Opuscula Math. 34, no. 1 (2014), 183–197
http://dx.doi.org/10.7494/OpMath.2014.34.1.183 Opuscula Mathematica

FIXED POINT THEOREMS FOR A SEMIGROUP OF TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Communicated by Alexander Gomilko

Abstract. In this paper, we provide existence and convergence theorems of common fixed points for left (or right) reversible semitopological semigroups of total asymptotically nonexpansive mappings in uniformly convex Banach spaces. The results presented in this paper extend and improve some recent results announced by other authors.

Keywords: fixed point, semitopological semigroup, reversible semigroup, total asymptotically nonexpansive semigroup, uniformly convex Banach spaces.

Mathematics Subject Classification: 47H09, 47H10.

1. INTRODUCTION

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $s \in S$, the mappings $s \mapsto ts$ and $s \mapsto st$ from S to S are continuous, and let $BC(S)$ be the Banach space of all bounded continuous real-valued functions with supremum norm. For $f \in BC(S)$ and $c \in \mathbb{R}$, we write $f(s) \to c$ as $s \to \infty_{\mathbb{R}}$ if for each $\varepsilon > 0$, there exists $w \in S$ such that $|f(tw) - c| < \varepsilon$ for all $t \in S$; see [1].

A semitopological semigroup S is said to be *left* (*resp. right*) *reversible* if any two closed right (resp. left) ideals of S have nonvoid intersection. If S is left reversible, $(S, \succeq)$ is a directed system when the binary relation "$\succeq$" on S is defined by $t \succeq s$ if and only if $\{t\} \cup \overline{tS} \subseteq \{s\} \cup \overline{sS}$ for $t, s \in S$. Similarly, we can define the binary relation "$\succeq$" on a right reversible semitopological semigroup S. Left reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are left amenable as discrete semigroups; see [2]. S is called *reversible* if it is both left and right reversible.

As is well known, the construction of fixed points of common fixed points of nonexpansive semigroups is an important problem in the theory of nonexpansive mappings and its applications, i.e., image recovery, convex feasibility problems, and signal processing problems (see, for example, [3–5]).

In 1969, Takahashi [6] proved the fixed point theorem for a noncommutative semigroup of nonexpansive mappings which generalizes De Marr's result [7]. He proved that any discrete left amenable semigroup has a common fixed point. In 1970, Mitchell [8] generalized Takahashi's result by showing that any discrete left reversible semigroup has a common fixed point; see also [9]. In 1981, Takahashi [10] proved a nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space. In 1987, Lau and Takahashi [11] considered the problem of weak convergence of a nonexpansive semigroup of a right reversible semitopological semigroup in a uniformly convex Banach space with Fréchet differentiable norm. After that, Lau [12–16] proved the existence of common fixed points for nonexpansive mappings related to reversibility or amenability of a semigroup. In [17], Kakavandi and Amini proved a nonlinear ergodic theorem for a nonexpansive semigroup in CAT(0) spaces as well as a strong convergence theorem for a commutative semitopological semigroup. In 2011, Anakkanmatee and Dhompongsa [18] extended Rodé's theorem [19] on common fixed points of semigroups of nonexpansive mappings in Hilbert spaces to the CAT(0) space setting. In 1988, Takahashi and Zhang [20, 21] established the weak convergence of an almost-orbit of Lipschitzian semigroups of a noncommutative semitopological semigroup. Later, Kim and Kim [22] proved weak convergence theorems for semigroups of asymptotically nonexpansive type of a right reversible semitopological semigroup and strong convergence theorems for a commutative case. For works related to semigroups of nonexpansive, asymptotically nonexpansive, and asymptotically nonexpansive type related to reversibility of a semigroup, we refer the reader to [23–30].

In this paper, we introduce a new semigroup for a left (or right) reversible semitopological semigroup on Banach spaces, called *a total asymptotically nonexpansive semigroup*, which is more general than the semigroups of nonexpansive, asymptotically nonexpansive, asymptotically nonexpansive type and generalized asymptotically nonexpansive. We also prove the existence and convergence theorems for such semigroup in uniformly convex Banach spaces. The results obtained in this paper extend and improve many recent results in [20–22, 24].

2. PRELIMINARIES

A Banach space X is called *uniformly convex* if for each $\varepsilon \in (0, 2]$ there is a $\delta > 0$ such that for $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. It is known that a uniformly convex Banach space is reflexive and strictly convex. The function $\delta_X : [0, 2] \to [0, 1]$ which called the *modulus of convexity* of X is defined as follows:

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in X, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}.$$

We now give some important properties of the modulus of convexity of Banach spaces.

Proposition 2.1 ([31]).

