Opuscula Math. 34, no. 2 (2014), 363–373
http://dx.doi.org/10.7494/OpMath.2014.34.2.363 Opuscula Mathematica

Dedicated to the Memory of Professor Zdzisław Kamont

UNIQUENESS AND PARAMETER DEPENDENCE OF POSITIVE DOUBLY PERIODIC SOLUTIONS OF NONLINEAR TELEGRAPH EQUATIONS

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Communicated by Vicentiu D. Radulescu

Abstract. The authors study a type of second order nonlinear telegraph equation. The existence and uniqueness of positive doubly periodic solutions are discussed. The parametric dependence of the solutions is also investigated. Two examples are given as applications of the results.

Keywords: telegraph equation, doubly periodic solution, Green's function.

Mathematics Subject Classification: 35B10, 35L70.

1. INTRODUCTION

In this paper, we consider the nonlinear telegraph equation

$$u_{tt} - u_{xx} + cu_t + a(t,x)u = \lambda F(t,x,u), \tag{1.1}$$

with the doubly periodic conditions

$$u(t,x) = u(t+2\pi,x) = u(t,x+2\pi), \quad (t,x) \in \mathbb{R}^2, \tag{1.2}$$

where $\mathbb{R}_+ = [0,\infty)$, c is a positive constant, λ is a positive parameter, $a \in C(\mathbb{R}^2, \mathbb{R}_+)$ and $F \in C(\mathbb{R}^2 \times \mathbb{R}_+, \mathbb{R}_+)$ are 2π-periodic in both t and x. Let $\mathbb{T}^2$ be the torus defined by $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z})$. Clearly, any doubly periodic functions defined on $\mathbb{R}^2$ can be identified to functions defined on $\mathbb{T}^2$. By a *positive doubly periodic solution* of Eq. (1.1) we understand a function $u \in L^1(\mathbb{T}^2)$ satisfying (1.1) in the distribution sense, i.e.,

$$\int\limits_{\mathbb{T}^2} u(\phi_{tt} - \phi_{xx} - c\phi_t + a(t,x)\phi)dtdx = \int\limits_{\mathbb{T}^2} \lambda F(t,x,u)\phi dtdx$$

for any $\phi \in C^\infty(\mathbb{T}^2)$ and $u(t,x) > 0$ on $\mathbb{T}^2$.

Second order telegraph equations are often used to model the mixture between diffusion and wave propagation by introducing a term accounting for effects of finite velocity into standard heat or mass transport equations. These equations have applications in various areas such as the study of electrical signals, heat transfer, chemical kinetics, biological population dispersal, and random walks; see, for example, [1, 5–7, 11, 15] and the references therein.

The existence of solutions of nonlinear telegraph equations has been considered by many authors; the reader is referred to [2, 3, 9, 10, 12–15] for some recent results. Ortega and Robles-Pérez [12] proved the existence of at least one doubly periodic solution of the equation

$$u_{tt} - u_{xx} + cu_t = F(t, x, u) \tag{1.3}$$

by using the upper and lower solution method. This method was also used by Li [9] to investigate the existence and uniqueness of time-periodic solution of Eq. (1.3). Using fixed point theory on cones, Li [10] studied the existence of positive doubly periodic solutions of the special case of Eq. (1.1) with $\lambda = 1$ and $F(t, x, u) = b(t, x)f(t, x, u)$. The multiplicity of positive solutions of Eq. (1.1) was considered by Wang and An[13]. For the Ambrosetti-Prodi-type results on nonlinear telegraph equations or systems, the reader is referred to [2, 3, 14].

To the best of our knowledge, very little is known on the existence of unique positive doubly periodic solutions of Eq. (1.1). In this paper, two theorems on the existence, uniqueness, and the parametric dependence of positive doubly periodic solutions of Eq. (1.1) are obtained by using mixed monotone operator theory. Our results reveal the relation between the solution and the parameter, and provide a method to approximate the unique solutions by the solutions of the associated linear equations.

This paper is organized as follows: after this introduction, the main results and examples are presented in Section 2. All the proofs are given in Section 3.

