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VULNERABILITY PARAMETERS OF TENSOR PRODUCT OF COMPLETE EQUIPARTITE GRAPHS

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Abstract. Let G_1 and G_2 be two simple graphs. The tensor product of G_1 and G_2, denoted by $G_1 \times G_2$, has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$. In this paper, we determine vulnerability parameters such as toughness, scattering number, integrity and tenacity of the tensor product of the graphs $K_{r(s)} \times K_{m(n)}$ for $r \geq 3, m \geq 3, s \geq 1$ and $n \geq 1$, where $K_{r(s)}$ denotes the complete r-partite graph in which each part has s vertices. Using the results obtained here the theorems proved in [Aygul Mamut, Elkin Vumar, *Vertex Vulnerability Parameters of Kronecker Products of Complete Graphs*, Information Processing Letters **106** (2008), 258–262] are obtained as corollaries.

Keywords: fault tolerance, tensor product, vulnerability parameters.

Mathematics Subject Classification: 05C76, 05C40.

1. INTRODUCTION

In this paper, all graphs considered are finite, undirected and simple. First we present the definitions of some vulnerability parameters of a graph such as toughness, scattering number, integrity, tenacity. Let $\omega(G)$ and $\tau(G)$ denote the number of components and the order of the largest component of G, respectively. Let $\kappa(G)$ denote the connectivity of G. A *separating set* or *vertex cut* of G is a set $S \subset V(G)$ such that $G - S$ has more than one component. The *toughness* of a graph G is defined by

$$t(G) = \min \left\{ \frac{|S|}{\omega(G-S)} : S \subset V(G) \text{ is a vertex cut of } G \right\}.$$

The *scattering number* of G is defined by

$$s(G) = \max\{\omega(G-S) - |S| : S \subset V(G) \text{ is a vertex cut of } G\}.$$

The *integrity* of a graph G is defined by

$$I(G) = \min\{|S| + \tau(G - S) : S \subset V(G) \text{ is a vertex cut of } G\}.$$

The *tenacity* of a graph G is given by

$$T(G) = \min\left\{\frac{|S| + \tau(G - S)}{\omega(G - S)} : S \subset V(G) \text{ is a vertex cut of } G\right\}.$$

The definitions and notation which are not defined here may be seen in [6] and [7].

In this paper, we determine vulnerability parameters, namely, connectivity, toughness, scattering number, integrity and tenacity of the tensor product of complete equipartite graphs $K_{r(s)} \times K_{m(n)}$ for $r \geq 3, m \geq 3, s \geq 1$ and $n \geq 1$, where $K_{r(s)}$ denotes the complete r-partite graph in which each part has s vertices.

A communication network is composed of processors and communication links. Network designers attach importance to reliability and stability of a network. If the network begins losing processors or communication links, then there is a loss in its effectiveness. This event is called the *vulnerability* of the communication network. In a communication network, vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. We may use graphs to model networks, as graph theoretical parameters can be used to describe the stability and reliability of communication networks. To measure the vulnerability of a graph we have parameters such as connectivity, toughness, scattering number, binding number, integrity and tenacity. The vulnerability parameters have been defined and studied in [1–5] and [9]. For a detailed account of these vulnerability parameters see [2, 4, 5, 8] and the references therein.

In [2] the vulnerability parameters have been studied for $K_r \times K_m$. Here we study the vulnerability parameters of $K_{r(s)} \times K_{m(n)}$, which coincides with $K_r \times K_m$ when $s = 1$ and $n = 1$.

Let G_1 and G_2 be two simple graphs. The tensor product of G_1 and G_2, denoted by $G_1 \times G_2$, has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set

$$E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\},$$

see Figure 1.

