

A NEUMANN BOUNDARY VALUE PROBLEM FOR A CLASS OF GRADIENT SYSTEMS

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Abstract. In this paper we study a class of two-point boundary value systems. Using very recent critical points theorems, we establish the existence of one non-trivial solution and infinitely many solutions of this problem, respectively.

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1. INTRODUCTION

In this paper, we study the Neumann boundary value problems:

$$\begin{cases} -(|u'_1(x)|^{p_1-2}u'_1(x))' + |u_1(x)|^{p_1-2}u_1(x) = \lambda F_{u_1}(x, u_1, \dots, u_m), & x \in (a, b), \\ -(|u'_2(x)|^{p_2-2}u'_2(x))' + |u_2(x)|^{p_2-2}u_2(x) = \lambda F_{u_2}(x, u_1, \dots, u_m), & x \in (a, b), \\ \dots \\ -(|u'_m(x)|^{p_m-2}u'_m(x))' + |u_m(x)|^{p_m-2}u_m(x) = \lambda F_{u_m}(x, u_1, \dots, u_m), & x \in (a, b), \\ u'_i(a) = u'_i(b) = 0, \end{cases} \quad (\mathcal{P}_\lambda)$$

where $p_i > 1$ are constants, for $1 \leq i \leq m$, λ is a positive parameter, $F : [a, b] \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a function such that $F(\cdot, t_1, \dots, t_m)$ is measurable in $[a, b]$ for all $(t_1, \dots, t_m) \in \mathbb{R}^m$, $F(x, \cdot, \dots, \cdot)$ is C^1 in \mathbb{R}^m for every $x \in [a, b]$ and for every $\varrho > 0$,

$$\sup_{|(t_1, \dots, t_m)| \leq \varrho} \sum_{i=1}^m |F_{t_i}(x, t_1, \dots, t_m)| \in L^1([a, b]),$$

and F_{u_i} denotes the partial derivative of F with respect to u_i for $1 \leq i \leq m$.

In the last decade or so, many authors applied variational methods to study the existence or multiplicity solutions of the Neumann problem of its variations; see, for

example, [6, 7, 9–13] and the references therein. We note that the main tools in these cited papers are several critical point theorems due to Bonanno [3], Bonanno and Bisci [4], Bonanno and Marano [8]. A Neumann boundary value problem for a class of gradient systems has already been studied by Afrouzi, Hadjian and Heidarkhani [1] and Hedarkhani and Tian [14] in the ODE case and Afrouzi, Heidarkhani and O’Regan [2] in the PDE case. In that papers at least three solutions are established. The aim of this article is to prove the existence of at least one non-trivial solution and infinitely many solutions for (\mathcal{P}_λ) for appropriate values of the parameter λ belonging to a precise real interval. Our motivation comes from the recent paper [4, 10]. We want to systematically study a class of gradient systems under a Neumann boundary using Bonanno’s critical point theorems. For basic notation and definitions, and also for a thorough account of the subject, we refer the reader to [15, 16].

2. PRELIMINARIES AND BASIC NOTATION

First we recall Bonanno’s critical point theorems which is our main tool to transfer the question of existence of weak solutions of (\mathcal{P}_λ) to the existence of critical points of the Euler functional.

For a given non-empty set X , and two functionals $\Phi, \Psi : X \rightarrow \mathbb{R}$, we define the following two functions:

$$\beta(r_1, r_2) = \inf_{v \in \Phi^{-1}((r_1, r_2))} \frac{\sup_{u \in \Phi^{-1}((r_1, r_2))} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}((r_1, r_2))} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}((-\infty, r_1))} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$.

Theorem 2.1 ([3, Theorem 5.1]). *Let X be a reflexive real Banach space, $\Phi : X \rightarrow \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* and $\Psi : X \rightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2).$$

Then, for each $\lambda \in \left(\frac{1}{\rho(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)}\right)$ there is $u_{0, \lambda} \in \Phi^{-1}((r_1, r_2))$ such that $I_\lambda(u_{0, \lambda}) \leq I_\lambda(u)$ for each $u \in \Phi^{-1}((r_1, r_2))$ and $I'_\lambda(u_{0, \lambda}) = 0$.

