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APPROXIMATING FIXED POINTS OF A COUNTABLE FAMILY OF STRICT PSEUDOCONTRACTIONS IN BANACH SPACES

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Abstract. We prove the strong convergence of the modified Mann-type iterative scheme for a countable family of strict pseudocontractions in q-uniformly smooth Banach spaces. Our results mainly improve and extend the results announced in [Y. Yao, H. Zhou, Y.-C. Liou, *Strong convergence of a modified Krasnoselski-Mann iterative algorithm for non-expansive mappings*, J. Appl. Math. Comput. **29** (2009), 383–389].

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1. INTRODUCTION

Let E and E^* be a real Banach space and the dual space of E, respectively. Let J_q $(q > 1)$ denote the generalized duality mapping from E into 2^{E^*} given by

$$J_q(x) = \{f \in E^* : \quad \langle x, f\rangle = \|x\|^q, \ \|f\| = \|x\|^{q-1}\}$$

for each $x \in E$, where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between E and E^*. It is well known that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \neq 0$. If E is smooth then J_q is single-valued, which is denoted by j_q. The duality mapping J from a smooth Banach space E into E^* is said to be weakly sequentially continuous if x_n weak convergent to x implies Jx_n weak* convergent to Jx. We denote the fixed point set of a nonlinear mapping $T : C \to E$ by $F(T) = \{x \in C : \ Tx = x\}$.

Definition 1.1. A mapping T with domain $D(T)$ and range $R(T)$ in E is called
(i) λ-*strictly pseudocontractive* [5] if there exists a constant $\lambda > 0$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \leq \|x - y\|^q - \lambda\|(I - T)x - (I - T)y\|^q$$

for all $x, y \in D(T)$ and for some $j_q(x-y) \in J_q(x-y)$; or equivalently to

$$\langle (I-T)x - (I-T)y, j_q(x-y)\rangle \geq \lambda \|(I-T)x - (I-T)y\|^q;$$

(ii) *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in D(T)$.

If $0 < L < 1$, then T is a contraction and if $L = 1$, then T is a nonexpansive mapping. By definition, we see that every λ-strictly pseudocontractive mapping is $(\frac{1+\lambda}{\lambda})$-Lipschitzian (see [10]).

Remark 1.2. Let C be a nonempty subset of a real Hilbert space and $T : C \to C$ be a mapping. Then T is said to be κ-strictly pseudocontractive [5] if there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I-T)x - (I-T)y\|^2 \tag{1.1}$$

for all $x, y \in D(T)$. It is known that (1.1) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-\kappa}{2}\|(I-T)x - (I-T)y\|^2.$$

In 1953, Mann [18] introduced the following iteration: $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 1, \tag{1.2}$$

where $\{\alpha_n\} \subset (0, 1)$. It is known as a *Mann iteration*. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen by $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then $\{x_n\}$ generated by (1.2) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [23]).

In 1967, Browder-Petryshyn [5] introduced the class of strict pseudocontractions and proved existence and weak convergence theorems in a real Hilbert setting by using the Mann iterative algorithm (1.2) with a constant sequence $\alpha_n = \alpha$ for all n. Recently, Marino-Xu [19] and Zhou [34] extended the results of Browder-Petryshyn [5] to Mann's iteration process (1.2). Zhou [36] also investigated the weak convergence in a 2-uniformly smooth Banach space. In a much more general setting, Osilike-Udomene [21], Zhang-Su [33], Zhang-Guo [32] and Zhou [37] investigated the weak convergence in a q-uniformly smooth Banach space. Since 1967, the construction of fixed points for pseudocontractions via the iterative process has been extensively investigated by many authors (see, e.g., [7–9, 22]).

In 1967, Halpern [15] introduced the following iteration which is the so-called *Halpern iteration*: $x_1 \in C$ and

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 1, \tag{1.3}$$

where $\{\alpha_n\} \subset (0, 1)$ and $u \in C$. It was proved, in a real Hilbert space, the convergence of $\{x_n\}$ to a fixed point of T, where $\alpha_n := n^{-a},\ a \in (0, 1)$.

