Opuscula Math. 34, no. 1 (2014), 139–160
http://dx.doi.org/10.7494/OpMath.2014.34.1.139 Opuscula Mathematica

# ASYMPTOTICS OF THE DISCRETE SPECTRUM FOR COMPLEX JACOBI MATRICES

Maria Malejki

*Communicated by Alexander Gomilko*

**Abstract.** The spectral properties and the asymptotic behaviour of the discrete spectrum for a special class of infinite tridiagonal matrices are given. We derive the asymptotic formulae for eigenvalues of unbounded complex Jacobi matrices acting in $l^2(\mathbb{N})$.

**Keywords:** tridiagonal matrix, complex Jacobi matrix, discrete spectrum, eigenvalue, asymptotic formula, unbounded operator, Riesz projection.

**Mathematics Subject Classification:** 47B36, 47B37, 47B06, 47A75, 15A18.

## 1. INTRODUCTION

Spectral properties of non-symmetric tridiagonal matrices and complex Jacobi matrices are investigated by several authors: Beckermann and Kaliaguine ([1, 2]), Djakov and Mityagin ([7, 8]), Egorova and Golinskii ([10, 11]) and others (see, e.g., [3,12–14,21,23]). The connections of tridiagonal matrices with formal orthogonal polynomials on the complex plane, Mathieu equation and functions, and Bessel functions can be found in [1–3, 7, 13, 21] and [25]. However, systematic research concerning spectral properties of non-selfadjoint tridiagonal operators is difficult because the structure of complex sequences can be more complicated then the structure of real sequences. Moreover, the spectral theorem and its consequences fail in this case. Nevertheless, some properties of real Jacobi matrices can be carried over to the complex tridiagonal matrices. We observe that effective research methods for non-selfadjoint operators use the Riesz projections instead of the spectral theorem (see [8, 16]).

The asymptotic behaviour of eigenvalues for selfadjoint Jacobi matrices was investigated with the use of several methods, which could be found for instance in [4, 5, 9, 15–17] and [24]. In this article we show that the asymptotic formulae for the point spectrum of unbounded discrete operators given by special classes of tridiagonal complex matrices are also true. We generalize the results obtained for selfadjoint Jacobi matrices in [18] and [19].

Consider a complex tridiagonal infinite matrix

$$J((d_n),(a_n),(b_n)) = \begin{pmatrix} d_1 & a_1 & 0 & \cdots & \cdots \\ b_1 & d_2 & a_2 & 0 & \ddots \\ 0 & b_2 & d_3 & a_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.1}$$

where $d_n$, $a_n$, $b_n \in \mathbb{C} \setminus \{0\}$. The matrix $J((d_n),(a_n),(b_n))$ defines a linear operator which acts on a maximal domain

$$Dom(J) = \{(f_n)_{n=1}^{\infty} \in l^2 : \ (b_{n-1}f_{n-1} + d_n f_n + a_n f_{n+1})_{n=1}^{\infty} \in l^2\}$$

and

$$(Jf)_n = b_{n-1}f_{n-1} + d_n f_n + a_n f_{n+1}, \quad n \geq 1,$$

for $f = (f_n)_{n=1}^{\infty} \in Dom(J)$ and $b_0 = 0$. In the second section we establish primary properties of tridiagonal operators.

Then the paper is organized as follows. In the third section we consider the symmetrization procedure of tridiagonal matrices (1.1) to complex Jacobi matrices

$$J_s = J((d_n),(c_n),(c_n)), \tag{1.2}$$

where

$$c_n^2 = a_n b_n, \ n \geq 1. \tag{1.3}$$

In [2] Beckermann and Kaliaguine proved that the resolvent set of the operator $J_s$ contains the resolvent set of the tridiagonal operator $J$, for which (1.3) holds. We prove that, for some classes of tridiagonal matrices, the symmetrized complex Jacobi matrices preserve the discrete spectrum.

The fourth section is devoted to a generalized version of the result, which was proved by Janas and Naboko in [16]. If discrete operators are near-similar in the sense of Rozenbljum ([20]), then we expect that their point spectra are asymptotically close. This result concerns asymptotic behaviour of the point spectrum for a compact perturbation of a diagonal discrete operator and it is essential for diagonalization methods.

In the last section we discuss that the method of diagonalization ([18]) can be easily applied for complex Jacobi matrices to obtain the asymptotic formulae for the point spectrum of some tridiagonal matrices.

## 2. PRELIMINARIES

Assume that the complex sequences $(d_n)_{n=1}^{\infty}$, $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$ satisfy the following conditions:

(A1) $|d_n| \to \infty$ as $n \to \infty$;

(A2)

$$\bigcup_{n=1}^{\infty} B(d_n, r_n) \neq \mathbb{C},$$

where $r_n = |a_n| + |b_n| + |a_{n-1}| + |b_{n-1}|$ and $B(d_n, r_n) = \{\lambda \in \mathbb{C} \colon |d_n - \lambda| \le r_n\}$ for $n \ge 1$, $a_0 = b_0 = 0$;

(A3)

$$\frac{1}{|d_n|} \sum_{k=-1}^{1} (|a_{n+k}| + |b_{n+k}|) \to 0 \text{ as } n \to \infty.$$

Let $D = \mathrm{Diag}((d_n))$ be an operator in $l^2$ given by a diagonal matrix

$$\begin{pmatrix} d_1 & 0 & \ddots & & \\ 0 & d_2 & 0 & \ddots & \\ 0 & 0 & d_3 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.1}$$

**Proposition 2.1.** *If* $J = J((d_n), (a_n), (b_n))$ *and* (A1)–(A3) *hold, then:*

(i) *Dom*$(J)$ = *Dom*$(D) = \{(f_n)_{n=1}^{\infty} \in l^2 : \ (d_n f_n)_{n=1}^{\infty} \in l^2\}$;

(ii) *$J$ is a densely defined and closed operator in $l^2$;*

(iii) *the spectrum of $J$ is discrete and*

$$\sigma(J) = \{\lambda_n(J) \in \mathbb{C} : \ n \ge 1\},$$

*where $\lambda_n(J)$ is an eigenvalue of $J$ $(n \ge 1)$ and the sequence $(\lambda_n(J))_{n=1}^{\infty}$ can be ordered such that*

$$|\lambda_n(J)| \to \infty, \ n \to \infty;$$

(iv)

$$\sigma(J) \subset \bigcup_{n=1}^{\infty} B(d_n, \rho_n),$$

*where $\rho_n = |a_n| + |b_{n-1}|$ and $B(d_n, \rho_n) = \{z \in \mathbb{C} : |z - d_n| \le \rho_n\}$;*

(v) *$J^*$ is given by the matrix $J((\overline{d_n}), (\overline{b_n}), (\overline{a_n}))$.*

*Proof.* First we prove that all eigenvalues of $J$ are contained in $\bigcup_{n=1}^{\infty} B(d_n, \rho_n)$. Indeed, let $\lambda \in \mathbb{C}$ be an eigenvalue for $J$ and $Jf = \lambda f$, where $f = (f_n) \in l^2 \setminus \{0\}$. There exists $k \in \mathbb{N}$ such that $|f_k| = \max\{|f_n| : \ n \ge 1\}$ and

$$b_{k-1} f_{k-1} + (d_k - \lambda) f_k + a_k f_{k+1} = 0.$$

Then

$$|d_k - \lambda| \leq |a_k||f_{k+1}|/|f_k| + |b_{k-1}||f_{k-1}|/|f_k| \leq |a_k| + |b_{k-1}|$$

and $\lambda \in B(d_k, \rho_k)$.

Conditions (A1)–(A3) entail

$$\mathbb{C} \neq \bigcup_{n=1}^{\infty} B(d_n, \rho_n).$$

Therefore, there exists

$$z \in \mathbb{C} \setminus \bigcup_{n=1}^{\infty} B(d_n, \rho_n) \tag{2.2}$$

and $(D - z)^{-1}$ is a compact operator.

From definition of $Dom(J)$ we observe that $Dom(D) \subset Dom(J)$. For $f \in Dom(D)$

$$(J - z)f = \left(I + (J - D)(D - z)^{-1}\right)(D - z)f. \tag{2.3}$$

Moreover, $(J - D)(D - z)^{-1}$ is also compact under (A1) and (A3).