(i) *A Banach space X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.*
(ii) *A Banach space X is strictly convex if and only if $\delta_X(2) = 1$.*
(iii) *Let X be a uniformly convex Banach space. For any r and ε with $r \geq \varepsilon > 0$ and $x, y \in X$ with $\|x\| \leq r$, $\|y\| \leq r$, $\|x - y\| \geq \varepsilon$, there exists a $\delta = \delta_X(\frac{\varepsilon}{r}) > 0$ such that*

$$\left\| \frac{x+y}{2} \right\| \leq r \left[1 - \delta_X \left(\frac{\varepsilon}{r} \right) \right].$$

A Banach space X is said to satisfy the *Opial property* [32] if for each weakly convergent net $\{x_\alpha\}$ in X with weak limit x, the inequality

$$\limsup_{\alpha} \|x_\alpha - x\| < \limsup_{\alpha} \|x_\alpha - y\|$$

holds for every $y \in X$ with $y \neq x$.

Let $\{x_\alpha\}$ be a bounded net in a nonempty closed convex subset C of a Banach space X. For $x \in X$, we set

$$r(x, \{x_\alpha\}) = \limsup_{\alpha} \|x - x_\alpha\|.$$

The *asymptotic radius* of $\{x_\alpha\}$ on C is given by

$$r(C, \{x_\alpha\}) = \inf_{x \in C} r(x, \{x_\alpha\}),$$

and the *asymptotic center* of $\{x_\alpha\}$ on C is given by

$$A(C, \{x_\alpha\}) = \{x \in C : r(x, \{x_\alpha\}) = r(C, \{x_\alpha\})\}.$$

It is known that, in a uniformly convex Banach space, $A(C, \{x_\alpha\})$ consists of exactly one point; see [33, 34].

The following lemmas are useful for our results.

Lemma 2.2 ([35]). *Let X be a uniformly convex Banach space, let $\{t_n\}$ be a sequence of real numbers such that $0 < a \leq t_n \leq b < 1$ for all $n \in \mathbb{N}$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and*

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r \text{ for some } r \geq 0.$$

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3 ([36]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers satisfing:*

$$a_{n+1} \leq (1 + b_n) a_n + c_n \text{ for all } n \in \mathbb{N},$$

where $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then:

(i) *$\lim_{n\to\infty} a_n$ exists,*
(ii) *if $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.*

3. EXISTENCE THEOREMS IN UNIFORMLY CONVEX BANACH SPACES

In this section, we first introduce the concept of a semigroup of total asymptotically nonexpansive mappings in Banach spaces as follows:

Let S be a semitopological semigroup and C be a nonempty closed subset of a Banach space X. A family $\mathfrak{T} = \{T_s : s \in S\}$ of mappings of C into itself is said to be a *semigroup* if it satisfies the following:

(S1) $T_{st}x = T_sT_tx$ for all $s, t \in S$ and $x \in C$,
(S2) for every $x \in C$, the mapping $s \mapsto T_sx$ from S into C is continuous.

We denote by $F(\mathfrak{T})$ the set of common fixed points of $\mathfrak{T}$, i.e.,

$$F(\mathfrak{T}) = \bigcap_{s\in S} F(T_s) = \bigcap_{s\in S} \{x \in C : T_sx = x\}.$$

Definition 3.1. Let S be a left (or right) reversible semitopological semigroup and C be a nonempty closed subset of a Banach space X. A semigroup $\mathfrak{T} = \{T_s : s \in S\}$ of mappings of C into itself is said to be *total asymptotically nonexpansive* if each T_s is continuous and there exist nonnegative real numbers k_s, μ_s with $\lim_s k_s = 0$, $\lim_s \mu_s = 0$ and a strictly increasing continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\|T_sx - T_sy\| \leq \|x - y\| + k_s\phi(\|x - y\|) + \mu_s$$

for each $x, y \in C$ and $s \in S$.

Remark 3.2. If $\phi(\lambda) = \lambda$, then a total asymptotically nonexpansive semigroup reduces to a generalized asymptotically nonexpansive semigroup. If $\phi(\lambda) = \lambda$ and $k_s = 0$ for all $s \in S$, then a total asymptotically nonexpansive semigroup reduces to an asymptotically nonexpansive semigroup. If $\phi(\lambda) = \lambda$ and $k_s = \mu_s = 0$ for all $s \in S$, then a total asymptotically nonexpansive semigroup reduces to a nonexpansive semigroup.

We now prove existence of common fixed points for total asymptotically nonexpansive semigroups in uniformly convex Banach spaces.

Theorem 3.3. *Let S be a left reversible semitopological semigroup, C be a nonempty closed convex subset of a uniformly convex Banach space X, and $\mathfrak{T} = \{T_s : s \in S\}$ be a total asymptotically nonexpansive semigroup of C into itself. Then $F(\mathfrak{T}) \neq \emptyset$ if and only if $\{T_sx : s \in S\}$ is bounded for some $x \in C$.*

Proof. The necessity is obvious. Conversely, we assume that $x \in C$ such that $\{T_sx : s \in S\}$ is bounded. We first show that

$$\text{if } T_sx \to y \text{ for } x, y \in C, \text{ then } y \in F(\mathfrak{T}). \tag{3.1}$$

Let $\varepsilon > 0$ be given. Fix $t \in S$. By the continuity of T_t at y, there exists $\delta > 0$ such that $\|x - y\| < \delta$ implies $\|T_tx - T_ty\| < \frac{\varepsilon}{2}$ for $x \in C$. Since $\|T_sx - y\| \to 0$ as

$s \to \infty_{\mathbb{R}}$, there exists $w \in S$ such that $\|T_{aw}x - y\| < \min\{\frac{\varepsilon}{2}, \delta\}$ for each $a \in S$. Then $\|T_t T_{aw}x - T_t y\| < \frac{\varepsilon}{2}$. So, we have

$$\|T_t y - y\| \leq \|T_t y - T_{taw}x\| + \|T_{taw}x - y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since ε is arbitrary, we get that $T_t y = y$ for each $t \in S$, so $y \in F(\mathfrak{T})$. Therefore, we obtain (3.1).

