

ASYMPTOTICALLY ISOMETRIC COPIES OF c_0 IN MUSIELAK-ORLICZ SPACES

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Abstract. Criteria in order that a Musielak-Orlicz function space L^Φ as well as Musielak-Orlicz sequence space l^Φ contains an asymptotically isometric copy of c_0 are given. These results extend some results of [Y.A. Cui, H. Hudzik, G. Lewicki, *Order asymptotically isometric copies of c_0 in the subspaces of order continuous elements in Orlicz spaces*, Journal of Convex Analysis **21** (2014)] to Musielak-Orlicz spaces.

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1. INTRODUCTION

Let \mathbb{R} , \mathbb{R}^+ and \mathbb{N} stand for the sets of reals, nonnegative reals and natural numbers, respectively. Let (T, Σ, μ) be an arbitrary σ -finite and complete measure space that does not reduce to a finite number of atoms only. A mapping $\Phi : T \times \mathbb{R} \rightarrow [0, +\infty]$ is said to be a *Musielak-Orlicz function* if:

1. There is a null set $T_0 \in \Sigma$ such that $\Phi(t, \cdot)$ is an Orlicz function for any $t \in T \setminus T_0$, that is, $\Phi(t, \cdot)$ is convex, even, vanishing at zero, left continuous on \mathbb{R}^+ and not identically equal to zero.
2. For any $u \in \mathbb{R}$, the function $\Phi(\cdot, u)$ is Σ -measurable.

Let $L^0 = L^0(T, \Sigma, \mu)$ denote the space of all (equivalence classes of) Σ -measurable real functions defined on T . Given any Musielak-Orlicz function Φ , we define on L^0 a convex modular I_Φ by the formula

$$I_\Phi(x) = \int_T \Phi(t, x(t)) d\mu.$$

The Musielak-Orlicz space L^Φ generated by a Musielak-Orlicz function Φ is defined by the formula

$$L^\Phi = \{x \in L^0 : I_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

We will consider this space under the Luxemburg norm (see [2, 9–12]):

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \leq 1\}$$

Let Ω denote the nonatomic part of T and \mathcal{N} denote the purely atomic part of T . Then the measure space (T, Σ, μ) can be written as the direct sum

$$(\Omega, \Sigma \cap \Omega, \mu/\Omega) \oplus (\mathcal{N}, 2^\mathcal{N}, \mu/2^\mathcal{N}).$$

In this paper we will consider two separate cases: μ nonatomic and μ purely atomic with $\mathcal{N} = \mathbb{N}$.

In a nonatomic case we say that Φ satisfies the growth condition Δ_2 ($\Phi \in \Delta_2$ for short) if there exist a null set $B \in \Sigma \cap \Omega$, a constant $K > 0$ and a nonnegative Σ -measurable function h on Ω such that $\int_\Omega \Phi(t, h(t)) d\mu < \infty$ and $\Phi(t, 2u) \leq K\Phi(t, u)$ for all $t \in \Omega \setminus B$ and $u \geq h(t)$ (see [2] and [11]).

In the purely atomic case we assume that $(T, \Sigma, \mu) = (\mathbb{N}, 2^\mathbb{N}, \text{card})$ and we will write $\Phi_n(u)$, l^Φ and x_n in place of $\Phi(n, u)$, L^Φ and $x(n)$, respectively. Then l^Φ is called the Musielak-Orlicz sequence space.

We say that $\Phi \in \delta_2^0$ if there are $K > 0$, $a > 0$ and a sequence $(c_n)_{n=1}^\infty$ in $[0, +\infty]$ such that $\sum_{n=m}^\infty c_n < \infty$ for some $m \in \mathbb{N}$ and the inequality

$$\Phi_n(2u) \leq K\Phi_n(u) + c_n$$

holds for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $\Phi_n(u) \leq a$ (see [11]).

Recall that if X is a Banach function lattice and $x \in X$, then x is said to be order continuous if $\|x_n\| \rightarrow 0$ for any sequence (x_n) in X such that $0 \leq x_n \leq |x|$ and $x_n \rightarrow 0$ μ -a.e. The subspace of all order continuous elements in X is denoted by X_a . It is possible that $X_a = \{0\}$. This is the case when X is equal to L^∞ or $L^1 \cap L^\infty$ for example. If the measure space (T, Σ, μ) is purely atomic, then $(L^\Phi)_a \neq \{0\}$ for any Musielak-Orlicz function Φ . However, if the measure space (T, Σ, μ) is nonatomic, we have $(L^\Phi)_a \neq \{0\}$ if and only if the set $\{t \in T : \Phi(t, \cdot) \text{ is finitely valued}\}$ has a positive measure, actually $\text{supp}(L^\Phi)_a = \{t \in T : \Phi(t, \cdot) \text{ is finitely valued}\}$ in this case. Consequently, if Φ does not depend on the parameter t and the measure μ is nonatomic, then $(L^\Phi)_a \neq \{0\}$ if and only if Φ is finitely valued (this is of course the case for Orlicz spaces).