If $I + (J - D)(D - z)^{-1}$ is not invertible then there exists an eigenvector $g \neq 0$ for this operator. Put $g' = (D - z)^{-1}g$. From (2.3) we derive

$$(J - z)g' = (I + (J - D)(D - z)^{-1})g = 0$$

but this relation is impossible because of (2.2).

Thus $I + (J - D)(D - z)^{-1}$ is an invertible operator on $l^2$ and from (2.3) we derive

$$(J - z)^{-1} = (D - z)^{-1}(I + (J - D)(D - z)^{-1})^{-1}. \tag{2.4}$$

Equation (2.4) implies that $(J - z)^{-1}$ is compact and $J$ is closed. Moreover, from (2.3) we deduce that $Dom(J) = Dom(D)$. So (i) and (ii) are proved.

Because $(J - z)^{-1}$ is compact, then we know the structure of its spectrum

$$\sigma((J - z)^{-1}) = \{0\} \cup \{z_n :, n \geq 1\},$$

where $z_n$, $n = 1, 2, \ldots$, is an eigenvalue of $(J - z)^{-1}$ and $z_n \to 0$ as $n \to \infty$. Also

$$\sigma(J) = \left\{\frac{1}{z_n} + z : \ n \geq 1\right\}$$

and the sequence of eigenvalues tends to infinity. So (iii) is satisfied. Notice that (iv) is also true because the spectrum of $J$ consists of the eigenvalues only.

To prove (v) denote $J^+ = J((\overline{d_n}), (\overline{b_n}), (\overline{a_n}))$ and notice that $(Jf, g) = (f, J^+g)$ for $f \in Dom(D)$ and $g \in l^2$. Then $g \in Dom(J^*)$ if and only if $J^+g \in {}^2$. Therefore, $Dom(J^*) = Dom(J^+)$ and $J^* = J^+$. $\square$

Similar results on tridiagonal matrices are included in [7, 8] and [23]. The Gershgorin type theorem for infinite matrices acting as operators in $l^\infty$ or $l^1$ can be find in [22].

## 3. COMPLEX JACOBI MATRICES AND SYMMETRIZATION

Let $J = J((d_n),(a_n),(b_n))$ and consider a complex Jacobi matrix

$$J_s = J((d_n),(c_n),(c_n)),$$

where $c_n \in \mathbb{C} \setminus \{0\}$ and $c_n^2 = a_n b_n,\ n \geq 1$. Choose a complex sequence $(\alpha_n)$ such that $\alpha_1 = 1$ and

$$\alpha_n^2 = \frac{a_{n-1}}{b_{n-1}}\alpha_{n-1}^2 = \frac{a_{n-1}a_{n-2}\cdot\ldots\cdot a_1}{b_{n-1}b_{n-2}\cdot\ldots\cdot b_1},\ \ n \geq 2, \tag{3.1}$$

and put

$$A = \operatorname{Diag}((\alpha_n)).$$

Then the formal matrix equation

$$AJ = J_s A. \tag{3.2}$$

is satisfied.

In [2] Beckermann and Kaliaguine proved that if $J$ is bounded then the resolvent set of $J$ is contained in the resolvent set of the symmetrized operator $J_s$.

**Proposition 3.1.** *If* (A1)–(A3) *are satisfied and* $J$, $J_s$ *are operators associated with* (1.1) *and* (1.2), *respectively, acting on the maximal domains, then*

$$\sigma(J) = \sigma(J_s).$$

*Proof.* Due to Proposition 2.1, under conditions (A1)–(A3), the spectra of $J$ and $J_s$ are discrete. Let $\lambda \in \sigma(J)$ and $f = (f_n) \in Dom(J)$ be an eigenvector of $J$ associated with $\lambda$. From (A1) and (A3) we deduce that $|d_n - \lambda| > |a_n|$ and $\frac{|b_{n-1}|}{|d_n-\lambda|-|a_n|} < 1$ for $n \geq n_0$, where $n_0$ is large enough.

Let $n \geq n_0$. The sequence $(f_k) \in l^2$ converges to 0, so there is $k \geq n$ such that $|f_{k+1}| \leq |f_k|$. Then from the spectral equation $Jf = \lambda f$ we derive

$$|b_{k-1}f_{k-1}| = |(d_k - \lambda)f_k + a_k f_{k+1}| \geq |(d_n - \lambda)f_k| - |a_k f_{k+1}| \geq (|d_n - \lambda| - |a_n|)|f_k|,$$

so

$$|f_k| \leq \frac{|b_{k-1}|}{|d_k - \lambda| - |a_k|}|f_{k-1}| \leq |f_{k-1}|.$$

Then by the mathematical induciton reasoning we deduce that

$$|f_j| \leq \frac{|b_{j-1}|}{|d_j - \lambda| - |a_j|}|f_{j-1}| \leq |f_{j-1}|$$

for $j \in \{n_0, \ldots, k\}$. Thus

$$|f_n| \leq \frac{|b_{n-1}|}{|d_n - \lambda| - |a_n|}|f_{n-1}|$$

and

$$|f_n| \le \frac{|b_{n-1}|}{|d_n-\lambda|-|a_n|}\cdot\frac{|b_{n-2}|}{|d_{n-1}-\lambda|-|a_{n-1}|}\cdot\ldots\cdot\frac{|b_{n_0-1}|}{|d_{n_0}-\lambda|-|a_{n_0}|}|f_{n_0-1}|, \tag{3.3}$$

for $n > n_0$.

Equation (3.2) entails also the formal matrix equality

$$J_sAf = AJf = \lambda Af \tag{3.4}$$

for the eigenvector $f$ of $J$; therefore, it is enough to prove that $Af \in l^2$. Notice that by (3.1) and (3.3) we obtain

$$|\alpha_n f_n|^2 \le \frac{|a_{n-1}||b_{n-1}|}{(|d_n-\lambda|-|a_n|)^2}\frac{|a_{n-2}||b_{n-2}|}{(|d_{n-1}-\lambda|-|a_{n-1}|)^2}\cdot\ldots\cdot\frac{|a_{n_0-1}||b_{n_0-1}|}{(|d_{n_0}-\lambda|-|a_{n_0}|)^2}\cdot C_0 =: P_n,$$

$n > n_0$. By (A1) and (A3),

$$P_{n+1}/P_n = \frac{|a_n||b_n|}{(|d_{n+1}-\lambda|-|a_{n+1}|)^2} \to 0,\ n\to\infty,$$

so $\sum_{n=n_0}^{\infty}|\alpha_n f_n|^2 < +\infty$ and $Af \in l^2$. Also from (3.4) we derive that $Af \in Dom(J_s)$, so $\lambda \in \sigma(J_s)$.

Now, assume $\lambda \in \mathbb{C}$ and $\lambda \in \sigma(J_s)$ and $J_sf = \lambda f$, where $f \in Dom(J_s)\setminus\{0\} \subset l^2$. Then the estimates

$$|f_n| \le \frac{|c_{n-1}|}{|d_n-\lambda|-|c_n|}|f_{n-1}| \le$$

$$\le \frac{|c_{n-1}|}{|d_n-\lambda|-|c_n|}\cdot\frac{|c_{n-2}|}{|d_{n-1}-\lambda|-|c_{n-1}|}\cdot\ldots\cdot\frac{|c_{n_0-1}|}{|d_{n_0}-\lambda|-|c_{n_0}|}|f_{n_0-1}|,$$

for $n \ge n_0$, where $n_0$ is large enough, can be obtained by the same method as inequality (3.3). Moreover, from (3.2) we derive the matrix equation

$$JA^{-1}f = A^{-1}J_sf = \lambda A^{-1}f,$$

where $A^{-1} = \mathrm{Diag}((\frac{1}{\alpha_n}))$. Next, the estimates

$$|\frac{1}{\alpha_n}f_n|^2 \le \frac{|b_{n-1}|^2}{(|d_n-\lambda|-|c_n|)^2}\cdot\frac{|b_{n-2}|^2}{(|d_{n-1}-\lambda|-|c_{n-1}|)^2}\cdot\ldots\cdot\frac{|b_{n_0-1}|^2}{(|d_{n_0}-\lambda|-|c_{n_0}|)^2}C_1,$$

satisfied for $n \ge n_0$, yield $A^{-1}f \in l^2$. Finally $\lambda \in \sigma(J)$ and $A^{-1}f \in Dom(J)$ is an eigenvector of $J$ corresponding to the eigenvalue $\lambda$. □

## 4. ASYMPTOTIC BEHAVIOUR OF THE POINT SPECTRUM OF A COMPACTLY PERTURBED DIAGONAL OPERATOR

Let us introduce the following condition.