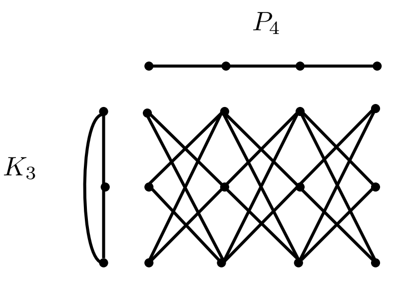

Fig. 1. $K_3 \times P_4$

First we introduce some notation. Let $G = K_{r(s)} \times K_{m(n)}$. Let $\{U_1, U_2, \ldots, U_r\}$ be the partite sets of $K_{r(s)}$ and let $\{V_1, V_2, \ldots, V_m\}$ be the partite sets of $K_{m(n)}$. Let $Z_{ij} = U_i \times V_j$. Let $S_i = U_i \times V(K_{m(n)})$; clearly S_i is an independent set in G and $S_i = \bigcup_{j=1}^{m} (U_i \times V_j) = \bigcup_{j=1}^{m} Z_{ij}$. Similarly, let $S'_j = V(K_{r(s)}) \times V_j$; it is clear that S'_j is an independent set in G and $S'_j = \bigcup_{i=1}^{r} (U_i \times V_j) = \bigcup_{i=1}^{r} Z_{ij}$. Clearly, the independence number of G is

$$\alpha(G) = \max_{1 \le i \le r;\ 1 \le j \le m} \left\{ |S_i|, \left|S'_j\right| \right\} = \max \{snr, snm\}.$$

2. MAIN RESULTS

Lemma 2.1. *Let r, s, m, n be integers with $r \ge 3, m \ge 3, s \ge 1$ and $n \ge 1$. If S is a minimal vertex cut of $G = K_{r(s)} \times K_{m(n)}$, then $G - S$ has $sn + 1$ components in which sn components are trivial.*

Proof. Let S be a minimal vertex cut of G. Assume that $G - S$ contains an isolated vertex x. As G is vertex transitive, we may assume that an isolated vertex say, x, of $G - S$ is in Z_{11}. Then S must contain $\bigcup_{2 \le i \le r;\ 2 \le j \le m} Z_{ij}$, since these are the vertices adjacent to x. If $S = \bigcup_{2 \le i \le r;\ 2 \le j \le m} Z_{ij}$, then $G - S$ contains all the vertices of Z_{11} as isolated vertices in $G - S$ and one component containing the vertices of $(S_1 \cup S'_1) - Z_{11}$. Thus we have $sn + 1$ components in $G - S$ in which sn components are isolated vertices. If $\bigcup_{2 \le i \le r;\ 2 \le j \le m} Z_{ij}$ is a proper subset of S, then S will not be a minimal vertex cut of G. Further, there is no minimal vertex cut S of G for which each component of $G - S$ has at least two vertices. Hence the result follows. □

Lemma 2.2. *Let r, s, m, n be integers with $r \ge 3, m \ge 3, s \ge 1$ and $n \ge 1$. If S is any vertex cut of $G = K_{r(s)} \times K_{m(n)}$, then $|S| \ge sn(r-1)(m-1)$ and for $T \subset V(G)$, if $G - T$ consists only of isolated vertices, then $|T| \ge \min \{snm(r-1), snr(m-1)\}$.*

Proof. If S is an arbitrary vertex cut of G, then S must contain $V(G) - (S_i \cup S'_j)$ for some i, j, where $1 \le i \le r, 1 \le j \le m$. If $S = V(G) - (S_i \cup S'_j)$, then $G - S$ consists of $sn + 1$ components in which there are sn isolated vertices, namely, the vertices in $S_i \cap S'_j$. Thus $|S| \ge sn(r-1)(m-1)$. However, for $T \subset V(G)$, if $G - T$ consists only of isolated vertices, then $V(G) - S_i \subseteq T$ for some i, or $V(G) - S'_j \subseteq T$ for some j and hence $|T| \ge \min \{snm(r-1), snr(m-1)\}$. □

The following lemma is trivial from the above lemmas.