In 1977, Lions [17] obtained a strong convergence of (1.3) still in a real Hilbert space provided the real sequence $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$,
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(C3) $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0$.

Reich [25] extended Halpern's result to a uniformly smooth Banach space. However, both Halpern's and Lions' conditions imposed on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n+1)$.

In 1992, Wittmann [27] proved, in a real Hilbert space, that the sequence $\{x_n\}$ converges strongly to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$,
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Shioji-Takahshi [26] extended Wittmann's result to real Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex and bounded subset has the fixed point property for nonexpansive mappings. The concept of a Halpern iterative scheme has been widely used to approximate the fixed points for nonexpansive mappings (see, e.g., [2,12,16,24,28,29] and the reference cited therein).

In 2009, Yao *et al.* introduced in [31] a new modified Mann iterative algorithm for a nonexpansive mapping in a real Hilbert space.

Algorithm 1.3. For given $x_1 \in H$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$\begin{aligned} y_n &= (1-\alpha_n)x_n, \\ x_{n+1} &= (1-\beta_n)y_n + \beta_n T y_n, \quad n \geq 1. \end{aligned} \tag{1.4}$$

They proved the following strong convergence theorem for a nonexpansive mapping in a real Hilbert space.

Theorem 1.4. *Let H be a real Hilbert space. Let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$. Assume the following conditions are satisfied:*

(C1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(C2) $\lim_{n\to\infty} \alpha_n = 0$,
(C3) $\beta_n \in [a,b] \subset (0,1)$.

Then the sequence $\{x_n\}$ and $\{y_n\}$ generated by (1.4) *strongly converge to a fixed point of T.*

Motivated and inspired by Marino-Xu [19], Zhang-Su [33], Zhou [34–37] and Yao *et al.* [31], we consider the following modified Mann-type iteration: $x_1 \in C$ and

$$\begin{aligned} y_n &= Q_C\big((1-\alpha_n)x_n\big), \\ x_{n+1} &= (1-\beta_n)y_n + \beta_n T_n y_n, \quad n \geq 1, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ and $\{T_n\}_{n=1}^{\infty}$ is a countable family of strict pseudocontractions on a nonempty, closed and convex subset C of a real Banach space E.

It is our purpose in this paper to prove a strong convergence of the modified Mann-type iteration process (1.5) in the framework of q-uniformly smooth Banach spaces for a countable family of strict pseudocontractions. The obtained results improve and extend those of Yao *et al.* [31] in several aspects.

We will use the notation:
$\rightharpoonup$ for weak convergence and $\to$ for strong convergence,
$\omega_\omega(x_n) = \{x : \ x_{n_i} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2. PRELIMINARIES

A Banach space E is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space E is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \ \|x\|, \|y\| \leq 1, \ \|x - y\| \geq \epsilon\right\},$$

for all $\epsilon \in [0, 2]$. E is uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. It is known that every uniformly convex Banach space is strictly convex and reflexive. Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E)$. In this case E is called *smooth*. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$. The norm of E is called *uniformly Fréchet differentiable*, if the limit is attained uniformly for $x, y \in S(E)$. It is well known that (uniformly) *Fréchet differentiability* of the norm of E implies (uniformly) *Gâteaux differentiability* of the norm of E.

Let $\rho_E : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \ x \in S(E), \ \|y\| \leq t\right\}.$$

A Banach space E is said to be *uniformly smooth* if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Let $q > 1$, then E is said to be *q-uniformly smooth* if there exists $c > 0$ such that $\rho_E(t) \leq ct^q$. It is easy to see that if E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth. It is well known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable, and hence the norm of E is Fréchet differentiable. For more details, we refer the reader to [1, 11].

In 1972, Gossez and Lami [14] gave some geometric properties related to the fixed point theory for nonexpansive mappings. They proved that a space with a weakly continuous duality map satisfies Opial's condition [20]. Conversely, if a space satisfies Opial's condition and has a uniformly Gâteaux differentiable norm, then it has a weakly continuous zero duality map.