(CΓ) Let $\Gamma_n$, $n \geq 1$, be a sequence of closed curves on $\mathbb{C}$ and there exists an increasing non-negative sequence $(p_n)$ such that

$$\{z \in \mathbb{C} : |z| \leq p_n\} \subset \mathrm{int}\Gamma_n, \quad n \geq 1,$$

and

$$\lim_{n\to\infty} \frac{|\Gamma_n|}{p_n^2} = 0, \quad \lim_{n\to\infty} p_n = +\infty,$$

where $\mathrm{int}\Gamma_n$ denotes the set surrounded by $\Gamma_n$ and $|\Gamma_n|$ means the length of $\Gamma_n$.

The following theorem generalize the lemma, which concerns asymptotic behaviour of the point spectrum for a compact perturbation of a diagonal discrete operator, given by Janas and Naboko ([16]).

**Theorem 4.1** ([16])**.** *Let $H$ be a separable complex Hilbert space and $\{e_n : n \geq 1\}$ be an orthonormal basis for $H$. Let $D$ be a diagonal operator in $H$ given by a diagonal matrix $Diag((d_n))$ with respect $\{e_n : n \geq 1\}$. Assume that the complex sequence $(d_n)_{n=1}^{\infty}$ satisfies:*

1. $|d_n - d_k| \geq \epsilon_0 > 0$ *for* $d_n \neq d_k$;
2. *there is a sequence of closed Jordan curves $\{\Gamma_n\}_{n\geq 1}$, satisfying* (CΓ)*, such that*

$$\mathrm{dist}(\Gamma_n, d_k) \geq \epsilon_0/4 \quad for \quad k, n \geq 1$$

*and*

$$\int_{\Gamma_n} \|(D-\lambda)^{-1}\|^2 |d\lambda| \leq C, \quad n \geq 1,$$

*where* $\mathrm{dist}(\Gamma_n, d_k) = \inf\{|\lambda - d_k| : \lambda \in \Gamma_n\}$;

*where $\epsilon_0$, $C > 0$ are independent on $n$.*

*Let $K$ be a compact operator in $H$ and $T = D + K$. Then the spectrum of $T$ is discrete and consists of the complex eigenvalues $\lambda_n(T)$, $n \geq 1$, which can be arranged such that*

$$\lambda_n(T) = d_n + O(\|K^* P_n\|) \quad \text{as} \quad n \to \infty,$$

*where $P_n$ is an orthogonal projection on the finite-dimensional space generated by $\{e_k : d_k = d_n\}$.*

The proof of Theorem 4.1 is a consequence of the next two lemmas.

**Lemma 4.2.** *Assume the operators $D$, $K$ and $T$ are operators described in Theorem 4.1. Let*

$$P_n f = \sum_{k:d_k=d_n} (f, e_k) e_k \quad for \quad f \in H, \tag{4.1}$$

$$r_n = 12\|K^* P_n\| \tag{4.2}$$

*and*

$$\gamma_n = \{z \in \mathbb{C} : |z - d_n| = r_n\}.$$

*Let*

$$P_{n,D} = \frac{1}{2\pi i}\int_{\gamma_n} (\lambda - D)^{-1} d\lambda, \quad P_{n,T} := \frac{1}{2\pi i}\int\limits_{\gamma_n} (\lambda - T)^{-1} d\lambda \tag{4.3}$$

*be Riesz projections. Then, under assumption* (1) *of Theorem 4.1, there exists* $n_0$ *such that* $\gamma_n \cap \sigma(T) = \emptyset$ *and*

$$\|P_{n,D} - P_{n,T}\| < 1$$

*for* $n \geq n_0$*.*

*Proof.* The idea of this proof directly comes from the paper [16]. $K^*$ is compact, so

$$K^* = \sum_{k=1}^{\infty} s_k(\cdot, \psi_k)\varphi_k, \tag{4.4}$$

where $\{\psi_k : k = 1, 2, \ldots\}$ and $\{\varphi_k : k = 1, 2, \ldots\}$ are suitable orthogonal bases in $H$ and the sequence of singular numbers $s_k$, $k \geq 1$, is decreasing and tends to 0. Moreover, $\|K^* P_n\| \to 0$ as $n \to \infty$, and

$$r_n = 12\|K^* P_n\| < \epsilon_0/2 \tag{4.5}$$

for $n \geq n_0$, where $n_0$ is large enough.

Let $P_n$ be given by (4.1) and define

$$P_n^{\perp} := I - P_n.$$

For $\lambda \in \gamma_n$ consider the following estimate

$$\|K^*(D^* - \bar{\lambda})^{-1}\| \leq \|K^*(D^* - \bar{\lambda})^{-1} P_n\| + \|K^*(D^* - \bar{\lambda})^{-1} P_n^{\perp}\|. \tag{4.6}$$

Then

$$\|K^*(D^* - \bar{\lambda})^{-1} P_n f\| = \|K^* P_n f\|/|d_n - \lambda| \leq (\|K^* P_n\|/|d_n - \lambda|)\|f\|,$$

so

$$\|K^*(D^* - \bar{\lambda})^{-1} P_n\| \leq \|K^* P_n\|/|d_n - \lambda| \leq 1/12 \tag{4.7}$$

for $\lambda \in \gamma_n$. Obviously,

$$\|(D - \lambda)^{-1}\| = \|(D^* - \bar{\lambda})^{-1}\| = 1/r_n$$

and

$$\|(D^* - \bar{\lambda})^{-1} P_n^{\perp}\| = 1/(\min_{d_k \neq d_n} |d_k - \lambda|) \leq 1/(\min_{d_k \neq d_n} |d_k - d_n| - r_n) \leq 2/\epsilon_0 \tag{4.8}$$

for $\lambda \in \gamma_n$.

For $f \in H$

$$\begin{aligned}\|K^*(D^*-\bar\lambda)^{-1}P_n^\perp f\|^2 &= \|\sum_{k=1}^{\infty} s_k((D^*-\bar\lambda)^{-1}P_n^\perp f,\psi_k)\varphi_k\|^2 = \\ &= \sum_{k=1}^{N} s_k^2|((D^*-\bar\lambda)^{-1}P_n^\perp f,\psi_k)|^2 + \\ &\quad + \sum_{k=N+1}^{\infty} s_k^2|((D^*-\bar\lambda)^{-1}P_n^\perp f,\psi_k)|^2\end{aligned}$$

and

$$\begin{aligned}B_2(\lambda,N,n) &:= \sum_{k=N+1}^{\infty} s_k^2|((D^*-\bar\lambda)^{-1}P_n^\perp f,\psi_k)|^2 \le \\ &\le s_{N+1}^2 \sum_{k=N+1}^{\infty} |((D^*-\bar\lambda)^{-1}P_n^\perp f,\psi_k)|^2 \le \\ &\le s_{N+1}^2\|(D^*-\bar\lambda)^{-1}P_n^\perp f\|^2 \le \\ &\le \frac{4s_{N+1}^2}{\epsilon_0^2}\|f\|^2 \le \frac{1}{32}\|f\|^2\end{aligned}$$

for large enough $N$ and $\lambda \in \gamma_n$, $n \ge n_0$.

Now we are going to prove that

$$B_1(\lambda,N,n) := \sum_{k=1}^{N} s_k^2|((D^*-\bar\lambda)^{-1}P_n^\perp f,\psi_k)|^2 \le \frac{1}{32}\|f\|^2$$

for $n \ge N_0$ and large $N_0$. Let

$$Q_l f = (f,e_l)e_l, \quad l \ge 1.$$

The following estimates are true:

$$\begin{aligned}&|((D^*-\bar\lambda)^{-1}P_n^\perp f,\psi_k)| \le \\ &\quad \le |((D^*-\bar\lambda)^{-1}P_n^\perp f, \sum_{l=1}^{L} P_l\psi_k)| + |((D^*-\bar\lambda)^{-1}P_n^\perp f,(I-\sum_{l=1}^{L} P_l)\psi_k)|\end{aligned}$$

and, by (4.8),

$$|((D^*-\bar\lambda)^{-1}P_n^\perp f,(I-\sum_{l=1}^{L} P_l)\psi_k)| \le 2/\epsilon_0\|f\|\|(I-\sum_{l=1}^{L} P_l)\psi_k\| \le \frac{1}{16s_1\sqrt{N}}\|f\|$$

for $k = 1, 2, \ldots, N$ and large enough $L$.