Lemma 2.3. *Let r, s, m, n be integers with $r \ge 3, m \ge 3, s \ge 1$ and $n \ge 1$. Let S be a vertex cut of $G = K_{r(s)} \times K_{m(n)}$.*

(i) *If $\omega(G - S) = sn + 1$ and $G_1, G_2, \ldots, G_{sn+1}$ are the components of $G - S$, then* $\min_i |V(G_i)| = 1$.
(ii) *If $\omega(G - S) \ge sn + 2$, then every component of $G - S$ is an isolated vertex and* $|S| \ge \min \{snm(r-1), snr(m-1)\}$.

Theorem 2.4. *For $r \ge 3, m \ge 3, s \ge 1$ and $n \ge 1$, the connectivity of $K_{r(s)} \times K_{m(n)}$ is equal to $sn(r-1)(m-1)$.*

Proof. Let $G = K_{r(s)} \times K_{m(n)}$. Since the degree of each vertex in G is $sn(r-1)(m-1)$, we have

$$\kappa(G) \leq sn(r-1)(m-1). \tag{2.1}$$

Further, by the nature of the graph, any minimal vertex cut S of G must contain all the vertices of $V(G) - (S_i \cup S'_j)$ for some i and j. Thus S has cardinality at least $sn(r-1)(m-1)$. Therefore,

$$\kappa(G) \geq sn(r-1)(m-1). \tag{2.2}$$

From (2.1) and (2.2), $\kappa(G) = sn(r-1)(m-1)$. □

Theorem 2.5. *For $r \geq 3, m \geq 3, s \geq 1$ and $n \geq 1$, the toughness of $K_{r(s)} \times K_{m(n)}$ is equal to* $\min\{(r-1),(m-1)\}$.

Proof. Let $G = K_{r(s)} \times K_{m(n)}$. Recall that the toughness of a graph G, $t(G)$, is defined as $t(G) = \min\{\frac{|S|}{\omega(G-S)} : S \subset V$ is a vertex cut of $G\}$. Let $\min\{r,m\} = r$. Let S be a vertex cut of G; then $|S| \geq sn(r-1)(m-1)$, by Theorem 2.4. If $\omega(G-S) = sn+1$, then

$$\frac{|S|}{\omega(G-S)} \geq \frac{sn(r-1)(m-1)}{sn+1}. \tag{2.3}$$

If $\omega(G-S) \geq sn+2$, then every component of $G-S$ is an isolated vertex and $|S| \geq \min\{snm(r-1), snr(m-1)\}$, by Lemma 2.3, and hence $\omega(G-S) \leq snm$, the independence number of G; consequently,

$$|S| = snrm - \omega(G-S) \geq snrm - snm = snm(r-1).$$

Therefore,

$$\frac{|S|}{\omega(G-S)} \geq \frac{snm(r-1)}{snm} = r-1. \tag{2.4}$$

Comparing inequalities (2.3) and (2.4), we see that $\frac{sn(r-1)(m-1)}{sn+1} \geq r-1$. Therefore,

$$t(G) \geq r-1. \tag{2.5}$$

On the other hand, if $T = \bigcup_{2\leq i\leq r;\ 1\leq j\leq m} Z_{ij}$, then T is a vertex cut of G with $|T| = snm(r-1)$ and $\omega(G-T) = snm$, hence

$$\frac{|T|}{\omega(G-T)} = \frac{snm(r-1)}{snm} = r-1. \tag{2.6}$$

Therefore,

$$t(G) \leq r-1. \tag{2.7}$$

From (2.5) and (2.7),

$$t(G) = r - 1. \tag{2.8}$$

If $\min\{r, m\} = m$, then we would have $t(G) = m - 1$ and hence $t(G) = \min\{r-1, m-1\}$. □

