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A NOTE ON k-ROMAN GRAPHS

Ahmed Bouchou, Mostafa Blidia, and Mustapha Chellali

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Abstract. Let $G = (V, E)$ be a graph and let k be a positive integer. A subset D of $V(G)$ is a k-dominating set of G if every vertex in $V(G) \setminus D$ has at least k neighbours in D. The k-domination number $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G. A Roman k-dominating function on G is a function $f\colon V(G) \longrightarrow \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least k vertices $v_1, v_2, \ldots, v_k$ with $f(v_i) = 2$ for $i = 1, 2, \ldots, k$. The weight of a Roman k-dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$ and the minimum weight of a Roman k-dominating function on G is called the Roman k-domination number $\gamma_{kR}(G)$ of G. A graph G is said to be a k-Roman graph if $\gamma_{kR}(G) = 2\gamma_k(G)$. In this note we study k-Roman graphs.

Keywords: Roman k-domination, k-Roman graph.

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1. INTRODUCTION

We consider finite, undirected, and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* $N_G(v)$ of a vertex v consists of the vertices adjacent to v, and $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighborhood.* The *degree* of v is $|N_G(v)|$. A *leaf* is a vertex of degree one. By $\Delta(G) = \Delta$ we denote the *maximum degree* of a graph G. A graph is *bipartite* if its vertex set can be partitioned into two independent sets. A *d-regular* graph is a graph with degree d for each vertex of G. A graph is called a *d-semiregular bipartite graph* if its vertex set can be partitioned in such a way that every vertex in one of the partite sets has degree d. The *subdivision graph* of a graph G is the graph obtained from G by replacing each edge uv of G by a vertex w and edges uw and vw. A graph G is called a *cactus graph* if each edge of G is contained in at most one cycle. A *unicyclic graph* is a connected graph containing exactly one cycle. A *tree* is a connected graph with no cycle. We denote by $K_{1,t}$ a *star* of order $t + 1$.

Let k be a positive integer. A subset $D \subseteq V(G)$ is a *k-dominating set* of a graph G if $|N_G(v) \cap D| \geq k$ for every $v \in V(G) \backslash D$. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality among the k-dominating sets of G. The concept of k-domination was introduced by Fink and Jacobson in [2].

A *Roman k-dominating function* on G is a function $f : V(G) \longrightarrow \{0,1,2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least k vertices $v_1, v_2, \ldots, v_k$ with $f(v_i) = 2$ for $i = 1, 2, \ldots, k$. The *weight* of a Roman k-dominating function is the value $f(V(G)) = \sum_{v \in V(G)} f(v)$. The minimum weight of a Roman k-dominating function on a graph G is called the *Roman k-domination number* $\gamma_{kR}(G)$. Note that if $k \geq \Delta + 1$, then clearly $\gamma_{kR}(G) = |V|$. Hence we may assume in the whole paper that $k \leq \Delta$. Also, if $f : V(G) \longrightarrow \{0,1,2\}$ is a Roman k-dominating function on G, then let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i = 0, 1, 2$. Note that there is a one to one correspondence between the functions $f : V(G) \rightarrow \{0,1,2\}$ and the ordered partitions (V_0, V_1, V_2) of $V(G)$. The Roman 1-domination number γ_{1R} corresponds to the well-known *Roman domination number* γ_R, which was given implicitly by Steward in [5] and by ReVelle and Rosing in [4].

2. KNOWN RESULTS

We begin by listing some known results that will be useful here. The first one gives a relation between the Roman k-domination and k-domination numbers for any graph.

Proposition 2.1 (Kämmerling and Volkmann [3])**.** *For any graph G,*

$$\gamma_k(G) \leq \gamma_{kR}(G) \leq 2\gamma_k(G).$$

According to [3], a graph G is said to be a *k-Roman graph* if $\gamma_{kR}(G) = 2\gamma_k(G)$. Kämmerling and Volkmann gave a necessary and sufficient condition for a graph to be k-Roman.

Proposition 2.2 (Kämmerling and Volkmann [3])**.** *A graph G is a k-Roman graph if and only if it has a γ_{kR}-function $f = (V_0, V_1, V_2)$ with $V_1 = \emptyset$.*

The following two results give sufficient conditions for G to have $\gamma_{kR}(G) = n$.

Proposition 2.3 (Kämmerling and Volkmann [3])**.** *If G is a graph with at most one cycle and $k \geq 2$, or G is a cactus graph and $k \geq 3$, then $\gamma_{kR}(G) = n$.*

Proposition 2.4 (Kämmerling and Volkmann [3])**.** *If G is a graph of order n and maximum degree $\Delta \geq 1$, then $\gamma_{\Delta R}(G) = n$.*

In [2], Fink and Jacobson have established a lower bound on the k-domination number of a graph.