2. MAIN RESULTS

In this paper, we let $F(t, x, u) = f(t, x, u, u) + r(t, x, u)$, where $f \in C(\mathbb{R}^2 \times \mathbb{R}_+^2, \mathbb{R}_+)$ and $r \in C(\mathbb{R}^2 \times \mathbb{R}_+, \mathbb{R}_+)$ are 2π-periodic in both t and x. The following assumptions will be needed:

(H1) $0 \le a(t, x) \le c^2/4$ on $\mathbb{R}^2$ and $\int_{\mathbb{T}^2} a(t, x)dtdx > 0$;
(H2) $f(t, x, \cdot, \mathrm{v})$ is increasing for any fixed $(t, x, \mathrm{v}) \in \mathbb{T}^2 \times \mathbb{R}_+$, and $f(t, x, \mathrm{u}, \cdot)$ is decreasing for any fixed $(t, x, \mathrm{u}) \in \mathbb{T}^2 \times \mathbb{R}_+$;
(H3) there exists $\theta \in (0, 1)$ such that

$$f(t, x, \kappa\mathrm{u}, \kappa^{-1}\mathrm{v}) \ge \kappa^\theta f(t, x, \mathrm{u}, \mathrm{v})$$

for $(t, x) \in \mathbb{T}^2$, $\kappa \in (0, 1)$, $\mathrm{u} \in \mathbb{R}_+$, and $\mathrm{v} \in \mathbb{R}_+$;
(H4) $r(t, x, \cdot)$ is increasing for any fixed (t, x) and there exists a constant $w > 0$ such that $r(t, x, w) \not\equiv 0$ on $\mathbb{T}^2$;

(H5) $r(t,x,\kappa \mathrm{u}) \geq \kappa r(t,x,\mathrm{u})$ for $(t,x) \in \mathbb{T}^2$, $\kappa \in (0,1)$, and $\mathrm{u} \in \mathbb{R}_+$;
(H6) there exists $\eta > 0$ such that

$$f(t,x,\mathrm{u},\mathrm{v}) \geq \eta r(t,x,\mathrm{u})$$

for $(t,x) \in \mathbb{T}^2$, $\mathrm{u} \in \mathbb{R}_+$, and $\mathrm{v} \in \mathbb{R}_+$;
(H7) for θ given in (H3), we have $\theta \in (0,1/2)$ and

$$r(t,x,\kappa \mathrm{u}) \geq \kappa^\theta r(t,x,\mathrm{u}), \quad (t,x) \in \mathbb{T}^2, \ \kappa \in (0,1), \ \text{ and } \mathrm{u} \in \mathbb{R}_+.$$

Remark 2.1. We would like to make a few comments on the form of the nonlinear term f above. The analysis in this paper mainly relies on mixed monotone operator theory. To apply such theory, one alternative way is to write the nonlinearity as $f(t,x,\mathrm{u})$ and assume that $f(t,x,\mathrm{u})$ can be decomposed as $f(t,x,\mathrm{u}) = g(t,x,\mathrm{u}) + h(t,x,\mathrm{u})$, where $g : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing in the third argument, $h : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nonincreasing in the third argument, and there exists $\theta \in (0,1)$ such that

$$g(t,x,\kappa \mathrm{u}) \geq \kappa^\theta g(t,x,\mathrm{u}) \tag{2.1}$$

and

$$h(t,x,\kappa^{-1} \mathrm{u}) \geq \kappa^\theta h(t,x,\mathrm{u}) \tag{2.2}$$

for $(t,x) \in \mathbb{R}^2$, $\kappa \in (0,1)$, and $\mathrm{u} \geq 0$. The reader may refer to [4] for a related discussion.

Here, the nonlinear term f is written as a function of four arguments. Then, to apply mixed monotone operator theory, we need to assume that the conditions (H2) and (H3) above are satisfied. By writing f this way, a larger class of functions can be covered. For instance, if $f(t,x,\mathrm{u},\mathrm{v}) = \sqrt[3]{\mathrm{u}}/\sqrt{\mathrm{v}+1}$, then, $f(t,x,\mathrm{u},\mathrm{u})$ cannot be decomposed into a summation of two functions g and h satisfying (2.1) and (2.2), but $f(t,x,\mathrm{u},\mathrm{v})$ does satisfy (H2) and (H3) with $\theta = 5/6$.