Theorem 2.6. *For $r \geq 3$, $m \geq 3$, $s \geq 1$ and $n \geq 1$, the scattering number of $K_{r(s)} \times K_{m(n)}$ is*

$$s(K_{r(s)} \times K_{m(n)}) = \begin{cases} (1+sn) - sn(m-1)^2 & \text{if } r = m, \\ \max\{snm(2-r), snr(2-m)\} & \text{if } r \neq m. \end{cases}$$

Proof. Let $G = K_{r(s)} \times K_{m(n)}$. Recall that the scattering number, $s(G)$, of a graph G is defined as $s(G) = \max\{\omega(G-S) - |S| : S \subset V \text{ is a vertex cut of } G\}$.
Case 1. $r = m$.
The set $S = \bigcup_{2 \leq i \leq r;\ 2 \leq j \leq m} Z_{ij}$ is a vertex cut of G with $|S| = sn(r-1)(m-1)$ and $\omega(G-S) = sn+1$; as $r = m$,

$$\omega(G-S) - |S| = sn + 1 - sn(r-1)(m-1) = sn + 1 - sn(m-1)^2$$

and hence

$$s(G) \geq sn + 1 - sn(m-1)^2. \tag{2.9}$$

Let S' be an arbitrary vertex cut of G. If $\omega(G - S') = sn + 1$, then $|S'| \geq sn(r-1)(m-1)$. Therefore,

$$\omega(G-S') - |S'| \leq (sn+1) - sn(r-1)(m-1) = \tag{2.10}$$

$$= sn(2m - m^2) + 1, \text{ as } r = m. \tag{2.11}$$

If $\omega(G - S') \geq sn + 2$, then every component of $G - S'$ is an isolated vertex, by Lemma 2.3, and hence $|S'| = snrm - \omega(G - S')$. Therefore,

$$\begin{aligned} \omega(G-S') - |S'| &= snrm - 2\,|S'| \leq \\ &\leq snrm - 2(snrm - snm), \text{ by Lemma 2.3 with } r = m, \qquad (2.12) \\ &= sn(2m - m^2), \text{ as } r = m\ . \end{aligned}$$

As $r = m$, from (2.11) and (2.12),

$$\omega(G-S') - |S'| \leq \begin{cases} sn(2m - m^2) + 1 & \text{if } \omega(G-S') = sn+1, \\ sn(2m-m^2) & \text{if } \omega(G-S') \geq sn+2. \end{cases}$$

But $sn(2m - m^2) + 1 > sn(2m - m^2)$. Consequently, an upper bound for $s(G)$ is attained corresponding to S' in (2.11). Therefore, by (2.10),

$$s(G) \leq (sn+1) - sn(m-1)^2, \text{ as } r = m. \tag{2.13}$$

From (2.9) and (2.13), if $r = m$, then

$$s(G) = sn + 1 - sn(m-1)^2. \tag{2.14}$$

Case 2. $r \neq m$.
Let $\min\{r, m\} = r$. Then $r \leq m - 1$.
Let $T = \bigcup_{2\leq i\leq r;\ 1\leq j\leq m} Z_{ij}$; T is a vertex cut of G with $|T| = snm(r-1)$ and $\omega(G-T) = snm$, the independence number of G. Therefore,

$$\begin{aligned} s(G) &\geq \omega(G-T) - |T| = \\ &= snm - snm(r-1) = \\ &= snm(2-r). \end{aligned} \tag{2.15}$$

Let S be a vertex cut of G; then $|S| \geq sn(r-1)(m-1)$, by the Lemma 2.2. If $\omega(G-S) = sn+1$, then

$$\omega(G-S) - |S| \leq (sn+1) - sn(r-1)(m-1) = sn(r+m-rm) + 1. \tag{2.16}$$

If $\omega(G-S) \geq sn+2$, then every component of $G-S$ is an isolated vertex, by Lemma 2.3, and $|S| \geq snrm - snm$, as $r = \min\{r, m\}$, so that

$$\omega(G-S) - |S| \leq snm(2-r), \quad \text{since } r < m. \tag{2.17}$$

But

$$\begin{aligned} sn(r+m-rm)+1 &\leq sn(m-1+m-rm)+1, \ \text{since } r \leq m-1, \\ &= sn(2m-1-rm)+1 = \\ &= sn(2m-rm) - (sn-1) \leq sn(2m-rm) = snm(2-r). \end{aligned}$$