Theorem 2.2 ([4, Theorem 2.1]). *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \rightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}((-\infty, r))} \frac{\left(\sup_{v \in \Phi^{-1}((-\infty, r))} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r \rightarrow +\infty} \varphi(r).$$

Under the above assumptions if $\gamma < +\infty$ then, for each $\lambda \in \left(0, \frac{1}{\gamma}\right)$, the following alternative holds:

either

(b₁) I_λ possesses a global minimum,

or

(b₂) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n \rightarrow +\infty} \Phi(u_n) = +\infty$.

Let us introduce notation that will be used later. Let Y_i be the Sobolev space $W^{1,p_i}([a, b])$ endowed with the norm

$$\|u\|_{p_i} := \left(\int_a^b |u'(x)|^{p_i} dx + \int_a^b |u(x)|^{p_i} dx \right)^{1/p_i},$$

and let

$$k_i = 2^{(p_i-1)/p_i} \max\{(b-a)^{-1/p_i}, (b-a)^{(p_i-1)/p_i}\},$$

we recall the following inequality which we use in the sequel

$$|u(x)| \leq k_i \|u\|_{p_i} \tag{2.1}$$

for all $u \in Y_i$, and for all $x \in [a, b]$. Let $K = \max\{k_i^{p_i}\}$, for $1 \leq i \leq m$. Here and in the sequel, $X := Y_1 \times \dots \times Y_m$.

We say that $u = (u_1, \dots, u_m)$ is a weak solution to the (\mathcal{P}_λ) if $u = (u_1, \dots, u_m) \in X$ and

$$\begin{aligned} & \sum_{i=1}^m \int_a^b (|u'_i(x)|^{p_i-2} u'_i(x) v'_i(x) + |u_i(x)|^{p_i-2} u_i(x) v_i(x)) dx - \\ & - \lambda \sum_{i=1}^m \int_a^b F_{u_i}(x, u_1, \dots, u_m) v_i(x) dx = 0 \end{aligned}$$

for every $v = (v_1, \dots, v_m) \in X$. For $\gamma > 0$ we denote the set

$$\Theta(\gamma) = \left\{ (t_1, \dots, t_m) \in \mathbb{R}^m : \sum_{i=1}^m \frac{|t_i|^{p_i}}{p_i} \leq \frac{\gamma}{\prod_{i=1}^m p_i} \right\}. \tag{2.2}$$

Let

$$\Phi(u) = \sum_{i=1}^m \frac{\|u_i\|_{p_i}^{p_i}}{p_i}, \tag{2.3}$$

$$\Psi(u) = \int_a^b F(x, u_1(x), \dots, u_m(x)) dx. \tag{2.4}$$

It is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u = (u_1, \dots, u_m) \in X$ are the functionals $\Phi'(u), \Psi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \sum_{i=1}^m \int_a^b (|u'_i(x)|^{p_i-2} u'_i(x) v'_i(x) + |u_i(x)|^{p_i-2} u_i(x) v_i(x)) dx,$$

$$\Psi'(u)(v) = \int_a^b \sum_{i=1}^m F_{u_i}(x, u_1(x), \dots, u_m(x)) v_i(x) dx$$

for every $v = (v_1, \dots, v_m) \in X$, respectively. Moreover, Φ is sequentially weakly lower semicontinuous, Φ' admits a continuous inverse on X^* as well as Ψ is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : X \rightarrow X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X . For this, for fixed $(u_1, \dots, u_m) \in X$, let $(u_{1n}, \dots, u_{mn}) \rightarrow (u_1, \dots, u_m)$ weakly in X as $n \rightarrow +\infty$, then we have (u_{1n}, \dots, u_{mn}) converges uniformly to (u_1, \dots, u_m) on $[a, b]$ as $n \rightarrow +\infty$ (see [16]). Since $F(x, \dots, \dots)$ is C^1 in \mathbb{R}^m for every $x \in [a, b]$, the derivatives of F are continuous in \mathbb{R}^m for every $x \in [a, b]$, so for $1 \leq i \leq m$, $F_{u_i}(x, u_{1n}, \dots, u_{mn}) \rightarrow F_{u_i}(x, u_1, \dots, u_m)$ strongly as $n \rightarrow +\infty$ which follows $\Psi'(u_{1n}, \dots, u_{mn}) \rightarrow \Psi'(u_1, \dots, u_m)$ strongly as $n \rightarrow +\infty$. Thus we proved that Ψ' is strongly continuous on X , which implies that Ψ' is a compact operator by Proposition 26.2 of [16].