Since $\{T_s x : s \in S\}$ is bounded, there exists a unique element $z \in C$ such that $z \in A(C, \{T_s x\})$. Then

$$R := r(z, \{T_s x\}) = r(C, \{T_s x\}) = \inf_{y \in C} r(y, \{T_s x\}).$$

If $R = 0$, then $\limsup_s \|z - T_s x\| = 0$ and so $T_s x \to z$. It implies by (3.1) that $z \in F(\mathfrak{T})$. Next, we assume $R > 0$. Suppose that $z \notin F(\mathfrak{T})$. This implies from (3.1) that $\{T_s z\}$ does not converge to z. Then, there exists $\varepsilon > 0$ and a subnet $\{s_\alpha\}$ in S such that

$$s_\alpha \succeq \alpha \text{ and } \|z - T_{s_\alpha} z\| > \varepsilon \text{ for each } \alpha \in S. \tag{3.2}$$

We choose a positive number η such that

$$(R + \eta)\left(1 - \delta\left(\frac{\varepsilon}{R + \eta}\right)\right) < R.$$

Since $\mathfrak{T}$ is a total asymptotically nonexpansive semigroup, there exists $s_0 \in S$ such that

$$\begin{aligned} \|T_s z - T_s y\| &\leq \limsup_a \|T_a z - T_a y\| + \frac{\eta}{2} \leq \\ &\leq \limsup_a (\|z - y\| + k_a \phi(\|z - y\|) + \mu_a) + \frac{\eta}{2} = \|z - y\| + \frac{\eta}{2} \end{aligned} \tag{3.3}$$

for each $s \in S$ with $s \succeq s_0$, and $y \in C$. It is known from [1] that

$$\inf_t \sup_s \|z - T_{ts}x\| = \limsup_u \|z - T_u x\|.$$

Then, $\inf_t \sup_s \|z - T_{ts}x\| = R$. So, there exists $t_0 \in S$ such that for all $t \in S$ with $t \succeq t_0$,

$$\|z - T_{ts}x\| < R + \frac{\eta}{2} \text{ for each } s \in S. \tag{3.4}$$

Since S is left reversible, there exists $\gamma \in S$ with $\gamma \succeq s_0$ and $\gamma \succeq t_0$. Then, by (3.2), we have $s_\gamma \succeq \gamma$ and

$$\|z - T_{s_\gamma} z\| > \varepsilon. \tag{3.5}$$

Let $t \succeq s_\gamma \gamma$. By the left reversibility of S, we get $t \in \{s_\gamma \gamma\} \cup \overline{s_\gamma \gamma S}$. Then we may assume $t \in \overline{s_\gamma \gamma S}$. So, there exists $\{t_\beta\}$ in S such that $s_\gamma \gamma t_\beta \to t$. It follows by (3.3) and (3.4) that

$$\|T_{s_\gamma} z - T_{s_\gamma} T_{\gamma t_\beta} x\| \leq \|z - T_{\gamma t_\beta} x\| + \frac{\eta}{2} \leq R + \eta \quad \text{for each } \beta.$$

By (3.4) and $s_\gamma \gamma t_\beta \to t$, we have

$$\|T_{s_\gamma} z - T_t x\| \leq R + \frac{\eta}{2} < R + \eta \quad \text{for all } t \succeq s_\gamma \gamma. \tag{3.6}$$

Again, by (3.4), we get that $\|z - T_{s_\gamma} T_{\gamma t_\beta} x\| < R + \frac{\eta}{2} < R + \eta$ for each β. Since $s_\gamma \gamma t_\beta \to t$, we have

$$\|z - T_t x\| \leq R + \frac{\eta}{2} < R + \eta \quad \text{for all } t \succeq s_\gamma \gamma. \tag{3.7}$$

So, by the uniform convexity of X, (3.5), (3.6), and (3.7), we have

$$\left\| \frac{z + T_{s_\gamma} z}{2} - T_t x \right\| \leq (R + \eta) \left[1 - \delta \left(\frac{\varepsilon}{R + \eta} \right) \right].$$

This implies that

$$r\left(\frac{z + T_{s_\gamma} z}{2}, \{T_t x\} \right) \leq (R + \eta) \left[1 - \delta \left(\frac{\varepsilon}{R + \eta} \right) \right] < R,$$

which is a contradiction. Hence, $z \in F(\mathfrak{T})$. □

As direct consequence of Theorem 3.3, we obtain the following lemma.

