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PICONE-TYPE IDENTITY AND COMPARISON RESULTS FOR A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS OF ORDER $4m$

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Abstract. In the paper, a Picone-type identity for the weighted p-polyharmonic operator is established and comparison theorems and other qualitative results for a class of half-linear partial differential equations of the $4m$th order based on this identity are derived.

Keywords: p-polyharmonic operator, Picone's identity.

Mathematics Subject Classification: 35B05, 35J70.

1. INTRODUCTION

Recently, an identity of the Picone type for the weighted p-biharmonic operator was established by the present author [5] and employed in proving comparison theorems and other qualitative results for a pair of fourth-order elliptic partial differential equations of the form

$$\Delta\big(a(x)|\Delta u|^{p-2}\Delta u\big) - c(x)|u|^{p-2}u = 0, \tag{1.1}$$

and

$$\Delta\big(A(x)|\Delta v|^{p-2}\Delta v\big) - C(x)|v|^{p-2}v = 0, \tag{1.2}$$

where $p > 1$, $a, A \in \mathrm{C}^2(\bar{G}, \mathbb{R}_+), c, C \in \mathrm{C}(\bar{G}, \mathbb{R})$, G is a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary ∂G, $\Delta = (\partial^2/\partial x_1^2) + \ldots + (\partial^2/\partial x_n^2)$ is the usual Laplace operator and $|\cdot|$ denotes the Euclidean length of a vector in $\mathbb{R}^n$.

Roughly speaking, the identity from [5] states that if u and v are (classical) solutions of (1.1) and (1.2), respectively, and $v(x) > 0$ in the domain under consideration, then:

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$$\begin{aligned}
&\operatorname{div}\Big[u\nabla(a|\Delta u|^{p-2}\Delta u)-a|\Delta u|^{p-2}\Delta u\nabla u-\\
&\qquad -\frac{|u|^p}{v^{p-1}}\nabla(A|\Delta v|^{p-2}\Delta v)+A|\Delta v|^{p-2}\Delta v\nabla\Big(\frac{|u|^p}{v^{p-1}}\Big)\Big]=\\
&=\big(A-a\big)|\Delta u|^p+\big(c-C\big)|u|^p-A\Phi_p\big(\Delta u,u\frac{\Delta v}{v}\big)+\\
&\quad +p(p-1)A|u|^{p-2}\frac{|\Delta v|^{p-2}\Delta v}{v^{p-1}}\big|\nabla u-\frac{u}{v}\nabla v\big|^2,
\end{aligned}$$

where $\Phi_p(X,Y)$ is a certain positive semi-definite form specified in Section 2.

A typical result that can be obtained by integrating the above identity on a given bounded domain G with a piecewise smooth boundary ∂G and using the divergence theorem combined with suitable boundary conditions is the following comparison theorem.

Theorem 1.1. *If there exists a nontrivial solution u of* (1.1) *satisfying*

$$u=\frac{\partial u}{\partial\nu}=0\quad on\ \partial G \tag{1.3}$$

and

$$\int_G\big[(a(x)-A(x))|\Delta u|^p+(C(x)-c(x))|u|^p\big]dx\geq 0, \tag{1.4}$$

then every solution v of (1.2) *which satisfies $\Delta v<0$ in G and $v(x)>0$ for some $x\in G$ has a zero in $\bar{G}$.*

Here and in what follows $\partial/\partial\nu$ denotes the exterior normal derivative.

The purpose of this paper is to extend Theorem A and other results from [5] to a class of $4m$th order elliptic equations of the form

$$\Delta^m\big(a(x)|\Delta^m u|^{p-2}\Delta^m u\big)+c(x)|u|^{p-2}u=0, \tag{1.5}$$

and

$$\Delta^m\big(A(x)|\Delta^m v|^{p-2}\Delta^m v\big)+C(x)|v|^{p-2}v=0, \tag{1.6}$$

where $m\geq 1$, $a,A\in \mathrm{C}^{2m}(\bar{G},\mathbb{R}_+),c,C\in\mathrm{C}(\bar{G},\mathbb{R})$ and Δ^m denotes the mth iteration of the Laplace operator (or the so-called polyharmonic operator) defined by $\Delta^m u=\Delta(\Delta^{m-1}u)$ for $m=1,2,\ldots$.