A Banach function lattice X is said to be order continuous ($X \in OC$ for short) if $X_a = X$. It is well known that order continuity of a Banach function lattice X as well as of an element $x \in X$ is preserved if we change a norm $\|\cdot\|$ in X into another one $|||\cdot|||$ which is equivalent to $\|\cdot\|$. It is also well known that $(L^\Phi)_a = E^\Phi$, where $E^\Phi = \{x \in L^0 : I_\Phi(\lambda x) < \infty \text{ for any } \lambda > 0\}$, when the measure space is nonatomic and that in the purely atomic case, we have $(l^\Phi)_a = h^\Phi$, where

$$h^\Phi = \left\{ x = (x_n)_{n=1}^\infty : \forall \lambda > 0 \exists n_\lambda \in \mathbb{N} \sum_{n=n_\lambda}^\infty \Phi_n(\lambda x_n) < \infty \right\}.$$

It is also known that h^Φ is the closure (in the norm topology in l^Φ) of the space of all real sequences $x = (x_n)$ with a finite number of coordinates different from zero. Moreover (see [2] and [11]), for a nonatomic measure, we have $L^\Phi = E^\Phi$ if and only if $\Phi \in \Delta_2$ and for the purely atomic measure the equality $l^\Phi = h^\Phi$ holds if and only if $\Phi \in \delta_2^0$.

We say that a Banach space $(X, \|\cdot\|)$ contains asymptotically isometric copy of c_0 if there exists a sequence (ϵ_n) of numbers in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \epsilon_n = 0$ and there exists a linear operator $P : c_0 \rightarrow X$ such that

$$\sup_n (1 - \epsilon_n) |x_n| \leq \|Px\| \leq \sup_n |x_n|$$

for every element $x = (x_n)$ of c_0 .

The notion of asymptotically isometric copy of c_0 was introduced in [6], where it is shown that if a Banach space X contains such a copy, then X fails the fixed-point property for nonexpansive self-mappings on closed bounded convex subsets of X .

2. RESULTS

Theorem 2.1. *h^Φ equipped with the Luxemburg norm contains an asymptotically isometric copy of c_0 if and only if Φ does not satisfy the δ_2^0 condition.*

Proof. Let $\Phi \notin \delta_2^0$ and for $\varepsilon > 0, k \in \mathbb{N}, i \in \mathbb{N}$ define the numbers

$$d_i^k = \sup\{\Phi_i((1 + \frac{1}{k})x) : \Phi_i(x) \leq \frac{1}{2^{k+1}} \text{ and } \Phi_i((1 + \varepsilon)x) \geq 2^{k+1}\Phi_i(x)\}.$$

It is known (see [1, 4, 7, 8]) that

$$\sum_{i=1}^{\infty} d_i^k = \infty \text{ for every } k \in \mathbb{N}.$$

Define i_1 as the largest natural number such that

$$\sum_{i=1}^{i_1} d_i^1 \leq 1,$$

whenever $d_1^1 \leq 1$ and $i_1 = 0$ otherwise. Then

$$\sum_{i=1}^{i_1+1} d_i^1 > 1.$$

Put $N_1 = \{1, 2, \dots, i_1 + 1\}$. Next define i_2 as the largest natural number such that

$$\sum_{i=i_1+2}^{i_2} d_i^2 \leq 1,$$

if $d_{i_1+2}^2 \leq 1$ and $i_2 = i_1 + 2$ otherwise. Then

$$\sum_{i=i_1+2}^{i_2+1} d_i^2 > 1.$$

Put $N_2 = \{i_1 + 2, \dots, i_2 + 1\}$. By induction we can construct the sets

$$N_k = \{i_{k-1} + 2, \dots, i_k, i_k + 1\} \quad (k \in \mathbb{N}, i_0 = -1)$$

such that

$$\sum_{i \in N_k \setminus \{i_k+1\}} d_i^k \leq 1 \quad \text{and} \quad \sum_{i \in N_k} d_i^k > 1.$$