Lemma 3.4. *Let S be a left reversible semitopological semigroup, C be a nonempty bounded closed convex subset of a uniformly convex Banach space X, and $\mathfrak{T} = \{T_s : s \in S\}$ be a total asymptotically nonexpansive semigroup of C into itself. Then $F(\mathfrak{T}) \neq \emptyset$.*

We present the following property of total asymptotically nonexpansive semigroups in uniformly convex Banach spaces.

Theorem 3.5. *Let S be a left or right reversible semitopological semigroup, C be a nonempty closed convex subset of a uniformly convex Banach space X. If $\mathfrak{T} = \{T_s : s \in S\}$ is a total asymptotically nonexpansive semigroup of C into itself with $F(\mathfrak{T}) \neq \emptyset$, then $F(\mathfrak{T})$ is a closed convex subset of C.*

Proof. To show that $F(\mathfrak{T})$ is closed, we let $\{x_t\}$ be a net in $F(\mathfrak{T})$ such that $x_t \to x$. Since T_t is total asymptotically nonexpansive, we have

$$\|T_t x - x\| \leq \|T_t x - x_t\| + \|x - x_t\| \leq 2\|x - x_t\| + k_t \phi(\|x - x_t\|) + \mu_t.$$

This implies that $T_t x \to x$. Hence, $x \in F(\mathfrak{T})$ so that $F(\mathfrak{T})$ is closed.

Next, we show that $F(\mathfrak{T})$ is convex. Let $x, y \in F(\mathfrak{T})$ and $z = \frac{x+y}{2}$. For $t \in S$, we have

$$\|T_t z - x\| \leq \|z - x\| + k_t \phi(\|z - x\|) + \mu_t = \frac{1}{2}\|x - y\| + k_t \phi\left(\frac{1}{2}\|x - y\| \right) + \mu_t$$

and

$$\|T_t z - y\| \leq \|z - y\| + k_t \phi(\|z - y\|) + \mu_t = \frac{1}{2}\|x - y\| + k_t \phi\left(\frac{1}{2}\|x - y\| \right) + \mu_t.$$

By the uniform convexity of X, we have

$$\|z - T_t z\| \le \xi_t \left[1 - \delta_X\left(\frac{\|x-y\|}{\xi_t}\right)\right],$$

where $\xi_t = \frac{1}{2}\|x-y\| + k_t\phi\left(\frac{1}{2}\|x-y\|\right) + \mu_t$. This implies that $T_t z \to z$. So, we have $z \in F(\mathfrak{T})$. Hence, $F(\mathfrak{T})$ is convex. □

Taking $S = \mathbb{N}$ in Theorems 3.3 and 3.5, we obtain the following existence theorem of total asymptotically nonexpansive mappings in uniformly convex Banach spaces.

Theorem 3.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and $T : C \to C$ be a continuous total asymptotically nonexpansive mapping. Then $F(T) \neq \emptyset$ if and only if $\{T^n x : n \in \mathbb{N}\}$ is bounded for some $x \in C$. Moreover, $F(T)$ is a closed convex subset of C.*

4. WEAK CONVERGENCE THEOREMS IN UNIFORMLY CONVEX BANACH SPACES

In this section, we study the weak convergence theorems for total asymptotically nonexpansive semigroups in uniformly convex Banach spaces.

Lemma 4.1. *Let S be a right reversible semitopological semigroup, C be a nonempty closed convex subset of a uniformly convex Banach space X, and $\mathfrak{T} = \{T_s : s \in S\}$ be a total asymptotically nonexpansive semigroup of C into itself with $F(\mathfrak{T}) \neq \emptyset$. Then $\lim_s \|T_s x - z\|$ exists for each $z \in F(\mathfrak{T})$.*

Proof. Let $z \in F(\mathfrak{T})$ and $R = \inf_s \|T_s x - z\|$. For $\varepsilon > 0$, there is $s_0 \in S$ such that

$$\|T_{s_0} x - z\| < R + \frac{\varepsilon}{2}.$$

Since $\mathfrak{T}$ is a total asymptotically nonexpansive semigroup, there exists $t_0 \in S$ such that

$$\begin{aligned}\|T_t T_{s_0} x - z\| &\le \limsup_u \|T_u T_{s_0} x - z\| + \frac{\varepsilon}{2} \le \\ &\le \limsup_u (\|T_{s_0} x - z\| + k_u\phi(\|T_{s_0} x - z\|) + \mu_u) + \frac{\varepsilon}{2} = \|T_{s_0} x - z\| + \frac{\varepsilon}{2}\end{aligned}$$

for each $t \succeq t_0$. Let $b \succeq t_0 s_0$. Since S is right reversible, we have $b \in \{t_0 s_0\} \cup \overline{S t_0 s_0}$. Then we may assume $b \in \overline{S t_0 s_0}$. So, there exists $\{s_\alpha\}$ in S such that $s_\alpha t_0 s_0 \to b$. Therefore, $\|T_{s_\alpha t_0 s_0} x - z\| \le \|T_{s_0} x - z\| + \frac{\varepsilon}{2}$ for each α. Hence, $\|T_b x - z\| \le \|T_{s_0} x - z\| + \frac{\varepsilon}{2}$. This implies that

$$R \le \inf_s \sup_{t \succeq s} \|T_t x - z\| \le \sup_{b \succeq t_0 s_0} \|T_b x - z\| \le \|T_{s_0} x - z\| + \frac{\varepsilon}{2} < R + \varepsilon.$$