The paper is organized as follows. In Section 2, we establish two forms of the desired extension of Picone's formula. The first one is suitable for obtaining results concerning a single equation of the form (1.6) (or the associated inequality) while the second one relates two functions u and v from the domains of the operators l and L defined by the left-hand sides of (1.5) and (1.6), respectively, and can be used for comparison purposes. In Section 3, we illustrate applications of the basic identities by deriving comparison theorems and other qualitative results (such as lower bounds for eigenvalues, the generalized Wirtinger inequality etc.) for partial differential equations and inequalities involving the weighted p-polyharmonic operator.

For related results in the particular case $m = 1$ see [3] ($p = 2$), [4] (general $p > 1$ and $n = 1$) and [8] ($p = 2$ and $n = 1$). The case of linear even order ordinary differential equations was studied in [6]. Picone identities for various classes of half-linear partial differential equations of the second order and their applications can be found in the monographs [1, 10].

2. IDENTITIES FOR ELLIPTIC OPERATORS OF THE $4m$-th ORDER

Let $m \geq 1$ and G be a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary ∂G and let $a, A \in \mathrm{C}^{2m}(\bar{G}, \mathbb{R}_+)$ and $c, C \in \mathrm{C}(\bar{G}, \mathbb{R})$, where $\mathbb{R}_+ = (0, \infty)$. For a fixed $p > 1$ define the odd power function $\varphi_p : \mathbb{R} \to \mathbb{R}$ by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, and consider partial differential operators of the form

$$l[u] = \Delta^m(a(x)\varphi_p(\Delta^m u)) + c(x)\varphi_p(u)$$

and

$$L[v] = \Delta^m(A(x)\varphi_p(\Delta^m v)) + C(x)\varphi_p(v)$$

with the domains $\mathcal{D}_l(G)$ (resp. $\mathcal{D}_L(G)$) defined to be the sets of all functions u (resp. v) of class $\mathrm{C}^{2m}(\bar{G}, \mathbb{R})$ such that $a(x)\varphi_p(\Delta^m u)$ (resp. $A(x)\varphi_p(\Delta^m v)$) are in $\mathrm{C}^{2m}(G, \mathbb{R}) \cap \mathrm{C}(\bar{G}, \mathbb{R})$.

Also, denote by Φ_p the form defined for $X, Y \in \mathbb{R}$ and $p > 1$ by

$$\Phi_p(X, Y) := X\varphi_p(X) + (p-1)Y\varphi_p(Y) - pX\varphi_p(Y).$$

From the Young inequality it follows that $\Phi_p(X, Y) \geq 0$ for all $X, Y \in \mathbb{R}$ and the equality holds if and only if $X = Y$.

The proof of the basic identity contained in the following lemma involves only routine differentiation and its verification is left to the reader. We call it a *weaker form of Picone's identity* because of the relative weak hypothesis that u is an arbitrary $2m$-times continuously differentiable function which does not need to satisfy any differential equation or inequality.

Lemma 2.1. *If* $u \in \mathrm{C}^{2m}(\bar{G}, \mathbb{R}), v \in \mathcal{D}_L(G)$ *and* $\Delta^k v, k = 0, \dots, m-1$, *do not vanish in* G, *then*

$$\begin{aligned}
&\operatorname{div}\Big\{ \sum_{k=0}^{m-1} \Big[-\frac{|\Delta^{m-k-1}u|^p}{\varphi_p(\Delta^{m-k-1}v)} \nabla\big(\Delta^k\big(A\varphi_p(\Delta^m v)\big)\big) + \nabla\Big(\frac{|\Delta^{m-k-1}u|^p}{\varphi_p(\Delta^{m-k-1}v)}\Big)\Delta^k\big(A\varphi_p(\Delta^m v)\big)\Big]\Big\} = \\
&= -\frac{|u|^p}{\varphi_p(v)} L[v] + C|u|^p + A|\Delta^m u|^p - A\Phi_p\big(\Delta^m u, \frac{\Delta^{m-1}u}{\Delta^{m-1}v}\Delta^m v\big) - \\
&\quad - \sum_{k=1}^{m-1} \frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k}v)} \Phi_p\big(\Delta^{m-k}u, \frac{\Delta^{m-k-1}u}{\Delta^{m-k-1}v}\Delta^{m-k}v\big) + \\
&\quad + p(p-1)\sum_{k=0}^{m-1} \frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k-1}v)} \big|\Delta^{m-k-1}u\big|^{p-2} \big|\nabla(\Delta^{m-k-1}u) - \frac{\Delta^{m-k-1}u}{\Delta^{m-k-1}v}\nabla(\Delta^{m-k-1}v)\big|^2.
\end{aligned} \tag{2.1}$$