For every $k \in \mathbb{N}$ and $i \in N_k$ there exist such numbers x_i that

$$\sum_{i \in N_k} \Phi_i\left(\left(1 + \frac{1}{k}\right)x_i\right) > 1, \quad \Phi_i(x_i) \leq \frac{1}{2^{k+1}} \quad \text{and} \quad \Phi_i\left(\left(1 + \frac{1}{k}\right)x_i\right) \geq 2^{k+1}\Phi_i(x_i).$$

Hence

$$\begin{aligned} \sum_{i \in N_k} \Phi_i(x_i) &\leq \sum_{i \in N_k \setminus \{i_k+1\}} \frac{1}{2^{k+1}} \Phi_i\left(\left(1 + \frac{1}{k}\right)x_i\right) + \frac{1}{2^{k+1}} \leq \\ &\leq \frac{1}{2^{k+1}} \sum_{i \in N_k \setminus \{i_k+1\}} d_i^k + \frac{1}{2^{k+1}} \leq \frac{1}{2^k}. \end{aligned}$$

Define $y_k = \sum_{i \in N_k} x_i e_i$ for $k \in \mathbb{N}$. Then

$$\begin{aligned} I_\Phi(y_k) &= \sum_{i \in N_k} \Phi_i(x_i) \leq \frac{1}{2^k}, \\ I_\Phi\left(\left(1 + \frac{1}{k}\right)y_k\right) &= \sum_{i \in N_k} \Phi\left(\left(1 + \frac{1}{k}\right)x_i\right) > 1. \end{aligned}$$

for any $k \in \mathbb{N}$. Now define an operator $P : c_0 \rightarrow h^\Phi$ by the formula

$$Pu = \sum_{k=1}^{\infty} u_k y_k \quad \text{for } u = (u_k) \in c_0.$$

We will show that P is well defined, i.e. $Pu \in h^\Phi$ for any $u \in c_0$. Take any $\lambda > 0$ and $l \in \mathbb{N}$ such that $\lambda|u_k| \leq 1$ for every $k \geq l$. Then

$$\begin{aligned} I_\Phi(\lambda \cdot Pu \cdot \chi_{N_l \cup N_{l+1} \cup \dots}) &= I_\Phi\left(\lambda \sum_{k=l}^{\infty} u_k y_k\right) = \sum_{k=l}^{\infty} I_\Phi(\lambda u_k y_k) \leq \\ &\leq \sum_{k=l}^{\infty} I_\Phi(y_k) \leq \sum_{k=l}^{\infty} \frac{1}{2^k} < \infty. \end{aligned}$$

Consequently, $Pu \in h^\Phi$.

Next we will show that $\|Pu\| \leq \|u\|_\infty$. For any nonzero $u \in c_0$ we have

$$\begin{aligned} I_\Phi \left(\frac{Pu}{\|u\|_\infty} \right) &= I_\Phi \left(\frac{1}{\|u\|_\infty} \sum_{k=1}^{\infty} u_k y_k \right) \leq \sum_{k=1}^{\infty} I_\Phi \left(\frac{1}{\|u\|_\infty} u_k y_k \right) \leq \\ &\leq \sum_{k=1}^{\infty} I_\Phi (y_k) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1. \end{aligned}$$

Consequently, $\|Pu\| \leq \|u\|_\infty$.

Finally, we will show that there exists a sequence (ε_n) such that

$$\varepsilon_n \downarrow 0 \text{ and } \sup_n (1 - \varepsilon_n) |u_n| \leq \|Pu\|.$$

Define for every $k \in \mathbb{N}$ the number $\varepsilon_k = \frac{1}{k+1}$. Observe that $\frac{1}{1-\varepsilon_k} = 1 + \frac{1}{k}$. Take any $\lambda > 1$. For every nonzero $u = (u_k) \in c_0$ there exists $m \in \mathbb{N}$ such that

$$\frac{(1 - \varepsilon_m) \lambda |u_m|}{\sup_n (1 - \varepsilon_n) |u_n|} \geq 1,$$

equivalently

$$\frac{\lambda |u_m|}{\sup_n (1 - \varepsilon_n) |u_n|} \geq \frac{1}{1 - \varepsilon_m}.$$

Then we have

$$\begin{aligned} I_\Phi \left(\frac{\lambda Pu}{\sup_n (1 - \varepsilon_n) |u_n|} \right) &= I_\Phi \left(\frac{\sum_{k=1}^{\infty} \lambda u_k y_k}{\sup_n (1 - \varepsilon_n) |u_n|} \right) \geq I_\Phi \left(\frac{\lambda u_m y_m}{\sup_n (1 - \varepsilon_n) |u_n|} \right) \geq \\ &\geq I_\Phi \left(\frac{1}{1 - \varepsilon_m} y_m \right) = I_\Phi \left(\frac{1}{1 - \varepsilon_m} \sum_{i \in N_m} x_i e_i \right) = \\ &= \sum_{i \in N_m} \Phi_i \left(\left(1 + \frac{1}{m}\right) x_i \right) > 1, \end{aligned}$$

whence

$$\frac{1}{\lambda} \sup_n (1 - \varepsilon_n) |u_n| \leq \|Pu\|$$

and from arbitrariness of $\lambda > 1$, we get the thesis.