3. RESULTS

Before our proof, we first list nonlinear term F which satisfies the following hypotheses, where μ_1, μ_2 and ν are some constants.

- (H1) $F(x, 0, \dots, 0) = 0$ for a.e. $x \in [a, b]$,
- (H2) $a_\nu(\mu_2) < a_\nu(\mu_1)$, where

$$a_\nu(\mu) := K \frac{\int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(\mu)} F(x, t_1, \dots, t_m) dx - \int_a^b F(x, \nu, \dots, \nu) dx}{\mu - K \sum_{i=1}^m (\prod_{j=1, j \neq i}^m p_j) \nu^{p_i}},$$

(H3)

$$\frac{\int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(\mu)} F(x, t_1, \dots, t_m) dx}{\mu} < \frac{\int_a^b F(x, \nu, \dots, \nu) dx}{K \sum_{i=1}^m (\prod_{j=1, j \neq i}^m p_j) \nu^{p_i}},$$

(H4)

$$\liminf_{\mu \rightarrow +\infty} \frac{\int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(\mu)} F(x, t_1, \dots, t_m) dx}{\mu} <$$

$$< \frac{1}{K \prod_{i=1}^m p_i (b-a)} \limsup_{|t_1| \rightarrow +\infty, \dots, |t_m| \rightarrow +\infty} \frac{\int_a^b F(x, t_1, \dots, t_m) dx}{\sum_{i=1}^m \frac{|t_i|^{p_i}}{p_i}}.$$

3.1. ONE NONTRIVIAL SOLUTION

We formulate our main result as follows:

Theorem 3.1. *Assume that there exist a non-negative constant c_1 and two positive constants c_2 and d with*

$$c_1 < K(b - a) \sum_{i=1}^m (\prod_{j=1, j \neq i}^m p_j) d^{p_i} < c_2$$

such that (H1) and (H2) are satisfied. Then, for each $\lambda \in (\frac{1}{a_d(c_1)}, \frac{1}{a_d(c_2)})$, system (P_λ) admits at least one non-trivial weak solution $u_0 = (u_{01}, \dots, u_{0m}) \in X$ such that

$$\frac{c_1}{K \prod_{i=1}^m p_i} < \sum_{i=1}^m \frac{\|u_{0i}\|_{p_i}^{p_i}}{p_i} < \frac{c_2}{K \prod_{i=1}^m p_i}.$$

Proof. To apply Theorem 2.1 to our problem, we introduce the functionals $\Phi, \Psi : X \rightarrow \mathbb{R}$ for each $u = (u_1, \dots, u_n) \in X$, as (2.3) and (2.4). Moreover, Φ is sequentially weakly lower semicontinuous, Φ' admits a continuous inverse on X^* as well as $\Psi' : X \rightarrow X^*$ is a compact operator. Set $w(x) = (w_1(x), \dots, w_m(x))$ such that for $1 \leq i \leq m$,

$$w_i(x) = d$$

$r_1 = \frac{c_1}{K \prod_{i=1}^m p_i}$ and $r_2 = \frac{c_2}{K \prod_{i=1}^m p_i}$. It is easy to verify that $w = (w_1, \dots, w_m) \in X$, and in particular, one has

$$\|w_i\|_{p_i}^{p_i} = (b - a)d^{p_i}$$

for $1 \leq i \leq m$. So, from the definition of Φ , we have

$$\Phi(w) = (b - a) \sum_{i=1}^m \frac{d^{p_i}}{p_i}.$$

From the conditions $c_1 < K \sum_{i=1}^m (\prod_{j=1, j \neq i}^m p_j)(b - a)d^{p_i} < c_2$, we obtain

$$r_1 < \Phi(w) < r_2.$$

Moreover, from (2.1) one has

$$\sup_{x \in [a, b]} |u_i(x)|^{p_i} \leq k_i^{p_i} \|u_i\|_{p_i}^{p_i}$$

and

$$\sup_{x \in [a, b]} |u_i(x)|^{p_i} \leq K \|u_i\|_{p_i}^{p_i}$$

for each $u = (u_1, \dots, u_m) \in X$, so from the definition of Φ , we observe that