Formally integrating (2.1) over G and applying the divergence theorem yields the generalized Picone identity in the integral form

$$\begin{aligned}
&-\int_{\partial G} \sum_{k=0}^{m-1} \frac{|\Delta^{m-k-1}u|^p}{\varphi_p(\Delta^{m-k-1}v)} \frac{\partial}{\partial \nu}\big(\Delta^k(A\varphi_p(\Delta^m v))\big) ds + \\
&+ (p-1)\int_{\partial G} \sum_{k=0}^{m-1} \frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k-1}v)} \Big[\frac{\varphi_p(\Delta^{m-k-1}u)}{\Delta^{m-k-1}v}\Big(\Delta^{m-k-1}v\frac{\partial(\Delta^{m-k-1}u)}{\partial\nu} - \\
&\qquad\qquad - \Delta^{m-k-1}u\frac{\partial(\Delta^{m-k-1}v)}{\partial\nu}\Big)\Big] ds + \\
&+ \int_{\partial G} \sum_{k=0}^{m-1} \frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k-1}v)} \varphi_p(\Delta^{m-k-1}u)\frac{\partial}{\partial\nu}\big(\Delta^{m-k-1}u\big) ds = \\
&= -\int_G \frac{|u|^p}{\varphi_p(v)} L[v] dx + \int_G \big[A|\Delta^m u|^p + C|u|^p\big] dx - \\
&\quad - \int_G \Big[A\Phi_p\big(\Delta^m u, \frac{\Delta^{m-1}u}{\Delta^{m-1}v}\Delta^m v\big) + \\
&\qquad + \sum_{k=1}^{m-1} \frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k}v)} \Phi_p\big(\Delta^{m-k}u, \frac{\Delta^{m-k-1}u}{\Delta^{m-k-1}v}\Delta^{m-k}v\big)\Big] dx + \\
&+ p(p-1)\int_G \sum_{k=0}^{m-1} \frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k-1}v)} \big|\Delta^{m-k-1}u\big|^{p-2} \big|\nabla(\Delta^{m-k-1}u) - \\
&\qquad - \frac{\Delta^{m-k-1}u}{\Delta^{m-k-1}v}\nabla(\Delta^{m-k-1}v)\big|^2 dx,
\end{aligned} \tag{2.2}$$

where $\partial/\partial\nu$ denotes the exterior normal derivative, which extends the formula in [9, Theorem 2.2].

Addition to (2.1) of the obvious identity

$$\operatorname{div}\Big\{\sum_{k=0}^{m-1}\big[\Delta^{m-k-1}\big(a\varphi_p(\Delta^m u)\big)\nabla(\Delta^k u)-\Delta^{m-k-1}u\nabla\big(\Delta^k(a\varphi_p(\Delta^m u))\big)\big]\Big\}=$$

$$=-ul[u]+a|\Delta^m u|^p-c|u|^p,$$

which holds for any $u\in\mathcal{D}_l(G)$, yields the following *stronger form* of the generalized Picone's formula:

Lemma 2.2. *If* $u\in\mathcal{D}_l(G), v\in\mathcal{D}_L(G)$ *and* $\Delta^k v(x)\neq 0$ *in* G *for* $k=0,\ldots,m-1$, *then*

$$\operatorname{div}\Big\{\sum_{k=0}^{m-1}\Big[\frac{|\Delta^{m-k-1}u|^p}{\varphi_p\big(\Delta^{m-k-1}v\big)}\nabla\big(\Delta^k\big(A\varphi_p(\Delta^m v)\big)\big)-$$

$$-\nabla\bigg(\frac{|\Delta^{m-k-1}u|^p}{\varphi_p\big(\Delta^{m-k-1}v\big)}\bigg)\Delta^k\big(A\varphi_p(\Delta^m v)\big)+$$