If $L$ is as large as above, then

$$\begin{aligned}|((D^*-\bar{\lambda})^{-1}P_n^{\perp}f,\sum_{l=1}^{L}P_l\psi_k)| &\leq \sum_{l=1}^{L}|(P_l(D^*-\bar{\lambda})^{-1}P_n^{\perp}f,\psi_k)| \leq \\ &\leq \sum_{l=1}^{L}|(P_l(D^*-\bar{\lambda})^{-1}\|\|f\| \leq \|f\|\sum_{l=1}^{L}\frac{1}{|\overline{d_l}-\bar{\lambda}|} \leq \\ &\leq \|f\|\sum_{l=1}^{L}\frac{1}{||d_n|-|d_l||-|\overline{d_n}-\bar{\lambda}|} \leq \\ &\leq \|f\|\sum_{l=1}^{L}\frac{1}{|d_n|-|d_l|-r_n} \leq \|f\|\frac{1}{16s_1\sqrt{N}}\end{aligned}$$

for $\lambda \in \gamma_n$, $n \geq N_0$ and large enough $N_0$ because $|d_k| \to +\infty$ as $k \to \infty$. Thus

$$|((D^*-\bar{\lambda})^{-1}P_n^{\perp}f,\psi_k)| \leq \frac{1}{8s_1\sqrt{N}}\|f\|$$

and

$$B_1(\lambda,N,n) \leq \sum_{k=1}^{N}s_k^2\left(\frac{1}{8s_1\sqrt{N}}\|f\|\right)^2 \leq \frac{1}{32}\|f\|^2.$$

Then for $n \geq N_0$ and $\lambda \in \gamma_n$

$$\|K^*(D^*-\bar{\lambda})^{-1}P_n^{\perp}f\|^2 \leq B_1(\lambda,N,n)+B_2(\lambda,N,n) \leq \frac{1}{16}\|f\|^2$$

so

$$\|K^*(D^*-\bar{\lambda})^{-1}P_n^{\perp}\| \leq 1/4 \tag{4.9}$$

and, by (4.6),(4.7) and (4.9),

$$\|K^*(D^*-\bar{\lambda})^{-1}\| \leq 1/3$$

for $n \leq N_0$ and $\lambda \in \gamma_n$.

Therefore,

$$\sup_{\lambda\in\gamma_n}\|(D-\lambda)^{-1}K\| = \sup_{\lambda\in\gamma_n}\|K^*(D^*-\bar{\lambda})^{-1}\| \leq 1/3$$

for $n \geq N_0$. Moreover, $T-\lambda = (D-\lambda)^{-1}(I+(D-\lambda)^{-1}K)$ is invertible for $\lambda \in \gamma_n$.

Finally we observe that the Riesz projections (4.3) satisfy the following estimates

$$\begin{aligned}\|P_{n,T}-P_{n,D}\| &= \frac{1}{2\pi}\Big\|\int_{\gamma_n}((\lambda-T)^{-1}-(\lambda-D)^{-1})d\lambda\Big\| = \\ &= \frac{1}{2\pi}\Big\|\int_{\gamma_n}((\lambda-D)^{-1}\left(I-K(\lambda-D)^{-1}\right)^{-1}-(\lambda-D)^{-1})d\lambda\Big\| =\end{aligned}$$

$$= \frac{1}{2\pi}\Big\| \int\limits_{\gamma_n} \sum_{k=1}^{\infty} [(\lambda - D)^{-1}K]^k (\lambda - D)^{-1} d\lambda \Big\| \le$$

$$\le \frac{1}{2\pi} \int\limits_{\gamma_n} \sum_{k=1}^{\infty} \|(\lambda - D)^{-1}K\|^k \|(\lambda - D)^{-1}\| |d\lambda| \le$$

$$\le \frac{1}{2\pi} \int\limits_{\gamma_n} \sum_{k=1}^{\infty} (1/3)^k \|(\lambda - D)^{-1}\| |d\lambda| \le \frac{1}{2\pi} |\gamma_n| \frac{1}{2r_n} = 1/2 < 1.$$

□

**Lemma 4.3.** *Under assumptions of Theorem 4.1*

$$\Gamma_n \cap \sigma(T) = \Gamma_n \cap \sigma(D) = \emptyset$$

*and the Riesz projections*

$$\widetilde{P_{n,T}} = \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - T)^{-1} d\lambda, \quad \widetilde{P_{n,D}} = \frac{1}{2\pi i} \int\limits_{\Gamma_n} (\lambda - D)^{-1} d\lambda \tag{4.10}$$

*satisfy*

$$\|\widetilde{P_{n,T}} - \widetilde{P_{n,D}}\| < 1$$

*for enough large n.*

*Proof.* For $\lambda \in \Gamma_n$ and $f \in H$ we can write that

$$\|K^*(D^* - \bar{\lambda})^{-1} f\|^2 = \sum_{k=1}^{N} s_k^2 |((D^* - \bar{\lambda})^{-1} f, \psi_k)|^2 + \sum_{k=N+1}^{\infty} s_k^2 |((D^* - \bar{\lambda})^{-1} f, \psi_k)|^2.$$

Notice that $|d_l - \lambda| \ge \epsilon_0/4$, $l \ge 1$, so $\|(D^* - \bar{\lambda})^{-1}\| \le 4/\epsilon_0$ because $\lambda \in \Gamma_n$. Then

$$\sum_{k=N+1}^{\infty} s_k^2 |((D^* - \bar{\lambda})^{-1} f, \psi_k)|^2 \le s_{N+1}^2 \|(D^* - \bar{\lambda})^{-1} f\|^2 \le$$

$$\le s_{N+1}^2 (4/\epsilon_0)^2 \|f\|^2 \le \frac{1}{18}\|f\|^2,$$

if $N$ is large enough because $s_N \to 0$ as $N \to \infty$.

Let $g \in H$. Then

$$(D^* - \bar{\lambda})^{-1} g = (D^* - \bar{\lambda})^{-1} \left( \sum_{l=1}^{\infty} (g, e_l) e_l \right) = \sum_{l=1}^{\infty} \frac{(g, e_l)}{\overline{d_l} - \bar{\lambda}} e_l, \tag{4.11}$$

$$\|(D^* - \bar{\lambda})^{-1} g\|^2 = \sum_{l=1}^{\infty} \frac{|(g, e_l)|^2}{|\overline{d_l} - \bar{\lambda}|^2} \le (4/\epsilon_0)^2 \|g\|^2. \tag{4.12}$$

For a fixed large $N$ we have

$$\sum_{k=1}^{N} s_k^2 |((D^* - \bar{\lambda})^{-1} f, \psi_k)|^2 = \sum_{k=1}^{N} s_k^2 |(f, (D-\lambda)^{-1}\psi_k)|^2 \le N s_1^2 \|f\| \max_{1\le k\le N} \|(D-\lambda)^{-1}\psi_k\|^2$$

and

$$\sup_{\lambda\in\Gamma_n} \|(D-\lambda)^{-1}\psi_k\|^2 \le \sum_{l=1}^{\infty} \left[ |(\psi_k, e_l)|^2 \left( \sup_{\lambda\in\Gamma_n} \frac{1}{|d_l - \lambda|^2} \right) \right].$$

It is clear that

$$\sup_{\lambda\in\Gamma_n} \left( \frac{1}{|d_l - \lambda|^2} \right) \to 0,\ n \to \infty, \tag{4.13}$$

for all $l \ge 1$, so

$$\Sigma(n,k) := \sum_{l=1}^{\infty} \left[ |(\psi_k, e_l)|^2 \left( \sup_{\lambda\in\Gamma_n} \frac{1}{|d_l - \lambda|^2} \right) \right] \le$$

$$\le \sum_{l=1}^{\infty} |(\psi_k, e_l)|^2 (4/\epsilon_0)^2 \le (4/\epsilon_0)^2 \|\psi_k\|^2 < +\infty$$

because of (4.11) and (4.12) applied to $\psi_k$. Moreover, by the dominated Lebesgue theorem, (4.13) and the above estimates,

$$\Sigma(n,k) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore,

$$\sup_{\lambda\in\Gamma_n} \sum_{k=1}^{N} s_k^2 |((D^* - \bar{\lambda})^{-1} f, \psi_k)|^2 \le N s_1^2 \|f\|^2 \epsilon_n,$$

where

$$\epsilon_n = \max_{1\le k\le N} \left( \sup_{\lambda\in\Gamma_n} \|(D-\lambda)^{-1}\psi_k\|^2 \right) \to 0,\ n \to \infty.$$

Let $n_0$ be such that $\epsilon_n \le \frac{1}{18 s_1^2 N}$ for $n \ge n_0$. Then

$$\|K^*(D^* - \bar{\lambda})^{-1} f\|^2 \le \frac{1}{9} \|f\|^2,$$

for $f \in H$ and $\lambda \in \Gamma_n$.