$$\begin{aligned} \Phi^{-1}((-\infty, r_2)) &= \{(u_1, \dots, u_n) \in X : \Phi(u_1, \dots, u_n) < r_2\} = \\ &= \{(u_1, \dots, u_n) \in X : \sum_{i=1}^m \frac{\|u_i\|_{p_i}^{p_i}}{p_i} < r_2\} \subseteq \\ &\subseteq \left\{ (u_1, \dots, u_n) \in X : \sum_{i=1}^m \frac{|u_i(x)|^{p_i}}{p_i} \leq \frac{c_2}{\prod_{i=1}^m p_i} \text{ for all } x \in [a, b] \right\}, \end{aligned}$$

from which it follows

$$\begin{aligned} \sup_{(u_1, \dots, u_m) \in \Phi^{-1}((-\infty, r_2))} \Psi(u) &= \sup_{(u_1, \dots, u_m) \in \Phi^{-1}((-\infty, r_2))} \int_a^b F(x, u_1(x), \dots, u_m(x)) dx \leq \\ &\leq \int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(c_2)} F(x, t_1, \dots, t_m) dx. \end{aligned}$$

Since for $1 \leq i \leq m$, for each $x \in [a, b]$, the condition (A1) ensures that

$$\begin{aligned} \beta(r_1, r_2) &\leq \frac{\sup_{u \in \Phi^{-1}((-\infty, r_2))} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \leq \\ &\leq \frac{\int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(c_2)} F(x, t_1, \dots, t_m) dx - \Psi(w)}{r_2 - \Phi(w)} \leq a_d(c_2). \end{aligned}$$

On the other hand, by similar reasoning as before, one has

$$\begin{aligned} \rho(r_1, r_2) &\geq \frac{\Psi(w) - \sup_{u \in \Phi^{-1}((-\infty, r_1))} \Psi(u)}{\Phi(w) - r_1} \geq \\ &\geq \frac{\Psi(w) - \int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(c_1)} F(x, t_1, \dots, t_m) dx}{\Phi(w) - r_1} \geq a_d(c_1). \end{aligned}$$

Hence, from Assumption (A2), one has $\beta(r_1, r_2) < \rho(r_1, r_2)$. Therefore, from Theorem 2.1, taking into account that the weak solutions of the system (\mathcal{P}_λ) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$, we have the conclusion. \square

Now we point out the following consequence of Theorem 3.1.

Theorem 3.2. *Suppose that there exist two positive constants c and d with*

$$c > K(b - a) \sum_{i=1}^m (\prod_{j=1, j \neq i}^m p_j) d^{p_i}$$

such that (H1) and (H3) hold. Then, for each

$$\lambda \in \left(\frac{K(b - a) \sum_{i=1}^m (\prod_{j=1, j \neq i}^m p_j) d^{p_i}}{\int_a^b F(x, d, \dots, d) dx}, \frac{c}{\int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(c)} F(x, t_1, \dots, t_m) dx} \right),$$

system (\mathcal{P}_λ) admits at least one non-trivial weak solution $u_0 = (u_{01}, \dots, u_{0n}) \in X$ such that

$$\sum_{i=1}^m \frac{\|u_{0i}\|_\infty^{p_i}}{p_i} < \frac{c}{K \prod_{i=1}^m p_i}.$$

Proof. The conclusion follows from Theorem 3.1, by taking $c_1 = 0$ and $c_2 = c$. Indeed, owing to our assumptions, one has

$$\begin{aligned}
 a_d(c_2) &= K \prod_{i=1}^m p_i \frac{\int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(c)} F(x, t_1, \dots, t_m) dx - \int_a^b F(x, d, \dots, d) dx}{c - K(b-a) \sum_{i=1}^m (\prod_{j=1, j \neq i}^m p_j) d^{p_i}} \leq \\
 &\leq K \prod_{i=1}^m p_i \frac{\int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(c)} F(x, t_1, \dots, t_m) dx - \frac{\int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(c)} F(x, t_1, \dots, t_m) dx}{K \sum_{i=1}^m (\prod_{j=1, j \neq i}^m p_j) d^{p_i}}}{c - K(b-a) \sum_{i=1}^m (\prod_{j=1, j \neq i}^m p_j) d^{p_i}} = \\
 &= \frac{\int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(c)} F(x, t_1, \dots, t_m) dx}{c}.
 \end{aligned}$$