For any $u \in C(\mathbb{T}^2)$, let $\|u\| = \max_{(t,x)\in\mathbb{T}^2} |u(t,x)|$. The following theorem is our main result.

Theorem 2.2. *Assume that* (H1)–(H6) *hold. Then:*

1. *for any $\lambda > 0$, Eq.* (1.1) *has a unique positive doubly periodic solution $u_\lambda \in C(\mathbb{T}^2)$;*
2. *for any functions u_0 and $v_0 \in C(\mathbb{T}^2)$ with $\min_{(t,x)\in\mathbb{T}^2} u_0(t,x) > 0$ and $\min_{(t,x)\in\mathbb{T}^2} v_0(t,x) > 0$, let $\{u_n\}$ and $\{v_n\}$ be the solutions of the linear telegraph equations*

$$(u_n)_{tt} - (u_n)_{xx} + c(u_n)_t + a(t,x)u_n = \\ = \lambda(f(t,x,u_{n-1},v_{n-1}) + r(t,x,u_{n-1})), \tag{2.3}$$

$$(v_n)_{tt} - (v_n)_{xx} + c(v_n)_t + a(t,x)v_n = \\ = \lambda(f(t,x,v_{n-1},u_{n-1}) + r(t,x,v_{n-1})), \tag{2.4}$$

$n = 1,2,\ldots.$ *Then $\|u_n - u_\lambda\| \to 0$ and $\|v_n - u_\lambda\| \to 0$ as $n \to \infty$.*

3. *If in addition,* (H7) *holds, then the unique solution* u_λ *satisfies the following properties:*
   (a) $u_\lambda(t,x)$ *is strictly increasing in* λ*, i.e.,* $\lambda_1 > \lambda_2 > 0$ *implies* $u_{\lambda_1}(t,x) > u_{\lambda_2}(t,x)$ *on* $\mathbb{R}^2$*;*
   (b) $\lim_{\lambda\to 0+}\|u_\lambda\| = 0$ *and* $\lim_{\lambda\to\infty}\|u_\lambda\| = \infty$*;*
   (c) $u_\lambda(t,x)$ *is continuous in* λ*, i.e.,* $\|u_\lambda - u_{\lambda_0}\| \to 0$ *if* $\lambda \to \lambda_0 > 0$.

When $r(t,x,\mathrm{u}) \equiv 0$ on $\mathbb{R}^2 \times \mathbb{R}_+$, i.e., $F(t,x,\mathrm{u}) = f(t,x,\mathrm{u},\mathrm{u})$, we obtain a similar result.

Theorem 2.3. *Assume that* $F(t,x,\mathrm{u}) = f(t,x,\mathrm{u},\mathrm{u})$ *with* (H1)–(H3) *hold. Furthermore, there exists* $w > 0$ *such that* $f(t,x,w,w) \not\equiv 0$ *on* $\mathbb{T}^2$*. Then:*

1. *For any* $\lambda > 0$*, Eq.* (1.1) *has a unique positive doubly periodic solution* $u_\lambda \in C(\mathbb{T}^2)$.
2. *For any functions* u_0 *and* $v_0 \in C(\mathbb{T}^2)$ *with* $\min_{(t,x)\in\mathbb{T}^2} u_0(t,x) > 0$ *and* $\min_{(t,x)\in\mathbb{T}^2} v_0(t,x) > 0$*, let* $\{u_n\}$ *and* $\{v_n\}$ *be the solutions of the linear telegraph equations*

$$(u_n)_{tt} - (u_n)_{xx} + c(u_n)_t + a(t,x)u_n = \lambda f(t,x,u_{n-1},v_{n-1}),$$
$$(v_n)_{tt} - (v_n)_{xx} + c(v_n)_t + a(t,x)v_n = \lambda f(t,x,v_{n-1},u_{n-1}),$$