Finally,

$$\sup_{\lambda\in\Gamma_n} \|(D-\lambda)^{-1} K\| \le 1/3$$

for $n \ge n_0$.

Then $I+(D-\lambda)^{-1}K$ $(\lambda\in\Gamma_n)$ is invertible in $H$ and the equation $T-\lambda=(D-\lambda)(I+(D-\lambda)^{-1}K)$ yields $\sigma(T)\cup\Gamma_n=\emptyset$ for large $n$. Moreover,

$$\begin{aligned}
&\|\widetilde{P_{n,T}}-\widetilde{P_{n,D}}\|\le\frac{1}{2\pi}\int\limits_{\Gamma_n}\|(\lambda-T)^{-1}-(\lambda-D)^{-1}\||d\lambda|=\\
&=\frac{1}{2\pi}\int\limits_{\Gamma_n}\|(\lambda-T)^{-1}K(\lambda-D)^{-1}\||d\lambda|=\\
&=\frac{1}{2\pi}\int\limits_{\Gamma_n}\|(I+\sum_{k=1}^{\infty}[(\lambda-D)^{-1}K]^k)(\lambda-D)^{-1}K(\lambda-D)^{-1}\||d\lambda|\le\\
&\le\frac{1}{2\pi}\int\limits_{\Gamma_n}\left(1+\sum_{k=1}^{\infty}\|(\lambda-D)^{-1}K\|^k\right)\|(\lambda-D)^{-1}K(\lambda-D)^{-1}\||d\lambda|\le\\
&\le\frac{1}{2\pi}\int\limits_{\Gamma_n}\left(\sum_{k=0}^{\infty}(1/3)^k\right)\|(\lambda-D)^{-1}K(\lambda-D)^{-1}\||d\lambda|\le\\
&\le\frac{3}{4\pi}\int\limits_{\Gamma_n}\|(\lambda-D)^{-1}K(\lambda-D)^{-1}\||d\lambda|.
\end{aligned}$$

From (4.4) we also derive

$$K=\sum_{k=1}^{\infty}s_k(\cdot,\varphi_k)\psi_k=K_1+\sum_{k=1}^{M}s_k(\cdot,\varphi_k)\psi_k,$$

where $\|K_1\|\le 1/C$. Then

$$\begin{aligned}
&\|\widetilde{P_T}-\widetilde{P_D}\|\le\\
&\le\frac{3}{4\pi}\left(\int\limits_{\Gamma_n}\|K_1\|\|(D-\lambda)^{-1}\|^2|d\lambda|+\right.\\
&\quad\left.+\sum_{k=1}^{M}\int\limits_{\Gamma_n}s_k\|(\cdot,(D^*-\overline{\lambda})^{-1}\varphi_k)(D-\lambda)^{-1}\psi_k\||d\lambda|\right)\le\\
&\le\frac{3}{4\pi}\|K_1\|C+\frac{3}{4\pi}\sum_{k=1}^{M}\int\limits_{\Gamma_n}s_k\|(D^*-\overline{\lambda})^{-1}\varphi_k\|\|(D-\lambda)^{-1}\psi_k\||d\lambda|\le\\
&\le\frac{3}{4\pi}+\\
&\quad+\frac{3s_1M}{4\pi}\max_{1\le k\le M}\left[\left(\int\limits_{\Gamma_n}\|(D^*-\overline{\lambda})^{-1}\varphi_k\|^2|d\lambda|\right)^{1/2}\left(\int\limits_{\Gamma_n}\|(D-\lambda)^{-1}\psi_k\|^2|d\lambda|\right)^{1/2}\right].
\end{aligned}$$

For $k \geq 1$ and $g \in H$, due to (CΓ) and assumption (2),

$$\int_{\Gamma_n} \frac{|(g,e_k)|^2}{|d_k-\lambda|^2}|d\lambda| \leq |(g,e_k)|^2 \int_{\Gamma_n} \|(D-\lambda)^{-1}\|^2|d\lambda| \leq C|(g,e_k)|^2 \tag{4.14}$$

and

$$\int_{\Gamma_n} \frac{|(g,e_k)|^2}{|d_k-\lambda|^2}|d\lambda| \leq \|g\|^2 \int_{\Gamma_n} \frac{1}{|d_k-\lambda|^2}|d\lambda| \leq \|g\|^2|\Gamma_n|/(p_n-|d_k|)^2 \to 0 \tag{4.15}$$

as $n \to \infty$.

Because of (4.14), (4.15) and the dominated Lebesgue theorem, we deduce

$$\int_{\Gamma_n} \|(D-\lambda)^{-1}g\|^2|d\lambda| = \int_{\Gamma_n} \|(D^*-\overline{\lambda})^{-1}g\|^2|d\lambda| = \sum_{k=1}^{\infty}\left(\int_{\Gamma_n} \frac{|(g,e_k)|^2}{|d_k-\lambda|^2}\right)|d\lambda| \to 0,$$

as $n \to \infty$ for all $g \in H$. Therefore, if

$$\widetilde{\epsilon_n} = \max_{1\leq k\leq M}\left[\left(\int_{\Gamma_n} \|(D^*-\overline{\lambda})^{-1}\varphi_k\|^2|d\lambda|\right)^{1/2}\left(\int_{\Gamma_n} \|(D-\lambda)^{-1}\psi_k\|^2|d\lambda|\right)^{1/2}\right],$$

then $\widetilde{\epsilon_n} \to 0$ as $n \to \infty$.

Then, finally,

$$\|\widetilde{P_T}-\widetilde{P_D}\| \leq \frac{3}{4\pi}(1+Ms_1\widetilde{\epsilon_0}) < 1$$

for $n \geq \widetilde{n_0}$, where $\widetilde{n_0}$ is large enough. □

*Proof of Theorem 4.1.* Denote rank$P$ = dim$P(H)$, where $P$ is a projection in $H$. Lemma 4.2 implies

$$\mathrm{rank}P_T = \mathrm{rank}P_D = M_n, \tag{4.16}$$

where

$$M_n = \sharp\{k : d_k = d_n\},$$

so the sum of the algebraic multiplicities of eigenvalues of $T$ in $\{z \in \mathbb{C} : |z-d_n| \leq r_n\}$ equals $M_n$ for $n \geq N_0$.

Moreover, from Lemma 4.3 for (4.10) we deduce

$$\mathrm{rank}\widetilde{P_{n,T}} = \mathrm{rank}\widetilde{P_{n,D}} < +\infty \tag{4.17}$$

for $n \geq \widetilde{n_0}$.

Taking into account (4.16) and (4.17) we see that the sequences, in which the algebraic multiplicities of the eigenvalues of $T$ and $D$ are taken into consideration, coincide and

$$|\lambda_n(T)-d_n| \leq r_n$$

for large $n$, where $r_n$ is given by (4.2). □

**Remark 4.4.** Consider a complex sequence $(d_n)_{n=1}^{\infty}$ and the following conditions:

1. $|d_{n+1}| > |d_n|$ for $n \geq n_0$, $\lim_{n\to\infty} |d_n| = +\infty$, and $\left(\frac{|d_{n+1}|+|d_n|}{(|d_{n+1}|-|d_n|)^2}\right)_{n=1}^{\infty}$ is bounded;
2. $d_n = \epsilon_n n^{\alpha}(1 + o(\frac{1}{n}))$, $|\epsilon_n| = \epsilon > 0$ for $n \geq 1$, where $\alpha \geq 2$;
3. $d_n = \epsilon_n \left([\frac{n+T-1}{T}]\right)^{\alpha} (1 + o(\frac{1}{n}))$, $|\epsilon_n| = \epsilon > 0$ for $n \geq 1$, where $\alpha \geq 1$, and $(\epsilon_n)$ is a periodic sequence with the period equal to $T$, $\epsilon_n \neq \epsilon_k$ for $k \neq n$, and $k, n = 1, \ldots, T$, and $[x] = \max\{k \in \mathbb{N} : \ k \leq x\}$ for $x \in \mathbb{R}$.