Since ε is arbitrary, we get

$$\inf_s \sup_{t\succeq s} \|T_t x - z\| = R = \inf_s \|T_s x - z\|.$$

Thus, $\lim_s \|T_s x - z\|$ exists. □

Theorem 4.2. *Let S be a right reversible semitopological semigroup, C be a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property, and $x \in C$. Assume that $\mathfrak{T} = \{T_s : s \in S\}$ is a total asymptotically nonexpansive semigroup of C into itself with $F(\mathfrak{T}) \neq \emptyset$. If $\lim_s \|T_s x - T_{ts} x\| = 0$ for all $t \in S$, then $\{T_s x : s \in S\}$ converges weakly to a common fixed point of the semigroup $\mathfrak{T}$.*

Proof. By Lemma 4.1, we have $\lim_s \|T_s x - z\|$ exists for each $z \in F(\mathfrak{T})$, and so $\{T_s x : s \in S\}$ is bounded. Since X is uniformly convex, we have that X is reflexive. Then, there exists a subnet $\{T_{s_\alpha} x\}$ of $\{T_s x\}$ such that $\{T_{s_\alpha} x\}$ converges weakly to $y \in C$. We will show that $y \in F(\mathfrak{T})$. Let $\varepsilon > 0$. Since $\mathfrak{T}$ is a total asymptotically nonexpansive semigroup, there exists $t_0 \in S$ such that

$$\begin{aligned}\|T_t T_{s_\alpha} x - T_t y\| &\leq \limsup_a \|T_a T_{s_\alpha} x - T_a y\| + \varepsilon \leq \\ &\leq \limsup_a (\|T_{s_\alpha} x - y\| + k_a \phi(\|T_{s_\alpha} x - y\|) + \mu_a) + \varepsilon \leq \|T_{s_\alpha} x - y\| + \varepsilon\end{aligned}$$

for each $t \succeq t_0$ and each β. This implies that

$$\|T_{s_\alpha} x - T_t y\| \leq \|T_{s_\alpha} x - T_{ts_\alpha} x\| + \|T_{ts_\alpha} x - T_t y\| \leq \|T_{s_\alpha} x - T_{ts_\alpha} x\| + \|T_{s_\alpha} x - y\| + \varepsilon$$

for each $t \succeq t_0$ and each β. By the assumption that $\lim_s \|T_s x - T_{ts} x\| = 0$ for all $t \in S$, we have $\limsup_\alpha \|T_{s_\alpha} x - T_t y\| \leq \limsup_\alpha \|T_{s_\alpha} x - y\| + \varepsilon$ for all $t \succeq t_0$. Since ε is arbitrary, we get

$$\limsup_\alpha \|T_{s_\alpha} x - T_t y\| \leq \limsup_\alpha \|T_{s_\alpha} x - y\|$$

for all $t \succeq t_0$. By the Opial property, we get that $T_t y = y$ for all $t \succeq t_0$. So, $T_t y \to y$. This implies by (3.1) that $y \in F(\mathfrak{T})$. By Lemma 4.1, $\lim_s \|T_s x - y\|$ exists. Now, we show that $\{T_s x\}$ converges weakly to y. To show this, suppose not. Then, there must exist a subnet $\{T_{s_\beta} x\}$ of $\{T_s x\}$ such that $\{T_{s_\beta} x\}$ converges weakly to $u \in C$ and $u \neq y$. Again, as above, we can conclude that $u \in F(\mathfrak{T})$. By Lemma 4.1 and the Opial property, we have

$$\begin{aligned}\limsup_s \|T_s x - y\| &= \limsup_\alpha \|T_{s_\alpha} x - y\| < \limsup_\alpha \|T_{s_\alpha} x - u\| = \\ &= \limsup_s \|T_s x - u\| = \limsup_\beta \|T_{s_\beta} x - u\| < \\ &< \limsup_\beta \|T_{s_\beta} x - y\| = \limsup_s \|T_s x - y\|\end{aligned}$$

which is a contradiction. Hence, $\{T_s x\}$ converges weakly to a common fixed point of the semigroup $\mathfrak{T}$. □

Taking $S = \mathbb{N}$ in Theorem 4.2, we obtain the following weak convergence theorem of total asymptotically nonexpansive mappings in uniformly convex Banach spaces.

Theorem 4.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property and $x \in C$. Assume that $T : C \to C$ is a continuous total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. If $\lim_{n\to\infty} \|T^n x - T^{n+1}x\| = 0$, then $\{T^n x : n \in \mathbb{N}\}$ converges weakly to a fixed point of T.*

5. STRONG CONVERGENCE THEOREMS IN UNIFORMLY CONVEX BANACH SPACES

In this section, we prove two strong convergence theorems for total asymptotically nonexpansive semigroups in uniformly convex Banach spaces.