Now assume that $\Phi \in \delta_2^0$. Then $h^\Phi = l^\Psi$ is the dual space of h^Ψ , where Ψ is the Orlicz function complementary in the sense of Young to Φ . Assume that h^Φ contains an asymptotically isometric copy of c_0 . Then it contains, as a dual space, an isometric copy of l^∞ (see [5]). But this contradicts the fact that h^Φ is order continuous. \square

Theorem 2.2. *If Φ takes only finite values then: E^Φ contains an asymptotically isometric copy of c_0 if and only if Φ does not satisfy the Δ_2 condition.*

Proof. If $\Phi < \infty$ and $\Phi \notin \Delta_2^0$ then there exist sequences of measurable functions (x_n) and measurable sets (E_n) such that:

$$\begin{aligned} E_m \cap E_n &= \emptyset \text{ for } m \neq n, \\ x_n(t) &< \infty \text{ for every } t \in E_n, n \in \mathbb{N}, \\ \int_{E_n} \Phi(t, x_n(t)) d\mu &= \frac{1}{2^n}, \\ \Phi(t, (1 + \frac{1}{n})x_n(t)) &\geq 2^{n+2}\Phi(t, x_n(t)) \text{ for every } t \in E_n, n \in \mathbb{N}. \end{aligned}$$

For details see [2].

Take any $n \in \mathbb{N}$ and define for every $k \in \mathbb{N}$ the set

$$E_{n,k} = \{t \in E_n : |x_n(t)| \leq k\} \cap T_k,$$

where (T_k) is a sequence of measurable sets satisfying: $T_1 \subset T_2 \subset \dots, \bigcup_{n=1}^{\infty} T_n = T$ and $\mu(T_k) < \infty$ for every $k \in \mathbb{N}$. Such sets exist by the assumption of σ -finiteness of the measure μ . Then, we have

$$\begin{aligned} E_{n,1} &\subset E_{n,2} \subset \dots, \\ \bigcup_{k=1}^{\infty} E_{n,k} &= E_n, \\ \mu(E_{n,k}) &< \infty \text{ for every } k \in \mathbb{N}. \end{aligned}$$

Consequently, we get that $|x_n|_{\chi_{E_{n,k}}} \uparrow |x_n|_{\chi_{E_n}}$ as $k \rightarrow \infty$. By the Beppo Levi monotone convergence theorem, we get

$$\lim_{k \rightarrow \infty} \int_{E_n} \Phi(t, x_n(t)\chi_{E_{n,k}}(t)) d\mu = \int_{E_n} \Phi(t, x_n(t)) d\mu.$$

Now, for every $n \in \mathbb{N}$ we can fix $k \in \mathbb{N}$ such that

$$\frac{1}{2^{n+1}} \leq \int_{E_n} \Phi(t, x_n(t)\chi_{E_{n,k}}(t)) d\mu = \int_{E_{n,k}} \Phi(t, x_n(t)) d\mu \leq \int_{E_n} \Phi(t, x_n(t)) d\mu = \frac{1}{2^n}.$$

Let us denote $E_{n,k}$ by F_k .

Summarizing, for now we have constructed a sequence of measurable functions (x_n) and a sequence of measurable sets (F_n) satisfying the following conditions:

$$\begin{aligned} F_m \cap F_n &= \emptyset \text{ for } m \neq n, \\ x_n &\text{ is bounded on } F_n, n \in \mathbb{N}, \\ \frac{1}{2^{n+1}} &\leq \int_{F_n} \Phi(t, x_n(t)) d\mu \leq \frac{1}{2^n}, \\ \Phi(t, (1 + \frac{1}{n})x_n(t)) &\geq 2^{n+2}\Phi(t, x_n(t)) \text{ for every } t \in F_n, n \in \mathbb{N}. \end{aligned}$$