If one of the above conditions is true then the assumptions of Theorem 4.1 are satisfied for the sequence $(d_n)_{n=1}^{\infty}$.

## 5. DIAGONALIZATION FOR COMPLEX JACOBI MATRICES

In this section we assume (A1)–(A3) and (1.3). Due to Proposition 3.1, we assume without loss of generality that $J = J((d_n), (c_n), (c_n))$ is a complex Jacobi matrix.

Let $1 \leq k \leq l$. Then denote

$$J_l^k = \begin{pmatrix} d_k & c_k & & \\ c_k & d_{k+1} & \ddots & \\ & \ddots & \ddots & c_{l-1} \\ & & c_{l-1} & d_l \end{pmatrix} \tag{5.1}$$

and

$$D_l^k(\lambda) = \det(J_l^k - \lambda). \tag{5.2}$$

Assume also $D_{k-1}^k(\lambda) = 1$.

For $n > q$ denote

$$D_n = \begin{pmatrix} d_{n-q} & 0 & \cdots \\ 0 & \ddots & 0 \\ \cdots & 0 & d_{n+q} \end{pmatrix}. \tag{5.3}$$

Let

$$c'_n(q) = \max\{|c_k| : \ k \leq n+q\}, \quad R_n = 6c'_n(q) \tag{5.4}$$

and let

$$C(d_n, R_n) = \{z \in \mathbb{C} : \ |z - d_n| = R_n\}$$

be positively oriented on the complex plane.

Define the Riesz projection for $J_{n+q}^{n-q}$ and $D_n$:

$$P_{1n} = \frac{1}{2\pi i} \int\limits_{C(d_n,R_n)} (\lambda - J_{n+q}^{n-q})^{-1} d\lambda, \quad P_{2n} = \frac{1}{2\pi i} \int\limits_{C(d_n,R_n)} (\lambda - D_n)^{-1} d\lambda. \tag{5.5}$$

**Lemma 5.1.** *If $|d_n - d_{n+j}| \geq 2R_n$ for $j \in \{\pm 1, \pm 2, \ldots, \pm q\}$, then $\|P_{1n} - P_{2n}\| < 1$ and there exists exactly one eigenvalue $\lambda_n$ of $J_{n+q}^{n-q}$ such that $|d_n - \lambda_n| \leq R_n$.*

*Proof.* At first notice that $\|(\lambda - D_n)^{-1}\| \leq 1/R_n$ and $\|C_n(\lambda - D_n)^{-1}\| \leq 1/3$, where

$$C_n = J_{n+q}^{n-q} - D_n, \tag{5.6}$$

provided that $|\lambda - d_n| = R_n$. Indeed,

$$C_n(\lambda - D_n)^{-1} = \begin{pmatrix} 0 & \frac{c_{n-q}}{\lambda - d_{n-q+1}} & & \\ \frac{c_{n-q}}{\lambda - d_{n-q}} & 0 & \ddots & \\ & \ddots & \ddots & \frac{c_{n+q-1}}{\lambda - d_{n+q}} \\ & & \frac{c_{n+q-1}}{\lambda - d_{n+q-1}} & 0 \end{pmatrix},$$

$$\frac{|c_{n+j}|}{|\lambda - d_{n+j+1}|} \leq \frac{c_n'(q)}{R_n}$$

and

$$\frac{|c_{n+j}|}{|\lambda - d_{n+j}|} \leq \frac{c_n'(q)}{R_n}$$

for $\lambda = d_n + R_n e^{it} \in C(d_n, R_n)$ and $-q \leq j \leq q-1$. Thus $\|C_n(\lambda - D_n)^{-1}\| \leq 2c_n'(q)/R_n \leq 1/3$.

Next, we observe that

$$\begin{aligned}(\lambda - J_{n+q}^{n-q})^{-1} &= (\lambda - D_n)^{-1}\left(I - C_n(\lambda - D_n)^{-1}\right)^{-1} = \\ &= (\lambda - D_n)^{-1}\left(I + \sum_{k=1}^{\infty}[C_n(\lambda - D_n)^{-1}]^k\right)\end{aligned}$$

and

$$(\lambda - J_{n+q}^{n-q})^{-1} - (\lambda - D_n)^{-1} = (\lambda - D_n)^{-1}\left(\sum_{k=1}^{\infty}[C_n(\lambda - D_n)^{-1}]^k\right).$$

So

$$\|(\lambda - J_{n+q}^{n-q})^{-1} - (\lambda - D_n)^{-1}\| \leq \frac{1}{R_n}\left(\sum_{k=1}^{\infty}(1/3)^k\right) = 1/(2R_n)$$

and

$$\|P_{1n} - P_{2n}\| \leq \frac{1}{2\pi}\int\limits_{C(d_n, R_n)} \|(\lambda - J_{n+q}^{n-q})^{-1} - (\lambda - D_n)^{-1}\||d\lambda| \leq 1/2 < 1.$$

Clearly, the Riesz projection $P_{2n}$ has the one-dimensional range generated by an eigenvector associated with the eigenvalue $d_n$. Then $\operatorname{rank} P_{1n} = \operatorname{rank} P_{2n} = 1$, so $J_{n+q}^{n-q}$ has also a unique eigenvalue $\lambda_n$ in the ball $\{z : \ |d_n - z| \leq R_n\}$. □

Assume that the complex sequences $(d_n)$ and $(c_n)$ satisfy the following properties. There exist $\alpha, \beta \geq 0$ and $p \in \{1, 2, \ldots\}$ such that $\alpha > \frac{p+1}{p}\beta + 1$ and

(C1) $|d_n - d_{n+j}| \geq \rho_p n^{\alpha-1}$ for $j = \pm 1, \pm 2, \dots, \pm p$, large $n$, where $\rho_p > 0$ is independent on $n$;

(C2) $|c_n| = O(n^\beta)$ as $n \to \infty$.

Choose the complex sequence $(\lambda_n)_{n=1}^\infty$, which satisfy the following condition:

(E) $\lambda_n$ is an eigenvalue of $J_{n+p-1}^{n-p+1}$ such that $|d_n - \lambda_n| \leq R_n$ for large enough $n$, where

$$R_n = 6 \max\{|c_k| : \ k \leq n+p\}.$$

(For small $n$ we assume that $\lambda_n$ is an eigenvalue of $J_{n+p-1}^1$.)

By Lemma 5.1, the sequence $(\lambda_n)_{n=1}^\infty$ is well defined because (C1) and (C2) imply that $|d_n - d_{n+j}| \geq 2R_n$ for $j = \pm 1, \pm 2, \dots \pm p$ if $n$ is large.

The method of diagonalization for real Jacobi matrices is described in [15, 16] and [18]. Conditions (C1), (C2) and (E) are sufficient to perform the diagonalization procedure presented in [18].

For fixed $n$ denote

$$\varphi^{(n)} = \begin{pmatrix} f_{n-p+1}^{(n)}(\lambda_n) \\ \vdots \\ f_n^{(n)}(\lambda_n) \\ \vdots \\ f_{n+p-1}^{(n)}(\lambda_n) \end{pmatrix} \tag{5.7}$$

an eigenvector of $J_{n+p-1}^{n-p+1}$ associated with $\lambda_n$. Notice that$f_n^{(n)}(\lambda_n) \neq 0$ if $n$ is large enough; therefore, we may assume

$$f_n^{(n)}(\lambda_n) = 1. \tag{5.8}$$

Further, put

$$f_{n-p}^{(n)}(\lambda_n) = \frac{-c_{n-p}}{d_{n-p} - \lambda_n} f_{n-p+1}^{(n)}(\lambda_n), \tag{5.9}$$

$$f_{n+p}^{(n)}(\lambda_n) = \frac{-c_{n+p-1}}{d_{n+p} - \lambda_n} f_{n+p-1}^{(n)}(\lambda_n) \tag{5.10}$$

and

$$f_{n\pm j}^{(n)}(\lambda_n) = 0, \ \text{ for } \ j \geq p+1. \tag{5.11}$$

Finally, for $n \geq 1$, we put

$$f^{(n)} = \left(\overline{f_k^{(n)}(\lambda_n)}\right)_{k=1}^\infty. \tag{5.12}$$

The properties of the system of the sequences $f^{(n)}$, $n \geq 1$, are expressed in the following lemma and proposition.