$$+\Delta^{m-k-1}\big(a\varphi_p(\Delta^m u)\big)\nabla(\Delta^k u)-$$

$$-\Delta^{m-k-1}u\nabla\big(\Delta^k(a\varphi_p(\Delta^m u))\big)\Big]\Big\}=$$

$$=\frac{|u|^p}{\varphi_p(v)}L[v]-ul[u]+(a-A)|\Delta^m u|^p+ \tag{2.3}$$

$$+(c-C)|u|^p+A\Phi_p\Big(\Delta^m u,\frac{\Delta^{m-1}u}{\Delta^{m-1}v}\Delta^m v\Big)+$$

$$+\sum_{k=1}^{m-1}\frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k}v)}\Phi_p\Big(\Delta^{m-k}u,\frac{\Delta^{m-k-1}u}{\Delta^{m-k-1}v}\Delta^{m-k}v\Big)-$$

$$-p(p-1)\sum_{k=0}^{m-1}\frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k-1}v)}\big|\Delta^{m-k-1}u\big|^{p-2}\Big|\nabla(\Delta^{m-k-1}u)-$$

$$-\frac{\Delta^{m-k-1}u}{\Delta^{m-k-1}v}\nabla(\Delta^{m-k-1}v)\Big|^2.$$

The integration of (2.3) with the use of the divergence theorem gives the integral formula of the Picone-type which generalizes the result from Yoshida [9, Theorem 2.1]:

$$\begin{aligned}
&\int_{\partial G} \sum_{k=0}^{m-1} \frac{\Delta^{m-k-1}u}{\varphi_p(\Delta^{m-k-1}v)} \Big[\varphi_p(\Delta^{m-k-1}v) \frac{\partial\big(\Delta^k(a\varphi_p(\Delta^m u))\big)}{\partial\nu} - \\
&\qquad - \varphi_p(\Delta^{m-k-1}u) \frac{\partial\big(\Delta^k(A\varphi_p(\Delta^m v))\big)}{\partial\nu} \Big] ds + \\
&+ (p-1) \int_{\partial G} \sum_{k=0}^{m-1} \frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k-1}v)} \Big[\frac{\varphi_p(\Delta^{m-k-1}u)}{\Delta^{m-k-1}v} \Big(\Delta^{m-k-1}v \frac{\partial(\Delta^{m-k-1}u)}{\partial\nu} - \\
&\qquad - \Delta^{m-k-1}u \frac{\partial(\Delta^{m-k-1}v)}{\partial\nu} \Big) \Big] ds + \\
&+ \int_{\partial G} \sum_{k=0}^{m-1} \frac{1}{\varphi_p(\Delta^{m-k-1}v)} \frac{\partial(\Delta^{m-k-1}u)}{\partial\nu} \Big[\Delta^k\big(A\varphi_p(\Delta^m v)\big)\varphi_p(\Delta^{m-k-1}u) \\
&\qquad - \Delta^{m-k-1}\big(a\varphi_p(\Delta^m u)\big)\varphi_p(\Delta^{m-k-1}v) \Big] ds = \\
&= \int_G \frac{u}{\varphi_p(v)} \big\{\varphi_p(v) l[u] - \varphi_p(u) L[v]\big\} dx + \int_G \big[(A-a)|\Delta^m u|^p + (C-c)|u|^p\big] dx - \\
&\quad - \int_G \Big[A\Phi_p\big(\Delta^m u, \frac{\Delta^{m-1}u}{\Delta^{m-1}v}\Delta^m v\big) + \\
&\qquad + \sum_{k=1}^{m-1} \frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k}v)} \Phi_p\big(\Delta^{m-k}u, \frac{\Delta^{m-k-1}u}{\Delta^{m-k-1}v}\Delta^{m-k}v\big) \Big] dx + \\
&\quad + p(p-1) \int_G \sum_{k=0}^{m-1} \frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k-1}v)} \big|\Delta^{m-k-1}u\big|^{p-2} \big|\nabla(\Delta^{m-k-1}u) - \\
&\qquad - \frac{\Delta^{m-k-1}u}{\Delta^{m-k-1}v} \nabla(\Delta^{m-k-1}v)\big|^2 dx.
\end{aligned} \tag{2.4}$$

3. APPLICATIONS

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index with nonnegative integer components and the norm $|\alpha| = \alpha_1 + \dots + \alpha_n$. Define the differential operator D^α by $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$.

As a first application of identity (2.2) we prove the following non-existence result.

