* **2** (1989), 91–110.
 [3] A. Al Twaty, P. Eloe, *The role of concavity in applications of Avery type fixed point theorems to higher order differential equations*, *J. Math. Inequal.* **6** (2012), 79–90.

- [4] D.R. Anderson, R.I. Avery, *Fixed point theorem of cone expansion and compression of functional type*, J. Difference Equations Appl. **8** (2002), 1073–1083.
- [5] D.R. Anderson, R.I. Avery, J. Henderson, *A topological proof and extension of the Leggett-Williams fixed point theorem*, Comm. Appl. Nonlinear Anal. **16** (2009), 39–44.
- [6] D.R. Anderson, R.I. Avery, J. Henderson, *Existence of a positive solution to a right focal boundary value problem*, Electron. J. Qual. Theory Differ. Equ. **16** (2010), 5, 6 pp.
- [7] R.I. Avery, P. Eloe, J. Henderson, *A Leggett-Williams type theorem applied to a fourth order problem*, Proc. of Dynamic Systems and Applications **6** (2012), 579–588.
- [8] R.I. Avery, J. Henderson, D. O'Regan, *Dual of the compression-expansion fixed point theorems*, Fixed Point Theory Appl. **2007** (2007), Article ID 90715, 11 pp.
- [9] Z. Bai, *The upper and lower solution method for some fourth-order boundary value problems*, Nonlinear Anal. **67** (2007), 1704–1709.
- [10] K.M. Das, A.S. Vatsala, *On Green's function of an n -point boundary value problem*, Trans. Amer. Math. Soc. **182** (1973), 469–480.
- [11] P. Eloe, J. Henderson, *Inequalities based on a generalization of concavity*, Proc. Amer. Math. Soc. **125** (1997), 2103–2107.
- [12] D. Guo, *A new fixed point theorem*, Acta Math. Sinica **24** (1981), 444–450.
- [13] D. Guo, *Some fixed point theorems on cone maps*, Kexue Tongbao **29** (1984), 575–578.
- [14] M.A. Krasnosel'skii, *Positive Solutions of Operator Equations*, P. Noordhoff, Groningen, The Netherlands, 1964.
- [15] J.R. Graef, B. Yang, *Positive solutions of a nonlinear fourth order boundary value problem*, Comm. Appl. Nonlinear Anal. **1** (2007), 61–73.
- [16] R.W. Leggett, L.R. Williams, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math. J. **28** (1979), 673–688.
- [17] R. Ma, H. Wang, *On the existence of positive solutions of fourth order-order ordinary differential equations*, Appl. Anal. **59** (1995), 225–231.
- [18] M. Pei, S.K. Chang, *Monotone iterative technique and symmetric positive solutions for a fourth-order boundary value problem*, Math. Comput. Modelling **51** (2010), 1260–1267.
- [19] J. Sun, G. Zhang, *A generalization of the cone expansion and compression fixed point theorem and applications*, Nonlinear Anal. **67** (2007), 579–586.
- [20] B. Yang, *Positive solutions for the beam equation under certain boundary conditions*, Electron. J. Differential Equations **2005** (78) (2005), 1–8.
- [21] B. Yang, *Upper estimate for positive solutions of the $(p, n - p)$ conjugate boundary value problem*, J. Math. Anal. Appl. **390** (2012), 535–548.
- [22] C. Zhai, R. Song, Q. Han, *The existence and uniqueness of symmetric positive solutions for a fourth-order boundary value problem*, Comput. Math. Appl. **62** (2011), 2639–2647.
- [23] E. Zeidler, *Nonlinear Functional Analysis and Its Applications I: Fixed-Point Theorems*, Springer-Verlag, New York, 1986.

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