Therefore, $\omega(G-S) - |S|$ attains its maximum only when every component of $G-S$ is an isolated vertex. Hence

$$s(G) \leq snm(2-r). \tag{2.18}$$

From (2.15) and (2.18),

$$s(G) = snm(2-r), \ \text{if } r < m. \tag{2.19}$$

If $m \leq r-1$, then we would get $s(G) = snr(2-m)$ and hence we have,

$$s(G) = \max\{snm(2-r), snr(2-m)\}, \ \text{if } r \neq m. \tag{2.20}$$

From (2.14) and (2.20),

$$s(G) = \begin{cases} (1+sn) - sn(m-1)^2 & \text{if } r = m, \\ \max\{snm(2-r), snr(2-m)\} & \text{if } r \neq m. \end{cases} \qquad \square$$

Theorem 2.7. *For $r \geq 3, m \geq 3, s \geq 1$ and $n \geq 1$, the integrity of $K_{r(s)} \times K_{m(n)}$ is equal to* $\min\{snm(r-1)+1, snr(m-1)+1\}$.

Proof. Let $G = K_{r(s)} \times K_{m(n)}$.Recall that the integrity $I(G)$, of a graph G is defined as $I(G) = \min\{|S| + \tau(G-S) : S \subset V$ is a vertex cut of $G\}$. Let $\max\{r, m\} = m$. Let S be a vertex cut of G.

First we suppose that $\tau(G-S) = 1$. Then, each component of $G-S$ is an isolated vertex and $\omega(G-S) \leq snm$, the independence number of G; hence

$$|S| = snrm - \omega(G-S) \geq snrm - snm = snm(r-1).$$

Consequently,

$$|S| + \tau(G-S) \geq snm(r-1) + 1. \tag{2.21}$$

Next we suppose that $\tau(G-S) > 1$. Then $\omega(G-S) = sn+1$. Let $G_1, G_2, \ldots, G_{sn+1}$ be the components of $G-S$. Then $\min_i |V(G_i)| = 1$, by Lemma 2.3, and $\tau(G-S) = snrm - |S| - sn$, since there are sn isolated vertices in $G-S$. Therefore,

$$|S| + \tau(G-S) = snrm - sn. \tag{2.22}$$

But

$$snrm - sn \geq snrm - snm + 1 = snm(r-1) + 1.$$

Therefore, from (2.21) and (2.22), $|S| + \tau(G-S)$ attains its minimum only when $G-S$ has only isolated vertices. Hence

$$I(G) \geq snm(r-1) + 1. \tag{2.23}$$

Now, $T = \bigcup_{2 \leq i \leq r; 1 \leq j \leq m} Z_{ij}$ is a vertex cut of G with $|T| = snm(r-1)$ and $\tau(G-T) = 1$. Therefore,

$$I(G) \leq |T| + \tau(G-T) = snm(r-1) + 1. \tag{2.24}$$

Combining (2.23) and (2.24),

$$I(G) = snm(r-1) + 1. \tag{2.25}$$

If $\max\{r, m\} = r$, then we would have $I(G) = snr(m-1)+1$ and hence $I(G) = \min\{snm(r-1)+1, snr(m-1)+1\}$. □

Theorem 2.8. *For $r \geq 3, m \geq 3, s \geq 1$ and $n \geq 1$, the tenacity of $K_{r(s)} \times K_{m(n)}$ is equal to* $\min\{r - 1 + \frac{1}{snm}, m - 1 + \frac{1}{snr}\}$.