Theorem 2.5 (Fink and Jacobson [2]). *If G has n vertices and $m(G)$ edges, then*

$$\gamma_k(G) \geq n - \frac{m(G)}{k} \quad for \quad k \geq 1.$$

Furthermore, if $m(G) \neq 0$, then $\gamma_k(G) = n - \frac{m(G)}{k}$ if and only if G is a k-semiregular bipartite graph.

Corollary 2.6 (Fink and Jacobson [2]). *If G is a graph with n vertices and $m(G) \neq 0$ edges, then*

$$\gamma_2(G) = n - \frac{m(G)}{2}.$$

if and only if G is the subdivision graph of another multigraph (graph with possibly parallel edges).

3. MAIN RESULTS

We begin by giving a necessary condition for a graph to be k-Roman.

Theorem 3.1. *If G is a k-Roman graph with $k \geq 2$, then every vertex of G is adjacent to at most $k-1$ leaves.*

Proof. Let G be a k-Roman graph with $k \geq 2$. Suppose that v is a vertex of G adjacent to at least k leaves. Let L_v be the set of leaves adjacent to v. Clearly, for every γ_{kR}-function every leaf is assigned a positive value. Also, by Proposition 2.2, G has a γ_{kR}-function $f = (V_0, V_1, V_2)$ with $V_1 = \emptyset$. Hence $f(w) = 2$ for every leaf $w \in L_v$. Now if $f(v) \neq 0$, then we can decrease the weight of f by assigning the value 1 instead of 2 to every leaf, contradicting the fact that f is a γ_{kR}-function. Thus $f(v) = 0$. Since $k \geq 2$, we can change $f(w) = 2$ to $f(w) = 1$ for every vertex $w \in L_v$ and $f(v) = 0$ to $f(v) = 1$. Clearly we obtain a Roman k-dominating function with weight less than $f(V(G))$, a contradiction. Therefore, $|L_v| \leq k-1$. □

We now give a characterization of k-Roman graphs when $k = \Delta$.

Theorem 3.2. *A graph G is Δ-Roman if and only if G is a bipartite regular graph.*

Proof. Let G be a graph with $\gamma_{\Delta R}(G) = 2\gamma_\Delta(G)$. Then by Proposition 2.4, $\gamma_{\Delta R}(G) = n = 2\gamma_\Delta(G)$, and so $\gamma_\Delta(G) = n/2$. Let S be a minimum Δ-dominating set of G. Clearly, since every vertex of $V \backslash S$ has Δ neighbours in S, the set $V \backslash S$ is independent. Now let m' be the number of edges between S and $V \backslash S$. Then $m' = \Delta\,|V \backslash S| = \Delta n/2$. Using the fact that $\Delta n \geq 2\,|E|$, it follows that $\Delta n = 2\,|E| = 2m' = \Delta n$, and so $|E| = m'$. Thus, every vertex of G has degree Δ and hence S is also independent. Therefore, G is a bipartite Δ-regular graph.

Conversely, assume that G is a bipartite Δ-regular graph. We know by Proposition 2.4 that $\gamma_{\Delta R}(G) = n$. Thus, it suffices to show that $\gamma_\Delta(G) = n/2$. By Proposition 2.1, we have $\gamma_\Delta(G) \geq n/2$. The equality is obtained from the fact that every partite set of G is a Δ-dominating set. □

Next we improve the upper bound in Proposition 2.1 for the class of trees. Moreover, we characterize all trees attaining this upper bound.

Theorem 3.3. *Let T be a tree of order $n \geq 3$ with $\Delta(T) \geq k \geq 2$. Then*

$$\gamma_{kR}(T) \leq 2\gamma_k(T) - k + 1,$$

with equality if and only if:

(i) *$k = 2$ and T is the subdivision graph of another tree, or*
(ii) *$k = n - 1$ and T is a star.*

Proof. We first prove the upper bound. Since $m = n - 1$ for trees, it follows from Theorem 2.5 that for every tree T and every positive integer k we have

$$\gamma_k(G) \geq \frac{(k-1)n+1}{k}.$$

Also, one can easily check that

$$\frac{(k-1)n+1}{k} \geq \frac{n+k-1}{2} \quad \text{for} \quad 2 \leq k \leq \Delta(T) \leq n-1.$$

Now using the fact that $\gamma_{kR}(T) = n$ (by Proposition 2.3) we obtain

$$\gamma_k(G) \geq \frac{(k-1)n+1}{k} \geq \frac{n+k-1}{2} = \frac{\gamma_{kR}(T)+k-1}{2},$$

and the bound is proved.