Let E be a real Banach space, C a nonempty, closed and convex subset of E, and K a nonempty subset of C. Let $Q : C \to K$. Then Q is said to be

– *sunny* if for each $x \in C$ and $t \in [0,1]$, we have

$$Q\big(tx + (1-t)Qx\big) = Qx;$$

– a *retraction* of C onto K if

$$Qx = x \quad \text{for each} \quad x \in K;$$

– a *sunny nonexpansive retraction* if Q is sunny, nonexpansive and retraction onto K. See also Bruck [6], Goebel-Reich [13] and Reich [24].

In the sequel, we shall need the following lemmas.

Lemma 2.1 ([30]). *Let E be a real q-uniformly smooth Banach space. Then the following inequality holds:*

$$\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q,$$

for all $x, y \in E$ and for some $C_q > 0$.

Lemma 2.2 ([37]). *Let C be a nonempty, closed and convex subset of a q-uniformly smooth Banach space E. Suppose that the generalized duality mapping $J_q : E \to E^*$ is weakly sequentially continuous at zero. Let $T : C \to E$ be a λ-strict pseudocontraction with $0 < \lambda < 1$. Then for any $\{x_n\} \subset C$, if $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \to y \in E$, then $x - Tx = y$.*

Lemma 2.3 ([29]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \ge 1,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that:

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(b) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

To deal with a family of mappings, the following conditions are introduced: Let C be a subset of a real Banach space E and let $\{T_n\}_{n=1}^{\infty}$ be a family of mappings of C such that $\bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$. Then $\{T_n\}$ is said to satisfy the *AKTT-condition* [2] if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in B\} < \infty.$$

Lemma 2.4 ([2]). *Let C be a nonempty and closed subset of a Banach space E and let $\{T_n\}$ be a family of mappings of C into itself which satisfies the AKTT-condition, then the mapping $T : C \to C$ defined by*

$$Tx = \lim_{n\to\infty} T_n x \quad \text{for each} \quad x \in K$$

satisfies

$$\limsup_{n\to\infty}\{\|Tz - T_n z\| : z \in B\} = 0$$

for each bounded subset B of C.

The following results can be found in [3, 4].

Lemma 2.5 ([3, 4]). *Let C be a closed and convex subset of a smooth Banach space E. Suppose that $\{T_n\}_{n=1}^{\infty}$ is a family of λ-strictly pseudocontractive mappings from C into E with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\mu_n\}_{n=1}^{\infty}$ is a real sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \mu_n = 1$. Then the following conclusions hold:*

(1) *$G := \sum_{n=1}^{\infty} \mu_n T_n : C \to E$ is a λ-strictly pseudocontractive mapping,*
(2) *$F(G) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.6 ([4]). *Let C be a closed and convex subset of a smooth Banach space E. Suppose that $\{S_k\}_{k=1}^{\infty}$ is a countable family of λ-strictly pseudocontractive mappings of C into itself with $\bigcap_{k=1}^{\infty} F(S_k) \neq \emptyset$. For each $n \in \mathbb{N}$, define $T_n : C \to C$ by*

$$T_n x = \sum_{k=1}^{n} \mu_n^k S_k x, \quad x \in C,$$

where $\{\mu_n^k\}$ is a family of nonnegative numbers satisfying:

(i) *$\sum_{k=1}^{n} \mu_n^k = 1$ for all $n \in \mathbb{N}$,*
(ii) *$\mu^k := \lim_{n\to\infty} \mu_n^k > 0$ for all $k \in \mathbb{N}$,*
(iii) *$\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\mu_{n+1}^k - \mu_n^k| < \infty$.*

Then:

(1) *Each T_n is a λ-strictly pseudocontractive mapping.*
(2) *$\{T_n\}$ satisfies AKTT-condition.*
(3) *If $T : C \to C$ is defined by*

$$Tx = \sum_{k=1}^{\infty} \mu^k S_k x, \quad x \in C,$$

then $Tx = \lim_{n\to\infty} T_n x$ and $F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{k=1}^{\infty} F(S_k)$.