On the other hand, taking Assumption (A1) into account, one has

$$\frac{\int_a^b F(x, d, \dots, d) dx}{K(b-a) \sum_{i=1}^m (\prod_{j=1, j \neq i}^m p_j) d^{p_i}} = a_d(c_1).$$

Moreover, since

$$\sup_{x \in [a, b]} |u_i(x)|^{p_i} \leq K \|u_i\|_{p_i}^{p_i}$$

for each $u = (u_1, \dots, u_m) \in X$, an easy computation ensures that

$$\sum_{i=1}^m \frac{\|u_{0i}\|_{\infty}^{p_i}}{p_i} < \frac{c}{K \prod_{i=1}^m p_i}$$

whenever $\Phi(u) < r_2$. Now, owing to Assumption (A3), it is sufficient to invoke Theorem 3.1 to conclude the proof. □

3.2. INFINITY MANY SOLUTIONS

Theorem 3.3. *Assume that (H1) and (H4) hold. Then, for every $\lambda \in \Lambda := (\lambda_1, \lambda_2)$, where*

$$\lambda_1 = \frac{(b-a)}{\limsup_{|t_1| \rightarrow +\infty, \dots, |t_m| \rightarrow +\infty} \frac{\int_a^b F(x, t_1, \dots, t_m) dx}{\sum_{i=1}^m \frac{|t_i|^{p_i}}{p_i}}}$$

and

$$\lambda_2 = \frac{1}{K \prod_{i=1}^m \liminf_{\mu \rightarrow +\infty} \frac{\int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(\mu)} F(x, t_1, \dots, t_m) dx}{\mu}}$$

the problem (\mathcal{P}_λ) admits an unbounded sequence of weak solutions which is unbounded in X .

Proof. Our goal is to apply Theorem 2.2. Now, as has been pointed out before, the functionals Φ and Ψ satisfy the regularity assumptions required in Theorem 2.2. Let $\{c_n\}$ be a real sequence such that $\lim_{n \rightarrow +\infty} c_n = +\infty$ and

$$\liminf_{n \rightarrow +\infty} \frac{\int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(c_n)} F(x, t_1, \dots, t_m) dx}{c_n} = \mathcal{A}. \tag{3.1}$$

Taking into account (2.1) for every $u \in X$ one has

$$|u(x)| \leq K \|u\|_{p_i}.$$

Also note

$$\sum_{i=1}^m \frac{|u_i(x)|^{p_i}}{p_i} \leq K \left(\sum_{i=1}^m \frac{\|u_i(x)\|_{p_i}^{p_i}}{p_i} \right).$$

Hence, an easy computation ensures that $\sum_{i=1}^m u \leq c_n$ when ever $u \in \Phi^{-1}((-\infty, r_n))$, where

$$r_n = \frac{1}{K} \frac{c_n}{\prod_{i=1}^m p_i}.$$

Taking into account $\|u_i^0\|_{p_i} = 0$ (where $u_i^0(x) = 0$ for every $x \in [a, b]$) and that $\int_a^b F(t, 0, \dots, 0) dx = 0$ for all $x \in [a, b]$, for every n large enough, one has

$$\begin{aligned} \varphi(r_n) &= \inf_{u \in \Phi^{-1}((-\infty, r_n))} \frac{\left(\sup_{v \in \Phi^{-1}((-\infty, r_n))} \Psi(v) \right) - \Psi(u)}{r_n - \Phi(u)} = \\ &= \inf_{\sum_{i=1}^m \frac{\|u_i\|_{p_i}^{p_i}}{p_i} < r_n} \frac{\sup_{\sum_{i=1}^m \frac{\|v_i\|_{p_i}^{p_i}}{p_i} < r_n} \int_a^b F(t, v_1(x), \dots, v_m(x)) dx - \int_a^b F(t, u_1(x), \dots, u_m(x)) dx}{r_n - \sum_{i=1}^m \frac{\|u_i\|_{p_i}^{p_i}}{p_i}} \leq \\ &\leq \frac{\sup_{\sum_{i=1}^m \frac{\|v_i\|_{p_i}^{p_i}}{p_i} < r_n} \int_a^b F(t, v_1(x), \dots, v_m(x)) dx}{r_n} \leq \\ &\leq K \prod_{i=1}^m p_i \liminf_{n \rightarrow +\infty} \frac{\int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(c_n)} F(x, t_1, \dots, t_m) dx}{c_n}. \end{aligned}$$