Theorem 5.1. *Let S be a right reversible semitopological semigroup, C be a nonempty closed convex subset of a uniformly convex Banach space X, and $x \in C$. Assume that $\mathfrak{T} = \{T_s : s \in S\}$ is a total asymptotically nonexpansive semigroup of C into itself with $F(\mathfrak{T}) \neq \emptyset$. Then $\{\pi T_s x\}$ converges strongly to a common fixed point of the semigroup $\mathfrak{T}$, where $\pi : C \to F(\mathfrak{T})$ is the metric projection. Moreover, if S is reversible, then $Px := \lim_s \pi T_s x$ is the unique asymptotic center of the net $\{T_s x : s \in S\}$.*

Proof. By Theorem 3.5, $F(\mathfrak{T})$ is closed and convex. So, the mapping π is well defined. Put $R = \inf_s \|T_s x - \pi T_s x\|$. As in the proof of Lemma 4.1, we have

$$R = \inf_s \|T_s x - \pi T_s x\| = \limsup_s \|T_s x - \pi T_s x\|.$$

We will show that $\{\pi T_s x\}$ is a Cauchy net. To show this, we divide into two cases.
Case 1. $R = 0$. For $\varepsilon > 0$, there exists $s_0 \in S$ such that

$$\|T_s x - \pi T_s x\| < \frac{\varepsilon}{4} \text{ for each } s \succeq s_0.$$

Since $\mathfrak{T}$ is a total asymptotically nonexpansive semigroup, there exists $t_0 \in S$ such that

$$\begin{aligned}\|T_{ts_0}x - \pi T_{s_0}x\| &\leq \limsup_u \|T_u T_{s_0}x - T_u \pi T_{s_0}x\| + \frac{\varepsilon}{4} = \\ &\leq \limsup_u (\|T_{s_0}x - \pi T_{s_0}x\| + k_u \phi(\|T_{s_0}x - \pi T_{s_0}x\|) + \mu_u) + \frac{\varepsilon}{4} = \\ &= \|T_{s_0}x - \pi T_{s_0}x\| + \frac{\varepsilon}{4}\end{aligned}$$

for each $t \succeq t_0$. Let $a, b \succeq t_0 s_0$. By the right reversibility of S, we have $a, b \in \{t_0 s_0\} \cup \overline{St_0 s_0}$. Then we may assume $a, b \in \overline{St_0 s_0}$. So, there exist $\{t_\alpha\}$ and $\{s_\beta\}$ in S such that $t_\alpha t_0 s_0 \to a$ and $s_\beta t_0 s_0 \to b$. Therefore, we have

$$\begin{aligned}\|T_{t_\alpha t_0 s_0}x - T_{s_\beta t_0 s_0}x\| &\leq \|T_{t_\alpha t_0 s_0}x - \pi T_{s_0}x\| + \|T_{s_\beta t_0 s_0}x - \pi T_{s_0}x\| \leq \\ &\leq 2\|T_{s_0}x - \pi T_{s_0}x\| + \frac{\varepsilon}{2}.\end{aligned}$$

This implies that

$$\|\pi T_a x - \pi T_b x\| \leq 2\|T_{s_0}x - \pi T_{s_0}x\| + \frac{\varepsilon}{2} < 2\left(\frac{\varepsilon}{4}\right) + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $\{\pi T_s x\}$ is a Cauchy net.
Case 2. $R > 0$. Suppose that $\{\pi T_s x\}$ is not a Cauchy net. Then, there exists $\varepsilon > 0$ such that for any $s \in S$, there are $a_s, b_s \in S$ with $a_s, b_s \succeq s$ and $\|\pi T_{a_s}x - \pi T_{b_s}x\| \geq \varepsilon$.

We choose a positive number η such that

$$(R+\eta)\left(1 - \delta\left(\frac{\varepsilon}{R+\eta}\right)\right) < R.$$

So, there exists $u_0 \in S$ such that

$$\|T_t x - \pi T_t x\| \leq R + \frac{\eta}{2} \text{ for each } t \succeq u_0. \tag{5.1}$$

Then $\|\pi T_{a_{u_0}}x - \pi T_{b_{u_0}}x\| \geq \varepsilon$. Since $\mathfrak{T}$ is a total asymptotically nonexpansive semigroup, there exists $v_0 \in S$ such that

$$\begin{aligned}\|T_t T_s x - \pi T_s x\| &\leq \limsup_u \|T_u T_s x - \pi T_s x\| + \frac{\eta}{2} \leq \\ &\leq \limsup_u (\|T_s x - \pi T_s x\| + k_u \phi(\|T_s x - \pi T_s x\|) + \mu_u) + \frac{\eta}{2} = \\ &= \|T_s x - \pi T_s x\| + \frac{\eta}{2}\end{aligned} \tag{5.2}$$

for each $t \succeq v_0$ and each $s \in S$.