Theorem 3.1. *If*

$$M_p[u] \equiv \int_G \big[A(x)|\Delta^m u|^p + C(x)|u|^p\big] dx \le 0, \tag{3.1}$$

for some nontrivial function $u \in \mathrm{C}^{2m}(\bar{G}, \mathbb{R})$ *such that*

$$D^\alpha u = 0 \quad \text{on } \partial G, \quad |\alpha| \le 2m-2, \tag{3.2}$$

then there does not exist a $v \in \mathcal{D}_L(G)$ *satisfying*

$$L[v] \geq 0 \quad in\, G, \tag{3.3}$$

$$v > 0 \quad on\ \partial G, \tag{3.4}$$

$$(-1)^k \Delta^k v > 0 \quad in\ \bar{G},\ k = 1, \dots, m-1, \tag{3.5}$$

$$(-1)^{m+k} \Delta^k \big(A(x)\varphi_p(\Delta^m v)\big) \geq 0 \quad in\ G, \quad k = 0, \dots, m-2, \tag{3.6}$$

$$\Delta^{m-1}\big(A(x)\varphi_p(\Delta^m v)\big) < 0 \quad in\ G. \tag{3.7}$$

Remark 3.2. When $m = 1$, then conditions (3.5) and (3.6) are redundant.

Proof. Suppose, by contradiction, that there exists a $v \in \mathcal{D}_L(G)$ which satisfies (3.3)–(3.7). Since $v > 0$ on ∂G and $\Delta v < 0$ in G, the maximum principle implies that $v > 0$ on $\bar{G}$. Thus, integral identity (2.2) is valid and it implies, in view of the hypotheses (3.2)-(3.7), that

$$
\begin{aligned}
0 &\geq M_p[u] - \int_G \frac{|u|^p}{\varphi_p(v)} L[v]dx = \\
&= \int_G \Bigg[A\Phi_p\big(\Delta^m u, \frac{\Delta^{m-1}u}{\Delta^{m-1}v}\Delta^m v\big) + \\
&\qquad + \sum_{k=1}^{m-1} \frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k}v)} \Phi_p\big(\Delta^{m-k}u, \frac{\Delta^{m-k-1}u}{\Delta^{m-k-1}v}\Delta^{m-k}v\big)\Bigg]dx - \\
&\quad - p(p-1)\int_G \sum_{k=0}^{m-1} \frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k-1}v)} \big|\Delta^{m-k-1}u\big|^{p-2}\big|\nabla(\Delta^{m-k-1}u) - \\
&\qquad - \frac{\Delta^{m-k-1}u}{\Delta^{m-k-1}v}\nabla(\Delta^{m-k-1}v)\big|^2 dx \geq \\
&\geq -p(p-1)\int_G \sum_{k=0}^{m-1} \frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k-1}v)} \big|\Delta^{m-k-1}u\big|^{p-2}\big|\nabla(\Delta^{m-k-1}u) - \\
&\qquad - \frac{\Delta^{m-k-1}u}{\Delta^{m-k-1}v}\nabla(\Delta^{m-k-1}v)\big|^2 dx \geq \\
&\geq -p(p-1)\int_G \frac{\Delta^{m-1}\big(A\varphi_p(\Delta^m v)\big)}{v^{p-1}} |u|^{p-2}\big|\nabla u - \frac{u}{v}\nabla v\big|^2 dx \ \geq\ 0.
\end{aligned}
$$

Consequently, $\nabla u - (u/v)\nabla v = 0$ in G, that is, $u/v = k$ in G, and hence on $\bar{G}$ by continuity, for some nonzero constant k. However, this cannot happen since $u = 0$ on ∂G whereas $v > 0$ on ∂G. This contradiction shows that no v satisfying (3.3)–(3.7) can exist. □

Theorem 3.3. *If there exists a nontrivial* $u \in \mathrm{C}^{2m}(\bar{G}, \mathbb{R})$ *which satisfies* (3.2) *and* (3.1), *then every solution* $v \in \mathcal{D}_L(G)$ *of the inequality* (3.3) *satisfying* (3.5)–(3.7) *and*

$$v(\hat{x}) > 0 \quad for\ some\ \hat{x} \in G \tag{3.8}$$

has a zero in $\bar{G}$.

Proof. If the solution of (3.3) satisfies (3.5)–(3.8), then either $v(x^1) < 0$ for some $x^1 \in \partial G$, and so v must vanish somewhere in G, or $v(x) \geq 0$ for all $x \in \partial G$. In the latter case, however, from Theorem 3.1 it follows that $v(x^0) = 0$ for some $x^0 \in \partial G$, and the proof is complete. □

From Theorem 3.3 we easily obtain the following integral inequality of the Wirtinger type.