In what follows, we will write $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}$ satisfies the AKTT-condition and T is defined by Lemma 2.4 with $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

3. MAIN RESULTS

Theorem 3.1. *Let E be a real q-uniformly smooth Banach space which satisfies Opial's condition and C a nonempty, closed and convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of λ-strict pseudocontractions $(0 < \lambda < 1)$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Assume that real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$ satisfy the following conditions:*

(C1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(C2) $\lim_{n\to\infty} \alpha_n = 0$,
(C3) $0 < a \leq \beta_n \leq \mu$, $\mu = \min\left\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\right\}$.

Suppose that $(\{T_n\}, T)$ satisfies the AKTT-condition. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.5) converge strongly to a common fixed point of $\{T_n\}_{n=1}^{\infty}$.

Proof. First, we prove that $\{x_n\}$ is bounded. For each $p \in F$, it follows from Lemma 2.1 that

$$\begin{aligned}\|x_{n+1} - p\|^q &= \|(y_n - p) + \beta_n(T_n y_n - y_n)\|^q \leq \\ &\leq \|y_n - p\|^q + q\beta_n\langle T_n y_n - y_n, j_q(y_n - p)\rangle + C_q\beta_n^q\|y_n - T_n y_n\|^q \leq \\ &\leq \|y_n - p\|^q - q\lambda\beta_n\|y_n - T_n y_n\|^q + C_q\beta_n^q\|y_n - T_n y_n\|^q = \\ &= \|y_n - p\|^q - \beta_n(q\lambda - C_q\beta_n^{q-1})\|y_n - T_n y_n\|^q.\end{aligned} \tag{3.1}$$

This implies by (C3) that

$$\begin{aligned}\|x_{n+1} - p\| \leq \|y_n - p\| = \|Q_C\big((1-\alpha_n)x_n\big) - Q_C p\| \leq \|(1-\alpha_n)x_n - p\| \leq \\ \leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|p\| \leq \max\{\|x_n - p\|, \|p\|\}.\end{aligned} \tag{3.2}$$

By induction, we get that $\{x_n\}$ is bounded, so is $\{y_n\}$. Observing

$$\|y_n - T_n y_n\| = \frac{1}{\beta_n}\|y_n - x_{n+1}\|, \tag{3.3}$$

we have from (3.1) that

$$\begin{aligned}\|x_{n+1} - p\|^q &\leq \|y_n - p\|^q - \beta_n(q\lambda - C_q\beta_n^{q-1})\|y_n - T_n y_n\|^q = \\ &= \|y_n - p\|^q - (q\lambda\beta_n^{1-q} - C_q)\|y_n - x_{n+1}\|^q \leq \\ &\leq \|x_n - p - \alpha_n x_n\|^q - (q\lambda\mu^{1-q} - C_q)\|y_n - x_{n+1}\|^q \leq \\ &\leq \|x_n - p\|^q - q\alpha_n\langle x_n, j_q(x_n - p)\rangle + C_q\alpha_n^q\|x_n\|^q - \\ &\quad - (q\lambda\mu^{1-q} - C_q)\|y_n - x_{n+1}\|^q.\end{aligned} \tag{3.4}$$

Since $\{x_n\}$ is bounded, there exists a constant $M \geq 0$ such that

$$\|x_{n+1} - p\|^q - \|x_n - p\|^q + k\|y_n - x_{n+1}\|^q \leq M\alpha_n, \tag{3.5}$$

where $k = q\lambda\mu^{1-q} - C_q$.

To this end, we divide the proof into two cases.

Case 1. Assume that the sequence $\{\|x_n - p\|\}_{n=1}^{\infty}$ is monotone decreasing. Then $\{\|x_n - p\|\}_{n=1}^{\infty}$ is convergent for all $p \in F$. So from (3.5) and (C2) we get that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{3.6}$$