Now assume that $\gamma_{kR}(T) = 2\gamma_k(T) - k + 1$. Then we have equality throughout the previous inequality chain. In particular, $((k-1)n+1)/k = (n+k-1)/2$ and $\gamma_k(G) = ((k-1)n+1)/k$. The first equality implies that $k = 2$ or $k = n-1$. Now, if $k = 2$, then $\gamma_2(G) = (n+1)/2$ and by Corollary 2.6 we obtain (i). If $k = n-1$, then T is the star $K_{1,n-1}$.

The converse is easy to show and we omit the details. □

The following corollary is an immediate consequence of Theorem 3.3.

Corollary 3.4. *There are no k-Roman trees for $k \geq 2$.*

Next we show that there are no k-Roman cactus graphs for $k \geq 3$. We need the following lemma, which can be found in [7] on p. 30.

Lemma 3.5. *If G is a cactus graph on n vertices and m edges, then*

$$2m \leq 3n - 3.$$

Proposition 3.6. *There are no k-Roman cactus graph for $k \geq 3$.*

Proof. Suppose that G is a k-Roman cactus graph for some $k \geq 3$. By Proposition 2.3 and Theorem 2.5 we have $n = \gamma_{kR}(T) = 2\gamma_k(G) \geq 2(n - m/k)$. Hence $kn \leq 2m$. Now, by Lemma 3.5 we get $kn \leq 3n - 3$, which is impossible since $k \geq 3$. □

Next we improve the upper bound in Proposition 2.1 for unicyclic graphs. We denote by $K_{1,p}+e$ the graph obtained from the star $K_{1,p}$ by adding an edge between two leaves of $K_{1,p}$. Let P_5 be the path on five vertices labeled in order $1,2,3,4,5$. Let F be the graph obtained from P_5 by adding a new vertex x and edges $x2$ and $x4$. Let G_1, G_2 and G_3 be three graphs obtained from P_5 by adding the edges 24, 35 and 25, respectively.

Theorem 3.7. *Let G be a unicyclic graph and $\Delta(G) \geq k \geq 3$. Then*

$$\gamma_{kR}(G) \leq 2\gamma_k(G) - k + 1,$$

with equality if and only if either $k \in \{3, 4, n-1\}$ and $G = K_{1,k} + e$, or $k = 3$ and $G = F$.

Proof. We first note that $n \geq 4$ since $\Delta \geq 3$. If $n = 4$, then $k = \Delta = 3$, $G = K_{1,3} + e$ and $\gamma_{kR}(G) = 2\gamma_k(G) - k + 1$. If $n = 5$, then $k \in \{3, 4\}$. If $k = 3$, then clearly $G \in \{G_1, G_2, G_3, K_{1,4} + e\}$ and $\gamma_{kR}(G) < 2\gamma_k(G) - k + 1$. If $k = 4$, then $G = K_{1,4} + e$ and $\gamma_{kR}(G) = 2\gamma_k(G) - k + 1$. Also if $n = k + 1$, then $k = \Delta$, $G = K_{1,n-1} + e$ and $\gamma_{kR}(G) = 2\gamma_k(G) - k + 1$.

Now let us suppose that $n \geq \max\{6, k+2\}$. It can be seen that

$$\frac{(k-1)n}{k} \geq \frac{n+k-1}{2} \tag{3.1}$$

and the upper bound follows from Proposition 2.3 and Theorem 2.5.

Now assume that $\gamma_{kR}(G) = 2\gamma_k(G) - k + 1$. Clearly, if $n \in \{4, 5, k+1\}$, then $G = K_{1,n-1} + e$. Hence we can assume that $n \geq \max\{6, k+2\}$. Then we have equality in (3.1), in particular $\gamma_k(G) = (n+k-1)/2 = (k-1)n/k$. It follows that $n = 6$, $k = 3$, $\gamma_3(G) = 4$, and so $G = F$. □

Theorem 3.8. *A unicyclic graph G is a 2-Roman graph if and only if G is the subdivided graph of another unicyclic graph (possibly with a cycle on two vertices).*

Proof. If $\gamma_{2R}(G) = 2\gamma_2(G)$, then by Proposition 2.3 we have $n = 2\gamma_2(G)$, and so $\gamma_2(G) = n/2$. By Corollary 2.6, G is the subdivided graph of another unicyclic graph. Now assume that G is the subdivided graph of another unicyclic graph. By Corollary 2.6, $\gamma_2(G) = n/2$ and by Proposition 2.3, $\gamma_{2R}(G) = n$. Therefore, $\gamma_{2R}(G) = 2\gamma_2(G)$. □

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Ahmed Bouchou
bouchou.ahmed@yahoo.fr

University Dr Yahia Farès
Médéa, Algeria

Mostafa Blidia
m_blidia@yahoo.fr

University of Blida
LAMDA-RO, Department of Mathematics
B.P. 270, Blida, Algeria

Mustapha Chellali
m_chellali@yahoo.com

University of Blida
LAMDA-RO, Department of Mathematics
B.P. 270, Blida, Algeria

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