**Lemma 5.2.** *Assume* (C1) *and* (C2) *hold. Let* $f_k^{(n)}(\lambda_n)$, $n,k \geq 1$, *be given by* (5.7)-(5.11) *and* $(\lambda_n)_{n=1}^{\infty}$ *satisfy* (E). *Then*

$$f_{n\pm j}^{(n)}(\lambda_n) = O\left(\frac{1}{n^{j(\alpha-\beta-1)}}\right), n \to \infty,$$

*for* $j \in \{1,2,\ldots,p\}$.

*Proof.* In [18] the analogous result is proved for real sequences $(d_n)$ and $(c_n)$ but the proof can be rewritten for a complex case. ☐

**Proposition 5.3.** *If* $n \geq 1$ *is large enough, then*

$$\begin{aligned}(J^* - \overline{\lambda_n})f^{(n)} =: R^{(n)} = \overline{c_{n-p-1}}\,\overline{f_{n-p}^{(n)}(\lambda_n)}e_{n-p-1} + \overline{c_{n-p}}\,\overline{f_{n-p}^{(n)}(\lambda_n)}e_{n-p+1} + \\ + \overline{c_{n+p-1}}\,\overline{f_{n+p}^{(n)}(\lambda_n)}e_{n+p-1} + \overline{c_{n+p}}\,\overline{f_{n+p}^{(n)}(\lambda_n)}e_{n+p+1},\end{aligned} \tag{5.13}$$

*where* $\{e_k : k \geq 1\}$ *is a canonical basis of* $l^2$.

*Proof.* Straightforward calculations lead to (5.13) because $J^* = J((\overline{d_n}), (\overline{c_n}), (\overline{c_n}))$. ☐

Let $f^{(n)}$, $n \geq 1$, be given by (5.12). Define

$$F = \left(f^{(1)}; f^{(2)}; \ldots\right), \tag{5.14}$$

i.e. $F$ is an infinite matrix, in which n-th column is given by the sequence $f^{(n)}$. We construct also an infinite matrix

$$R = \left(R^{(1)}; R^{(2)}; R^{(3)}; \ldots\right), \tag{5.15}$$

where $R^{(n)}$ is treated as an n-th column of $R$.

By $S$ we denote the shift operator on $l^2$ given on the basis vectors as follows

$$Se_k = e_{k+1}, \quad k \geq 1.$$

Then $S^*$ stands for the adjoint operator to $S$.

Notice that the structure of $F$ has a band diagonal shape

$$F = I + G,$$

where $G = \sum_{j=1}^{p}(S^j W_j + V_j S^{*j})$ and $W_j$, $V_j$ are diagonal operators

$$W_j = \mathrm{Diag}\left(\left(\overline{f_{n+j}^{(n)}(\lambda_n)}\right)_{n=1}^{\infty}\right), \quad V_j = \mathrm{Diag}\left(\left(\overline{f_n^{(n+j)}(\lambda_{n+j})}\right)_{n=1}^{\infty}\right).$$

Lemma 5.2 yields $G$ which is a compact operator because sequences $\left(\overline{f_{n+j}^{(n)}(\lambda_n)}\right)_{n=1}^{\infty}$ and $\left(\overline{f_n^{(n+j)}(\lambda_{n+j})}\right)_{n=1}^{\infty}$ ($j \in \{1,2,\ldots,p\}$) converge to 0.

Moreover, the matrix $R$ may also be written in a 4-diagonal form

$$R = S^{p-1}A_1 + S^{p+1}A_2 + B_1 S^{*p-1} + B_2^{*p+1},$$

where $A_1, A_2, B_1, B_2$ are diagonal operators

$$A_1 = \operatorname{Diag}\left(\left(\overline{\overline{c_{n+p-1}} f^{(n)}_{n+p}(\lambda_n)}\right)_{n=1}^{\infty}\right), \quad \operatorname{diag} A_2 = \operatorname{Diag}\left(\left(\overline{\overline{c_{n+p}} f^{(n)}_{n+p}(\lambda_n)}\right)_{n=1}^{\infty}\right).$$

$$B_1 = \operatorname{Diag}\left(\left(\overline{\overline{c_{n-1}} f^{(n-1+p)}_{n-1}(\lambda_{n-1+p})}\right)_{n=1}^{\infty}\right), \ (c_0 = 0),$$

$$B_2 = \operatorname{Diag}\left(\left(\overline{\overline{c_n} f^{(n+p+1)}_{n+1}(\lambda_{n+p+1})}\right)_{n=1}^{\infty}\right).$$

Then we observe that $R$ is a compact operator in $l^2$. Indeed, from Lemma 5.2 we derive

$$c_{n\pm p} f^{(n)}_{n\pm p}(\lambda_n) = O\left(\frac{n^{\beta}}{n^{p(\alpha-\beta-1)}}\right) = O\left(n^{-p(\alpha - \frac{p+1}{p}\beta - 1)}\right)$$

and

$$c_{n\pm p-1} f^{(n)}_{n\pm p}(\lambda_n) = O\left(n^{-p(\alpha - \frac{p+1}{p}\beta - 1)}\right), \ n \to \infty,$$

so the sequences above converge to 0, because of the choice of the constant $p$ in (C1) and (C2). Moreover,

$$\|Re_n\| = \|R^{(n)}\| = O\left(n^{-p(\alpha - \frac{p+1}{p}\beta - 1)}\right), \ n \to \infty. \tag{5.16}$$

Let

$$\Lambda = \operatorname{Diag}\left((\lambda_n)_{n=1}^{\infty}\right) \tag{5.17}$$

be a diagonal operator such that the sequence $(\lambda_n)$ satisfies (E).

**Proposition 5.4.** *If* (C1) *and* (C2) *hold then $J$ is similar to $\Lambda + K$, where $K$ is a compact operator such that* $\|K^* e_n\| = O\left(n^{-p(\alpha - \frac{p+1}{p}\beta - 1)}\right)$, $n \to \infty$.

*Proof.* We can rewrite (5.13) with use of an operator form

$$J^* F - F\Lambda^* = R + R', \tag{5.18}$$

where $F$ is given by (5.14), $R$ by (5.15) and $R'$ is a finite dimensional operator, which is represented in the canonical basis by a matrix with a finite number of non-zero entries only. There exists an invertible in $l^2$ operator $\tilde{F}$ such that the matrix representation of $F - \tilde{F}$ has a finite number of non-zero entries. Therefore, (5.18) implies

$$J^* \tilde{F} = \tilde{F}\Lambda^* + R + M, \tag{5.19}$$

where $M = R' + J^*(\tilde{F} - F) - (\tilde{F} - F)\Lambda^*$ is such that the entries of the matrix $M$, except for a finite number, are equal to zero. From (5.19) we derive

$$\tilde{F}^* J = \Lambda \tilde{F}^* + R^* + M^*.$$

$\tilde{F}^*$ is invertible, so

$$J = (\tilde{F}^*)^{-1} \left( \Lambda + (R^* + M^*)(\tilde{F}^*)^{-1} \right) \tilde{F}^*,$$

i.e., $J$ is a similar operator to $\Lambda + K$, where $K = (R^* + M^*)(\tilde{F}^*)^{-1}$. Obviously $K$ is a compact operator because $R^*$ and $M^*$ are compact and $(\tilde{F}^*)^{-1}$ is bounded. Moreover,

$$\|K^* e_n\| = \|\tilde{F}^{-1}(R + M)e_n\| \leq C\|Re_n\|,$$

where $C > 0$ is a constant independent on $n$. Then we apply equation (5.16). □

**Remark 5.5.** The condition of similarity of operators preserves the structure of spectra, so

$$\lambda_n(J) = \lambda_n(\Lambda + K), \;\; n \geq 1.$$

**Remark 5.6.** If $(d_n)$ satisfies one of the conditions given in Remark 4.4 and $\lambda_n = d_n + O(n^\beta)$ according to (E), then Remark 4.4 can be applied also to $(\lambda_n)$. Therefore, from Theorem 4.1, we derive that the sequence of the eigenvalues of $\Lambda + K$ has the asymptotic

$$\lambda_n(\Lambda + K) = \lambda_n + O(\|K^* e_n\|) = \lambda_n + O\left(n^{-p(\alpha - \frac{p+1}{p}\beta - 1)}\right)$$

as $n \to \infty$.