Combining (3.3) and (3.6), it follows from (C3) that

$$\lim_{n\to\infty} \|y_n - T_n y_n\| = 0. \tag{3.7}$$

On the other hand, we see that

$$\begin{aligned} \|y_n - x_n\| &= \|Q_C\big((1-\alpha_n)x_n\big) - Q_C x_n\| \leq \\ &\leq \|(1-\alpha_n)x_n - x_n\| = \alpha_n\|x_n\| \to 0, \end{aligned} \tag{3.8}$$

as $n \to \infty$. Hence

$$\begin{aligned} \|x_n - T_n x_n\| &\leq \|x_n - y_n\| + \|y_n - T_n y_n\| + \|T_n y_n - T_n x_n\| \leq \\ &\leq \Big(1 + \frac{1+\lambda}{\lambda}\Big)\|x_n - y_n\| + \|y_n - T_n y_n\|. \end{aligned}$$

From (3.7) and (3.8) we obtain that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.9}$$

Since $(\{T_n\}, T)$ satisfies the AKTT-condition, it follows from Lemma 2.4 and (3.9) that

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\| \leq \\ &\leq \|x_n - T_n x_n\| + \sup_{z\in\{x_n\}} \|T_n z - Tz\| \to 0, \end{aligned}$$

as $n \to \infty$. Note that E satisfies Opial's condition. Since E is q-uniformly smooth, the norm of E is uniformly Fréchet differentiable and hence the norm of E is uniformly Gâteaux differentiable. By Gossez and Dozo [14], we know that j_q is weakly sequentially continuous at zero. So by Lemma 2.2 we get that $\omega_\omega(x_n) \subset F(T) = F$. Opial's condition ensures that $\omega_\omega(x_n)$ is a singleton. Without lost of generality, we assume that $x_n \rightharpoonup x^* \in F$.

Next, we prove that $x_n \to x^* \in F$. From (3.2) we see that

$$\begin{aligned}
\|x_{n+1} - x^*\|^q &\leq \|y_n - x^*\|^q \leq \|(1-\alpha_n)(x_n - x^*) - \alpha_n x^*\|^q \leq \\
&\leq (1-\alpha_n)^q \|x_n - x^*\|^q + C_q \alpha_n^q \|x^*\|^q - \\
&\quad - q\alpha_n \Big\langle x^*, j_q\big((1-\alpha_n)(x_n - x^*)\big)\Big\rangle = \\
&= (1-\alpha_n)^q \|x_n - x^*\|^q + C_q \alpha_n^q \|x^*\|^q - \\
&\quad - q\alpha_n \|(1-\alpha_n)(x_n - x^*)\|^{q-2} \Big\langle x^*, j\big((1-\alpha_n)(x_n - x^*)\big)\Big\rangle = \\
&= (1-\alpha_n)^q \|x_n - x^*\|^q + C_q \alpha_n^q \|x^*\|^q - \\
&\quad - q\alpha_n (1-\alpha_n)^{q-1} \|x_n - x^*\|^{q-2} \big\langle x^*, j(x_n - x^*)\big\rangle = \\
&= (1-\alpha_n)^q \|x_n - x^*\|^q + C_q \alpha_n^q \|x^*\|^q - \\
&\quad - q\alpha_n (1-\alpha_n)^{q-1} \big\langle x^*, j_q(x_n - x^*)\big\rangle \leq \\
&\leq (1-\alpha_n) \|x_n - x^*\|^q + \\
&\quad + \alpha_n \Big(C_q \alpha_n^{q-1} \|x^*\|^q - q(1-\alpha_n)^{q-1} \big\langle x^*, j_q(x_n - x^*)\big\rangle \Big) = \\
&= (1-\alpha_n) \|x_n - x^*\|^q + \alpha_n \delta_n,
\end{aligned} \tag{3.10}$$

where

$$\delta_n = C_q \alpha_n^{q-1} \|x^*\|^q - q(1-\alpha_n)^{q-1} \big\langle x^*, j_q(x_n - x^*)\big\rangle.$$

It is easy to see that $\delta_n \to 0$ as $n \to \infty$. By Lemma 2.3, we conclude that $x_n \to x^*$ as $n \to \infty$ and hence $y_n \to x^*$ as $n \to \infty$.