Proof. Let $G = K_{r(s)} \times K_{m(n)}$. Recall that the tenacity $T(G)$, of a graph G is defined as $T(G) = \min\{\frac{|S| + \tau(G-S)}{\omega(G-S)} : S \subset V$ is a vertex cut of $G\}$. Let $\max\{r, m\} = m$. Let S be a vertex cut of G.

First we suppose that $\tau(G-S) = 1$. Then, each component of $G-S$ is an isolated vertex and hence $\omega(G-S) \leq snm$, the independence number of G; hence

$$|S| = snrm - \omega(G-S) \geq snrm - snm = snm(r-1).$$

Therefore,

$$\frac{|S|+\tau(G-S)}{\omega(G-S)} \geq \frac{snm(r-1)+1}{snm}. \tag{2.26}$$

Next we suppose that $\tau(G-S)>1$. Then $\omega(G-S)=sn+1$. Let $G_1, G_2, \ldots, G_{sn+1}$ be the components of $G-S$. Then $\min_i |V(G_i)|=1$, by Lemma 2.3, and $\tau(G-S)=snrm-|S|-sn$, since there are sn isolated vertices in $G-S$. Therefore,

$$\frac{|S|+\tau(G-S)}{\omega(G-S)} = \frac{sn(rm-1)}{sn+1}. \tag{2.27}$$

Comparing (2.26) and (2.27), we have

$$\frac{sn(rm-1)}{sn+1} \geq \frac{snm(r-1)+1}{snm} = r-1+\frac{1}{snm}$$

and hence

$$T(G) \geq r-1+\frac{1}{snm}. \tag{2.28}$$

Now let $T=\bigcup_{2\leq i\leq r;\ 1\leq j\leq m} Z_{ij}$. Then T is a vertex cut of G with $|T|=snm(r-1)$ and $\tau(G-T)=1$. Therefore,

$$T(G) \leq \frac{|T|+\tau(G-T)}{\omega(G-T)} \leq \frac{snm(r-1)+1}{snm} = r-1+\frac{1}{snm}. \tag{2.29}$$

Combining (2.28) and (2.29),

$$T(G) = r-1+\frac{1}{snm}. \tag{2.30}$$

If $\max\{r,m\}=r$, then we would have $T(G)=m-1+\frac{1}{snr}$ and hence

$$T(G)=\min\left\{r-1+\frac{1}{snm}, m-1+\frac{1}{snr}\right\}. \qquad \square$$

Applying the same proof technique as in above theorems, we have the following theorem and we omit its proof, as it resembles the proofs of Theorems 2.4 to 2.8 above.

Theorem 2.9. *Let m, n, a, b be integers. Let $K_{a,b}$ denote the complete bipartite graph in which the partite sets have size a and b. Let $m \geq 3$ and $a \geq b$. Then:*

1. $\kappa(K_{m(n)} \times K_{a,b}) = (m-1)nb$,
2. $t(K_{m(n)} \times K_{a,b}) = \frac{b}{a}$,
3. $s(K_{m(n)} \times K_{a,b}) = mn(a-b)$,
4. $I(K_{m(n)} \times K_{a,b}) = mnb+1$,
5. $T(K_{m(n)} \times K_{a,b}) = \frac{b}{a}+\frac{1}{mna}$.

Corollary 2.10 ([2]). *Let m, n be integers with $n \geq m \geq 2$ and $n \geq 3$. Then:*

1. $\kappa(K_m \times K_n) = (m-1)(n-1)$,
2. $t(K_m \times K_n) = m - 1$,
3. $s(K_m \times K_n) = \begin{cases} 2-(m-1)(n-1) & \text{if } m = n, \\ n(2-m) & \text{otherwise}, \end{cases}$
4. $I(K_m \times K_n) = mn - n + 1$,
5. $T(K_m \times K_n) = m + \frac{1}{n} - 1$.

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