Define an operator $P : c_0 \rightarrow E^\Phi$ by the formula

$$Pu = \sum_{n=1}^{\infty} u_n x_n \chi_{F_n} \text{ for } u = (u_n) \in c_0.$$

We will show that $I_\Phi(\lambda Pu) < \infty$ for any $u \in c_0$ and $\lambda > 0$. Fix any $u = (u_n) \in c_0$ and take any $\lambda > 0$. There exists $l_0 \in \mathbb{N}$ such that $\lambda|u_n| \leq 1$ for every $n \geq l_0$. We have

$$\begin{aligned} I_\Phi(\lambda Pu) &= \int_T \Phi \left(t, \lambda \sum_{n=1}^{\infty} u_n x_n(t) \chi_{F_n}(t) \right) d\mu = \sum_{n=1}^{\infty} \int_{F_n} \Phi(t, \lambda u_n x_n(t)) d\mu = \\ &= \sum_{n=1}^{l_0-1} \int_{F_n} \Phi(t, \lambda u_n x_n(t)) d\mu + \sum_{n=l_0}^{\infty} \int_{F_n} \Phi(t, x_n(t)) d\mu \leq \\ &\leq \sum_{n=1}^{l_0-1} \int_{F_n} \Phi(t, \lambda u_n x_n(t)) d\mu + \sum_{n=l_0}^{\infty} \frac{1}{2^n} < \infty, \end{aligned}$$

since $\int_{F_n} \Phi(t, \lambda u_n x_n(t)) d\mu$ is finite for every $n \in \mathbb{N}$. Consequently, $Pu \in E^\Phi$.

Next we will show that $\|Pu\| \leq \|u\|_\infty$. For any nonzero $u \in c_0$ we have

$$\begin{aligned} I_\Phi \left(\frac{Pu}{\|u\|_\infty} \right) &= \int_T \Phi \left(t, \sum_{n=1}^{\infty} \frac{1}{\|u\|_\infty} u_n x_n(t) \chi_{F_n}(t) \right) d\mu \leq \\ &\leq \int_T \Phi \left(t, \sum_{n=1}^{\infty} x_n(t) \chi_{F_n}(t) \right) d\mu = \sum_{n=1}^{\infty} \int_{F_n} \Phi(t, x_n(t)) d\mu \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1. \end{aligned}$$

Consequently, $\|Pu\| \leq \|u\|_\infty$.

Finally, we will show that there exists a sequence (ε_n) such that $\varepsilon_n \downarrow 0$ and $\sup_n (1 - \varepsilon_n) |u_n| \leq \|Pu\|$ for every $u \in c_0$. Define $\varepsilon_n = \frac{1}{n+1}$ for every $n \in \mathbb{N}$ and notice that $\frac{1}{1-\varepsilon_n} = 1 + \frac{1}{n}$ ($n \in \mathbb{N}$). Fix any nonzero $u \in c_0$ and $\lambda > 1$. Since $1 - \varepsilon_n \rightarrow 1$ as $n \rightarrow \infty$, then there exists $m \in \mathbb{N}$ such that

$$\frac{\lambda(1 - \varepsilon_m) |u_m|}{\sup_n (1 - \varepsilon_n) |u_n|} \geq 1$$

and equivalently

$$\frac{\lambda |u_m|}{\sup_n (1 - \varepsilon_n) |u_n|} \geq \frac{1}{(1 - \varepsilon_m)}.$$

Now we have

$$\begin{aligned}
 I_{\Phi} \left(\frac{\lambda Pu}{\sup_n (1-\varepsilon_n) |u_n|} \right) &= \int_T \Phi \left(t, \sum_{n=1}^{\infty} \lambda u_n x_n(t) \frac{1}{\sup_n (1-\varepsilon_n) |u_n|} \chi_{F_n}(t) \right) d\mu \geq \\
 &\geq \int_T \Phi \left(t, \lambda u_m x_m(t) \frac{1}{\sup_n (1-\varepsilon_n) |u_n|} \chi_{F_m}(t) \right) d\mu = \\
 &= \int_{F_m} \Phi \left(t, \lambda u_m x_m(t) \frac{1}{\sup_n (1-\varepsilon_n) |u_n|} \right) d\mu \geq \\
 &\geq \int_{F_m} \Phi \left(t, \frac{1}{1-\varepsilon_m} x_m(t) \right) d\mu = \int_{F_m} \Phi \left(t, \left(1 + \frac{1}{m}\right) x_m(t) \right) d\mu \geq \\
 &\geq 2^{m+2} \int_{F_m} \Phi(t, x_m(t)) d\mu \geq 2^{m+2} \cdot \frac{1}{2^{m+1}} = 2 > 1.
 \end{aligned}$$

Consequently, $\sup_n (1 - \varepsilon_n) |u_n| \leq \|Pu\|$.

The proof of the conversion is similar to in the previous theorem. \square

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