Since S is right reversible, there exists $c \in S$ such that $c \succeq v_0 a_{u_0}$ and $c \succeq v_0 b_{u_0}$. Then, there exist $\{t_\alpha\}$ and $\{s_\beta\}$ in S such that $t_\alpha v_0 a_{u_0} \to c$ and $s_\beta v_0 b_{u_0} \to c$. So, by (5.1) and (5.2), we have

$$\|T_{t_\alpha v_0 a_{u_0}}x - \pi T_{a_{u_0}}x\| \leq \|T_{a_{u_0}}x - \pi T_{a_{u_0}}x\| + \frac{\eta}{2} \leq R + \eta$$

and

$$\|T_{s_\beta v_0 b_{u_0}}x - \pi T_{b_{u_0}}x\| \leq \|T_{b_{u_0}}x - \pi T_{b_{u_0}}x\| + \frac{\eta}{2} \leq R + \eta.$$

This implies

$$\|T_c x - \pi T_{a_{u_0}}x\| \leq R + \eta \text{ and } \|T_c x - \pi T_{b_{u_0}}x\| \leq R + \eta.$$

By the uniform convexity of X, we get

$$\left\|T_c x - \frac{\pi T_{a_{u_0}}x + \pi T_{b_{u_0}}x}{2}\right\| \leq (R+\eta)\left(1 - \delta\left(\frac{\varepsilon}{R+\eta}\right)\right) < R.$$

Since π is the metric projection of C onto $F(\mathfrak{T})$, we have

$$\|T_c x - \pi T_c x\| \leq \left\| T_c x - \frac{\pi T_{a_{u_0}} x + \pi T_{b_{u_0}} x}{2} \right\| < R.$$

This contradicts with $R = \inf_s \|T_s x - \pi T_s x\|$. Then $\{\pi T_s x\}$ is a Cauchy net.

By Case 1 and Case 2, we get that $\{\pi T_s x\}$ is Cauchy in a closed subset $F(\mathfrak{T})$ of a Banach space X, hence it converges to some point in $F(\mathfrak{T})$, say Px.

Finally, by Lemma 4.1, we have $\{T_s x : s \in S\}$ is bounded. So, let $z \in A(C, \{T_s x\})$. Since S is reversible and by the same arguments as in the proof of Theorem 3.3, we have $z \in F(\mathfrak{T})$. Thus, by the property of π, we obtain

$$\begin{aligned} \limsup_s \|T_s x - Px\| &\leq \limsup_s (\|T_s x - \pi T_s x\| + \|\pi T_s x - Px\|) = \\ &= \limsup_s \|T_s x - \pi T_s x\| \leq \limsup_s \|T_s x - z\|. \end{aligned}$$

This implies, by the uniqueness of asymptotic centers, that $Px = z$. □

Taking $S = \mathbb{N}$ in Theorem 5.1, we obtain the following strong convergence theorem of total asymptotically nonexpansive mappings in uniformly convex Banach spaces.

Theorem 5.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and $x \in C$. Assume that $T : C \to C$ is a continuous total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Then $\{\pi T^n x\}$ converges strongly to a fixed point of T, where $\pi : C \to F(T)$ is the metric projection. Moreover, $Px := \lim_{n\to\infty} \pi T^n x$ is the unique asymptotic center of the sequence $\{T^n x : n \in \mathbb{N}\}$.*

In the case $S = [0, \infty)$, we obtain a strong convergence theorem of modified Mann iteration for uniformly Lipschitzian and total asymptotically nonexpansive semigroups in uniformly convex Banach spaces. Recall that a semigroup $\mathfrak{T} = \{T_t : t \in [0, \infty)\}$ is said to be *uniformly Lipschitzian* if there exists a nonnegative real number L_t for $t \in [0, \infty)$ such that

$$\|T_t x - T_t y\| \leq L_t \|x - y\|$$

for all $x, y \in C$ and $t \in [0, \infty)$.

Theorem 5.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let $\mathfrak{T} = \{T_t : t \in [0, \infty)\}$ be a uniformly Lipschitzian with L_t such that $L = \sup_{t\in[0,\infty)} L_t < \infty$ and be a total asymptotically nonexpansive semigroup of C into itself with k_{t_n}, μ_{t_n} such that $\sum_{n=1}^{\infty} k_{t_n} < \infty$ and $\sum_{n=1}^{\infty} \mu_{t_n} < \infty$. Assume that $F(\mathfrak{T}) \neq \emptyset$ and there exist positive constants M and M^* such that $\phi(\lambda) \leq M^* \lambda$ for all $\lambda \geq M$. Let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{t_n} x_n, \;\; n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\alpha_n \in [a, b] \subset (0, 1)$ and $\{t_n\}$ is an increasing sequence in $[0, \infty)$. If the following conditions are satisfied:

(C1) *for any bounded subset B of C,*

$$\lim_{n\to\infty} \sup_{x\in B, s\in[0,\infty)} \|T_{s+t_n}x - T_{t_n}x\| = 0,$$

(C2) *there exist a compact subset K of X such that $\bigcap_{t\in[0,\infty)} T_t(C) \subset K$,*

then the sequence $\{x_n\}$ converges strongly to a common fixed point of the semigroup $\mathfrak{T}$.