Corollary 3.4. *If there exists a $v \in \mathcal{D}_L(G)$ such that $L[v] = 0$, $v > 0$ in G and* (3.5)–(3.7) *are satisfied, then for any nontrivial function $u \in \mathrm{C}^{2m}(\bar{G}, \mathbb{R})$ satisfying*

$$D^\alpha u = 0 \quad \text{on } \partial G, \quad |\alpha| \leq 2m - 1,$$

we have

$$\int\limits_G \big[A(x)|\Delta^m u|^p + C(x)|u|^p\big]\, dx \geq 0. \tag{3.9}$$

The next result which readily follows from integral identity (2.4) derived in Section 2 is the following general comparison theorem. It belongs to weak comparison results in the sense that the existence of a zero of at least one of the functions $v, \Delta v, \ldots, \Delta^{m-1} v$ is established in $\bar{G}$ rather than in G.

Theorem 3.5. *If there exists a nontrivial $u \in \mathcal{D}_l(G)$ such that*

$$\int\limits_G u l[u] dx \leq 0, \tag{3.10}$$

$$D^\alpha u = 0 \quad \text{on } \partial G, \quad |\alpha| \leq 2m - 1, \tag{3.11}$$

and

$$V_p[u] \equiv \int\limits_G \big[\big(a(x) - A(x)\big)|\Delta^m u|^p + \big(c(x) - C(x)\big)|u|^p\big] dx \geq 0, \tag{3.12}$$

then for any $v \in \mathcal{D}_L(G)$ which satisfies (3.3),

$$(-1)^j \Delta^j v(x^j) > 0 \quad \text{at some points } x^j \in G,\ 0 \leq j \leq m-1, \tag{3.13}$$

$$(-1)^{m+j} \Delta^j \big(\big(A(x)\varphi_p(\Delta^m v)\big)\big) \geq 0 \quad \text{in } G,\ j = 0, \ldots, m-2, \tag{3.14}$$

and

$$\Delta^{m-1}\big(A(x)\varphi_p(\Delta^m)\big)\big) < 0 \quad \text{in } G, \tag{3.15}$$

at least one of $v, \Delta v, \ldots, \Delta^{m-1} v$ must vanish somewhere in $\bar{G}$.

Proof. Suppose that $\Delta^j v(x) \neq 0$ in $\bar{G}$ for all $j = 0, \ldots, m-1$. Then, condition (3.13) implies that $(-1)^j \Delta^j v(x) > 0$ for all $x \in \bar{G}$ and $j = 0, \ldots, m-1$, and from integral Picone identity (2.4) we obtain, in view of (3.3), (3.10)–(3.15), that

$$\begin{aligned}
0 = V_p[u] &+ \int_G \frac{|u|^p}{v^{p-1}} L[v] dx - \int_G u l[u] dx + \\
&+ \int_G \Big[A\Phi_p\big(\Delta^m u, \frac{\Delta^{m-1} u}{\Delta^{m-1} v} \Delta^m v\big) + \\
&\quad + \sum_{k=1}^{m-1} \frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k} v)} \Phi_p\big(\Delta^{m-k} u, \frac{\Delta^{m-k-1} u}{\Delta^{m-k-1} v} \Delta^{m-k} v\big) \Big] dx - \\
&- p(p-1) \int_G \sum_{k=0}^{m-1} \frac{\Delta^k\big(A\varphi_p(\Delta^m v)\big)}{\varphi_p(\Delta^{m-k-1} v)} \big|\Delta^{m-k-1} u\big|^{p-2} \big|\nabla(\Delta^{m-k-1} u) - \\
&\qquad - \frac{\Delta^{m-k-1} u}{\Delta^{m-k-1} v} \nabla(\Delta^{m-k-1} v)\big|^2 dx \geq \\
\geq -p(p-1) &\int_G \frac{\Delta^{m-1}\big(A\varphi_p(\Delta^m v)\big)}{v^{p-1}} \big|u\big|^{p-2} \big|\nabla u - \frac{u}{v} \nabla v\big|^2 dx \ \geq \ 0.
\end{aligned}$$