Therefore, since from assumption (H4) one has $\mathcal{A} < +\infty$, we obtain

$$\gamma = \liminf_{n \rightarrow +\infty} \varphi(r_n) \leq K \prod_{i=1}^m p_i \mathcal{A} < +\infty. \tag{3.2}$$

Now, fix $\lambda \in (\lambda_1, \lambda_2)$ and let us verify that the functional I_λ is unbounded from below. Let $\{\xi_{i,n}\}$ be m positive real sequences such that $\lim_{n \rightarrow +\infty} \sqrt{\sum_{i=1}^m \xi_{i,n}^2} = +\infty$, and

$$\limsup_{n \rightarrow +\infty} \frac{\int_a^b F(x, \xi_{1,n}, \dots, \xi_{m,n}) dx}{\sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i}} = \mathcal{B}. \tag{3.3}$$

For each $n \in \mathbb{N}$ define

$$w_{i,n}(x) := \xi_{i,n}$$

and put $w_n := (w_{1,n}, \dots, w_{m,n})$.

We easily get that

$$\|w_{i,n}\|_{p_i}^{p_i} = (b-a)|\xi_{i,n}|^{p_i}.$$

At this point, bearing in mind (i), we infer

$$\Phi(w_n) - \lambda\Psi(w_n) = \sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i} - \lambda \int_a^b F(x, \xi_{1,n}, \dots, \xi_{m,n}) dx, \quad n \in \mathbb{N}.$$

If $\mathcal{B} < +\infty$, let $\epsilon \in (\frac{1}{\lambda\mathcal{B}}, 1)$. By (3.3), there exists v_ϵ such that

$$\int_a^b F(x, \xi_{1,n}, \dots, \xi_{m,n}) dx > \epsilon\mathcal{B} \sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i}, \quad n > v_\epsilon.$$

Moreover,

$$\Phi(w_n) - \lambda\Psi(w_n) \leq \sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i} - \lambda\epsilon\mathcal{B} \sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i}, \quad n > v_\epsilon.$$

Taking into account the choice of ϵ , one has

$$\lim_{n \rightarrow +\infty} [\Phi(w_n) - \Psi(w_n)] = -\infty.$$

If $\mathcal{B} = +\infty$, let us consider $M > \frac{1}{\lambda}$. By (3.3), there exist v_m such that

$$\int_a^b F(x, \xi_{1,n}, \dots, \xi_{m,n}) dx > M \sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i}, \quad n > v_m.$$

Moreover,

$$\Phi(w_n) - \lambda\Psi(w_n) \leq \sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i} - \lambda M \sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i}, \quad n > v_\epsilon.$$

Taking into account the choice of M , also in this case, one has

$$\lim_{n \rightarrow +\infty} [\Phi(w_n) - \Psi(w_n)] = -\infty.$$

Applying Theorem 2.2, we deduce that the functional $\Phi - \lambda\Psi$ admits a sequence of critical points which is unbounded in X . Hence, our claim is proved and the conclusion is achieved. \square

Remark 3.4. If

$$\liminf_{\mu \rightarrow +\infty} \frac{\int_a^b \sup_{(t_1, \dots, t_m) \in \Theta(\mu)} F(x, t_1, \dots, t_m) dx}{\mu} = 0$$

and

$$\limsup_{|t_1| \rightarrow +\infty, \dots, |t_m| \rightarrow +\infty} \frac{\int_a^b F(x, t_1, \dots, t_m) dx}{\sum_{i=1}^m \frac{|t_i|^{p_i}}{p_i}} = +\infty,$$

clearly, hypothesis (H4) is verified and Theorem 3.3 guarantees the existence of infinitely many weak solutions for problem (\mathcal{P}_λ) , for every $\lambda \in (0, +\infty)$, the main result ensures the existence of infinitely many weak solutions for problem (\mathcal{P}_λ) .

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