Case 2. Assume that $\{\|x_n - p\|\}_{n=1}^\infty$ is not monotone decreasing. Set $\Gamma_n^p = \|x_n - p\|^q$ for each $p \in F$ and $n \in \mathbb{N}$, and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \in \mathbb{N} :\ k \leq n,\ \Gamma_k^p \leq \Gamma_{k+1}^p\}.$$

Clearly τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)}^p \leq \Gamma_{\tau(n)+1}^p$ for $n \geq n_0$ and $p \in F$. From (3.5) we see that

$$\|y_{\tau(n)} - x_{\tau(n)+1}\|^q \leq \frac{M\alpha_{\tau(n)}}{k} \to 0,$$

as $n \to \infty$. Hence

$$\lim_{n\to\infty} \|y_{\tau(n)} - x_{\tau(n)+1}\| = 0.$$

By the same argument as the proof in Case 1, we conclude that $x_{\tau(n)} \rightharpoonup x^* \in F$ as $\tau(n) \to \infty$. From (3.10), we see that, for all $n \geq n_0$,

$$\begin{aligned}
0 &\leq \|x_{\tau(n)+1} - x^*\|^q - \|x_{\tau(n)} - x^*\|^q \leq \\
&\leq \alpha_{\tau(n)} \Big(C_q \alpha_{\tau(n)}^{q-1} \|x^*\|^q - q(1-\alpha_{\tau(n)})^{q-1} \big\langle x^*, j_q(x_{\tau(n)} - x^*)\big\rangle - \|x_{\tau(n)} - x^*\|^q \Big).
\end{aligned}$$

This implies that

$$\|x_{\tau(n)} - x^*\|^q \leq C_q \alpha_{\tau(n)}^{q-1} \|x^*\|^q - q(1-\alpha_{\tau(n)})^{q-1} \big\langle x^*, j_q(x_{\tau(n)} - x^*)\big\rangle.$$

Hence

$$\lim_{n\to\infty} \|x_{\tau(n)} - x^*\| = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma^{x^*}_{\tau(n)} = \lim_{n\to\infty} \Gamma^{x^*}_{\tau(n)+1} = 0.$$

Moreover, for $n \geq n_0$, we see that $\Gamma^{x^*}_n \leq \Gamma^{x^*}_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma^{x^*}_j > \Gamma^{x^*}_{j+1}$ for $\tau(n)+1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma^{x^*}_n \leq \max\{\Gamma^{x^*}_{\tau(n)},\ \Gamma^{x^*}_{\tau(n)+1}\} = \Gamma^{x^*}_{\tau(n)+1}.$$

It follows that $\lim_{n\to\infty} \Gamma^{x^*}_n = 0$ and hence $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$. This completes the proof. □

As a direct consequence of Lemma 2.5, Lemma 2.6 and Theorem 3.1, we obtain the following result.

Theorem 3.2. *Let E be a real q-uniformly smooth Banach space which satisfies Opial's condition and C a nonempty, closed and convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C. Let $\{S_k\}_{k=1}^{\infty}$ be a sequence of λ_k-strict pseudocontractions of C into itself such that $\bigcap_{k=1}^{\infty} F(S_k) \neq \emptyset$ and $\inf\{\lambda_k : k \in \mathbb{N}\} = \lambda > 0$. Define the sequence $\{x_n\}$ by $x_1 \in C$,*

$$\begin{aligned} y_n &= Q_C\big((1-\alpha_n)x_n\big), \\ x_{n+1} &= (1-\beta_n)y_n + \beta_n \sum_{k=1}^{n} \mu_n^k S_k y_n, \quad n \geq 1, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ satisfying (C1)-(C3) of Theorem 3.1 and $\{\mu_n^k\}$ is a real sequence satisfying (i)-(iii) of Lemma 2.6. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to a common fixed point of $\{S_k\}_{k=1}^{\infty}$.

Remark 3.3. Since every Hilbert space is a q-uniformly smooth Banach space and satisfies Opial's condition, Theorems 3.1 and 3.2 hold in real Hilbert spaces.

Remark 3.4. Theorems 3.1 and 3.2 extend and improve the main result of Yao *et al.* [31] in the following senses:

(i) from real Hilbert spaces to real q-uniformly smooth Banach spaces which satisfy Opial's condition,

(ii) from a nonexpansive mapping to an infinitely countable family of strict pseudocontractions.

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