It follows that $\nabla u - (u/v)\nabla v = 0$ in G and therefore $u/v = k$ in $\bar{G}$ for some nonzero constant k. Since $u = 0$ on ∂G and $v > 0$ on ∂G, we have a contradiction. Thus, at least one of $v, \Delta v, \ldots, \Delta^{m-1} v$ must have a zero in $\bar{G}$. □

Corollary 3.6 (Pointwise comparison)**.** *Suppose that*

$$a(x) \geq A(x) \geq 0, \quad c(x) \geq C(x) \quad \textit{in } G.$$

If $u \in \mathcal{D}_l(G)$ *is a nontrivial solution of* $l[u] = 0$ *in* G *such that*

$$D^\alpha u = 0 \quad \textit{on } \partial G, \ |\alpha| \leq 2m-1,$$

then for any solution v *of* $L[v] = 0$ *which satisfies* (3.13)–(3.15) *at least one of* $v, \Delta v, \ldots, \Delta^{m-1} v$ *must have a zero in* $\bar{G}$.

As a final application of integral identity (2.4) we establish a lower bound for the first eigenvalue of the nonlinear eigenvalue problem

$$l[u] = \lambda |u|^{p-2} u \quad \text{in } G, \tag{3.16}$$

$$\Delta^j u = \Delta^{m-j-1}\big(a\varphi_p(\Delta^m u)\big) + \sigma_j \frac{\partial}{\partial \nu}(\Delta^j u) = 0 \quad \text{on } \partial G, \ j = 0, \ldots, m-1, \tag{3.17}$$

where $0 \leq \sigma_j \leq +\infty$ (the case $\sigma_j = +\infty$ corresponds to the condition $\partial(\Delta^j u)/\partial \nu = 0$).

A special case of (3.16)–(3.17) with $m = 1, \sigma_0 = 0$ and $a \equiv 1$ is the Navier eigenvalue problem

$$\Delta\big(|\Delta u|^{p-2}\Delta u\big) = \lambda |u|^{p-2}u \quad \text{in } G, \tag{3.18}$$

$$u = \Delta u = 0 \quad \text{on } \partial G, \tag{3.19}$$

investigated by Drábek and Ôtani [2]. They proved that for any $p > 1$ the problem (3.18)–(3.19) considered on a bounded $G \in \mathbb{R}^n$ with a smooth boundary ∂G, has a principal eigenvalue λ_1 which is simple and isolated and that there exists strictly positive eigenfunction u_1 in G associated with λ_1 and satisfying $\partial u_1/\partial\nu < 0$ on ∂G.

As far as we know, no similar results concerning a more general boundary value problem (3.16)–(3.17) with $m > 1$ are available in the literature. (For some results concerning nonlinear equations involving the p-polyharmonic operator of the form $\Delta^m\big(|\Delta^m u|^{p-2}\Delta^m u\big)$, $m \geq 1$, we refer the reader to [7].) However, if we assume that a solution (λ_1, u_1) to (3.16)–(3.17) does exist, then λ_1 can be estimated from below as follows.

Theorem 3.7. *Let λ_1 be the eigenvalue of problem* (3.16)–(3.17) *and $u_1 \in \mathcal{D}_l(G)$ be the corresponding eigenfunction. If there exists a $v \in \mathcal{D}_L(G)$ such that*

$$(-1)^j \Delta^j v > 0 \quad \textit{in } \bar{G},\ j = 0, \ldots, m-1, \tag{3.20}$$

$$(-1)^{m+j}\Delta^j\big(A(x)\varphi_p(\Delta^m v)\big) \geq 0 \quad \textit{in } G,\ j = 0, \ldots, m-1, \tag{3.21}$$

and if $V_p[u_1] \geq 0$, then

$$\lambda_1 \geq \inf_{x \in G}\left[\frac{L[v]}{v^{p-1}}\right]. \tag{3.22}$$

Proof. From identity (2.4), in view of the above hypotheses, we obtain

$$\lambda_1 \int\limits_G |u_1|^p dx - \int\limits_G |u_1|^p \frac{L[v]}{v^{p-1}} dx \geq 0, \tag{3.23}$$

from which the conclusion of the theorem readily follows. □

Remark 3.8. Our approach based on identity (2.4) allows to replace the boundary conditions in Theorem 3.7 by the conditions

$$D^\alpha u = 0 \quad \text{on } \partial G, \quad |\alpha| \leq 2m-1.$$

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