Proof. Let $p \in F(\mathfrak{T})$. Then, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|T_{t_n}x_n - p\| \leq \\ &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)(\|x_n - p\| + k_{t_n}\phi(\|x_n - p\|) + \mu_{t_n}) = \\ &= \|x_n - p\| + (1-\alpha_n)(k_{t_n}\phi(\|x_n - p\|) + \mu_{t_n}). \end{aligned} \tag{5.3}$$

Since ϕ is an increasing function, it results that $\phi(\lambda) \leq \phi(M)$ if $\lambda \leq M$ and $\phi(\lambda) \leq M^*\lambda$ if $\lambda \geq M$. In either case, we can obtain that $\phi(\lambda) \leq \phi(M) + M^*\lambda$. Then, (5.3) becomes

$$\begin{aligned} \|x_{n+1} - p\| &\leq \|x_n - p\| + (1-\alpha_n)(k_{t_n}(\phi(M) + M^*\|x_n - p\|) + \mu_{t_n}) = \\ &= (1 + (1-a)M^*k_{t_n})\|x_n - p\| + \theta_n \end{aligned} \tag{5.4}$$

where $\theta_n = (1-a)(k_{t_n}\phi(M) + \mu_{t_n})$. By $\sum_{n=1}^{\infty} k_{t_n} < \infty$, $\sum_{n=1}^{\infty} \mu_{t_n} < \infty$ and Lemma 2.3 we have that $\lim_{n\to\infty} \|x_n - p\|$ exists. So, the sequence $\{x_n\}$ is bounded, and there is $c \geq 0$ such that

$$\lim_{n\to\infty} \|x_n - p\| = c. \tag{5.5}$$

Since $\|T_{t_n}x_n - p\| \leq \|x_n - p\| + k_{t_n}\phi(\|x_n - p\|) + \mu_{t_n}$ and ϕ is a continuous function, we have

$$\limsup_{n\to\infty} \|T_{t_n}x_n - p\| \leq c. \tag{5.6}$$

From $\lim_{n\to\infty} \|x_{n+1} - p\| = c$, we get that

$$\lim_{n\to\infty} \|\alpha_n(x_n - p) + (1-\alpha_n)(T_{t_n}x_n - p)\| = c. \tag{5.7}$$

It follows by (5.5), (5.6), (5.7) and Lemma 2.2, we can conclude that

$$\lim_{n\to\infty} \|x_n - T_{t_n}x_n\| = 0. \tag{5.8}$$

By condition (C2), there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\to\infty} T_{t_{n_i}}x_{n_i} = q$ for some $q \in K$. Since $\|x_{n_i} - q\| \leq \|x_{n_i} - T_{t_{n_i}}x_{n_i}\| + \|T_{t_{n_i}}x_{n_i} - q\|$, by (5.8) we see that $\lim_{i\to\infty} x_{n_i} = q$. For any $t \in S$, we have

$$\begin{aligned} \|x_{n_i} - T_t x_{n_i}\| &\leq \|x_{n_i} - T_{t_{n_i}}x_{n_i}\| + \|T_{t_{n_i}}x_{n_i} - T_{t+t_{n_i}}x_{n_i}\| + \|T_{t+t_{n_i}}x_{n_i} - T_t x_{n_i}\| = \\ &= \|x_{n_i} - T_{t_{n_i}}x_{n_i}\| + \|T_{t_{n_i}}x_{n_i} - T_{t+t_{n_i}}x_{n_i}\| + \|T_t T_{t_{n_i}}x_{n_i} - T_t x_{n_i}\| \leq \\ &\leq (1+L)\|x_{n_i} - T_{t_{n_i}}x_{n_i}\| + \sup_{y\in\{x_{n_i}\}, s\in[0,\infty)} \|T_{t_{n_i}}y - T_{s+t_{n_i}}y\|. \end{aligned}$$

By condition (C1) and (5.8), we get

$$\lim_{i\to\infty} \|x_{n_i} - T_t x_{n_i}\| = 0.$$

This implies that

$$\begin{aligned}\|q - T_t q\| &\leq \|q - x_{n_i}\| + \|x_{n_i} - T_t x_{n_i}\| + \|T_t x_{n_i} - T_t q\| \leq \\ &\leq (1+L)\|q - x_{n_i}\| + \|x_{n_i} - T_t x_{n_i}\| \to 0 \text{ as } i \to \infty.\end{aligned}$$

Thus, $q = T_t q$ for all $t \in S$. So, $q \in F(\mathfrak{T})$. Since $\lim_{n\to\infty} \|x_n - q\|$ exists, we have that $\lim_{n\to\infty} x_n = q \in F(\mathfrak{T})$. □

Remark 5.4.

(i) It is known that every commutative semigroup is both left and right reversible and every discrete amenable semigroup is reversible. Then Theorems 3.3, 3.5, 4.2, and 5.1 are also obtained for a class of commutative and discrete amenable semigroups.

(ii) Theorem 4.2 extends and generalizes the results of [22,24] to total asymptotically nonexpansive semigroups.

Acknowledgments
The authors would like to thank the referees for valuable suggestions on the paper. The first author would like to thank Chiang Mai University, Thailand for financial support.

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Received: July 22, 2013.
Revised: November 4, 2013.
Accepted: November 8, 2013.