To complete the diagonalization procedure for the complex Jacobi matrices we show that under (C1) and (C2) the explicit asymptotic formulae for the sequence $(\lambda_n)$, chosen by (E), can be found. We refer also to [18]. If $p \geq 2$, from the Laplace formula we derive

$$D_{n+p-1}^{n-p+1}(\lambda) = (d_n - \lambda) D_{n-1}^{n-p+1}(\lambda) D_{n+p-1}^{n+1}(\lambda) +$$
$$-c_{n-1}^2 D_{n-2}^{n-p+1}(\lambda) D_{n+p-1}^{n+1}(\lambda) - c_n^2 D_{n-1}^{n-p+1}(\lambda) D_{n+p-1}^{n+2}(\lambda).$$

Notice that if $n$ is large enough and $|\lambda - d_n| \leq R_n$, where $R_n$ is fixed by (5.4), then the main diagonals of the matrices $J_{n-1}^{n-p+1} - \lambda$ and $J_{n+p-1}^{n+1} - \lambda$ dominate, so $D_{n-1}^{n-p+1}(\lambda) \neq 0$ and $D_{n+p-1}^{n+1}(\lambda) \neq 0$. Therefore, $\lambda_n$ satisfies

$$\lambda_n = d_n - c_{n-1}^2 \frac{D_{n-2}^{n-p+1}(\lambda_n)}{D_{n-1}^{n-p+1}(\lambda_n)} - c_n^2 \frac{D_{n+p-1}^{n+2}(\lambda_n)}{D_{n+p-1}^{n+1}(\lambda_n)}. \tag{5.20}$$

For $n \geq 1$, denote

$$w_{p,n}(\lambda) = d_n - c_{n-1}^2 \frac{D_{n-2}^{n-p+1}(\lambda)}{D_{n-1}^{n-p+1}(\lambda)} - c_n^2 \frac{D_{n+p-1}^{n+2}(\lambda)}{D_{n+p-1}^{n+1}(\lambda)}. \tag{5.21}$$

Next denote

$$\lambda_n^{(1)} = d_n, \quad n \geq 1, \tag{5.22}$$

$$\lambda_n^{(j)} = w_{p,n}(\lambda_n^{(j-1)}), \quad n \geq 1, \quad j \geq 2. \tag{5.23}$$

(For $p = 1$ we simply take $\lambda_n = d_n$ and $w_{1,n}(\lambda) = d_n$.)

**Proposition 5.7.** *Let* $\lambda_n$ *satisfy* (E) *and* $\lambda_n^{(k)}$ *for* $n \geq 1$ *be given by* (5.22) *and* (5.23), *where* $k = \min\{j \in \mathbb{N} : \ j \geq \frac{p+1}{2}\}$. *Under conditions* (C1) *and* (C2),

$$\lambda_n = \lambda_n^{(k)} + O\left(n^{-p(\alpha - \frac{p+1}{p}\beta - 1)}\right)$$

*as* $n \to \infty$.

*Proof.* The proof of the similar result can be found in [18]. □

## Acknowledgments

*The research is supported by the NCN grant, No. 2013/09/B/ST1/04319.*

## REFERENCES

[1] B. Beckermann, *Complex Jacobi matrices*, J. Comput. Appl. Math. **127** (2001), 17–65.

[2] B. Beckermann, V. Kaliaguine, *The diagonal of the Padé table and the approximation of the Weyl function of second-order difference operators*, Constr. Approx. **13** (1997), 481–510.

[3] Yu.M. Berezansky, L.Ya. Ivasiuk, O.A. Mokhonko, *Recursion relation for orthogonal polynomials on the complex plane*, Methods Funct. Anal. Topol. **14** (2008) 2, 108–116.

[4] A. Boutet de Monvel, S. Naboko, L. Silva, *The asympthotic behaviour of eigenvalues of modified Jaynes-Cummings model,* Asymptot. Anal. **47** (2006) 3–4, 291–315.

[5] A. Boutet de Monvel, L. Zielinski, *Explicit error estimates for eigenvalues of some unbounded Jacobi matrices*, Operator Theory: Adv. Appl. **221** (2012), 189–217.

[6] P. Cojuhari, J. Janas, *Discreteness of the spectrum for some unbounded Jacobi matrices*, Acta Sci. Math. (Szeged) **73** (2007), 649–667.

[7] P. Djakov, B. Mityagin, *Simple and double eigenvalues of the Hill operator with a two-term potential*, J. Approx. Theory **135** (2005), 70–104.

[8] P. Djakov, B. Mityagin, *Trace formula and spectral Riemann surface for a class of tri-diagonal matrices*, J. Approx. Theory **139** (2006), 293–326.

[9] J. Edward, *Spectra of Jacobi matrices, differential equations on the circle, and the* $su(1,1)$ *Lie algebra*, SIAM J. Math. Anal. **24** (1993) 3, 824–831.

[10] I. Egorova, L. Golinskii, *Discrete spectrum for complex perturbations of periodic Jacobi matrices*, J. Difference Equ. Appl. **11** (2005) 14, 1185–1203.

[11] I. Egorova, L. Golinskii, *On the location of the discrete spectrum for complex Jacobi matrices*, Proc. Am. Math. Soc. **133** (2005) 12, 3635–3641.

[12] L. Golinskii, M. Kudryavtsev, *On the discrete spectrum of complex banded matrices*, Zh. Mat. Fiz. Anal. Geom. **2** (2006) 4, 396–423.

[13] Y. Ikebe, N. Asai, Y. Miyazaki, D. Cai, *The eigenvalue problem for infinite complex symmetric tridiagonal matrices with application,* Linear Algebra Appl. **241–243** (1996), 599–618.

[14] J. Janas, M. Malejki, Y. Mykytyuk, *Similarity and the point spectrum of some non-selfadjoint Jacobi matrices*, Proc. Edinb. Math. Soc., II. Ser. **46** (2003) 3, 575–595.

[15] J. Janas, M. Malejki, *Alternative approaches to asymptotic behaviour of eigenvalues of some unbounded Jacobi matrices*, J. Comput. Appl. Math. **200** (2007), 342–356.

[16] J. Janas, S. Naboko, *Infinite Jacobi matrices with unbounded entries: asymptotics of eigenvalues and the transformation operator approach,* SIAM J. Math. Anal. **36** (2004) 2, 643–658.

[17] J. Janas, S. Naboko, *Multithreshold spectral phase transitions for a class of Jacobi matrices,* Operator Theory: Adv. Appl. **124** (2001), 267–285.

[18] M. Malejki, *Asymptotics of large eigenvalues for some discrete unbounded Jacobi matrices*, Linear Algebra Appl. **431** (2009), 1952–1970.

[19] M. Malejki, *Approximation and asymptotics of eigenvalues of unbounded self-adjoint Jacobi matrices acting in* $l^2$ *by the use of finite submatrices*, Cent. Eur. J. Math. **8** (2010) 1, 114–128.

[20] G.V. Rozenbljum, *Near-similarity of operators and the spectral asymptotic behaviour of pseudodifferential operators on the circle,* [in Russian] Trudy Maskov. Mat. Obshch. **36** (1978), 59–84.

[21] P.N. Shivakumar, Chuanxiang Li, *Upper and lower bounds for inverse elements of infinite tridiagonal matrices*, Linear Algebra Appl. **247** (1996), 297-316.

[22] P.N. Shivakumar, J.J. Wiliams, N. Rudraiah, *Eigenvalues for infinite matrices*, Linear Algebra Appl. **96** (1987), 35–63.

[23] H. Volkmer, *Approximation of eigenvalues of some differential equations by zeros of orthogonal polynomials*, J. Comput. Appl. Math. **213** (2008), 488–500.

[24] L. Zielinski, *Eigenvalue asymptotics for a class of Jacobi matrices*, Hot topics in operator theory, Theta Ser. Adv. Math., **9**, Theta, Bucharest, 2008, 217–229.

[25] C.H. Ziener, M. Rückl, T. Kampf, W.R. Bauer, H.P. Schlemmer, *Mathieu functions for purely imaginary parameters*, J. Comput. Appl. Math. **236** (2012), 4513–4524.

Maria Malejki
malejki@uci.agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Krakow, Poland

*Received: October 25, 2013.*
*Revised: December 4, 2013.*
*Accepted: December 4, 2013.*