   $n = 1,2,\ldots$ *. Then* $\|u_n - u_\lambda\| \to 0$ *and* $\|v_n - u_\lambda\| \to 0$ *as* $n \to \infty$.
3. *If in addition,* $\theta \in (0,1/2)$*, then the unique solution* u_λ *satisfies the following properties:*
   (a) $u_\lambda(t,x)$ *is strictly increasing in* λ*, i.e.,* $\lambda_1 > \lambda_2 > 0$ *implies* $u_{\lambda_1}(t,x) > u_{\lambda_2}(t,x)$ *on* $\mathbb{R}^2$*;*
   (b) $\lim_{\lambda\to 0+}\|u_\lambda\| = 0$ *and* $\lim_{\lambda\to\infty}\|u_\lambda\| = \infty$*;*
   (c) $u_\lambda(t,x)$ *is continuous in* λ*, i.e.,* $\|u_\lambda - u_{\lambda_0}\| \to 0$ *if* $\lambda \to \lambda_0 > 0$.

Remark 2.4. In Theorem 2.2 (2) and Theorem 2.3 (2), if we let $u_0 = v_0$, then it is easy to see that $u_n = v_n$ for any $n > 0$. Hence, we only need to solve one linear equation

$$(u_n)_{tt} - (u_n)_{xx} + c(u_n)_t + a(t,x)u_n = \lambda F(t,x,u_{n-1}) \tag{2.5}$$

in each step. Since Theorem 2.2 (2) and Theorem 2.3 (2) guarantee the convergence of $\{u_n\}$, we may use this iteration to approximate the unique positive doubly periodic solution of Eq. (1.1).

To demonstrate the application of our results, let us consider the following examples.

Example 2.5. Consider the equation

$$u_{tt} - u_{xx} + cu_t + \frac{c^2(\cos t + 1)}{8}u = \lambda\left(u^\theta + \arctan(u) + m\right). \tag{2.6}$$

We claim that Eq. (2.6) has a unique positive doubly periodic solution for any $c > 0$, $\lambda > 0$, $m > 0$, and $\theta \in (0,1)$. In fact, let

$$f(t,x,\mathrm{u},\mathrm{v}) = u^\theta + m \quad \text{and} \quad r(t,x,\mathrm{u}) = \arctan(\mathrm{u}) = r(u).$$

It is easy to verify that (H1)–(H4) and (H6) hold with $\eta = 2m/\pi$. For $\kappa \in (0,1)$ and $\mathrm{u} \in \mathbb{R}_+$, it is easy to see that

$$[r(\kappa \mathrm{u}) - \kappa r(\mathrm{u})]' = \frac{\kappa}{1+\kappa^2\mathrm{u}^2} - \frac{\kappa}{1+\mathrm{u}^2} \geq 0.$$

Hence, $r(\kappa \mathrm{u}) \geq \kappa r(\mathrm{u})$ for $\kappa \in (0,1)$ and $\mathrm{u} \in \mathbb{R}_+$, i.e., (H5) holds. Thus, by Theorem 2.2 (1), Eq.(2.6) has a unique positive doubly periodic solution u_λ. Note that for $\kappa \in (0,1)$ and $\theta \in (0,1)$, $[r(\kappa \mathrm{u}) - \kappa^\theta r(\mathrm{u})]'\big|_{\mathrm{u}=0} = \kappa - \kappa^\theta < 0$. Hence, $u^\theta + \arctan(u) + m$ does not satisfies (H3), i.e., we cannot use this function as the function f needed in our theorems.

The numerical solution of Eq. (2.6) with $c = 7$, $\lambda = 2$, $\theta = 1/2$, and $m = 11$ is computed by using (2.5) with $u_0 \equiv 1$. The maximum absolute errors $E_n = \|u_n - u_{n-1}\|$ between u_n and u_{n-1} for the first 10 iterations are given in Table 1, which confirm our results. The graph of u_{10} is given in Figure 1.

Table 1. The maximum absolute error for Eq. (2.6)

n	1	2	3	4	5
E_n	8.4149	1.4462	0.1082	0.0082	7.1787e-04
i	6	7	8	9	10
E_n	6.8257e-05	6.5136e-06	5.9842e-07	5.2871e-08	4.5926e-09

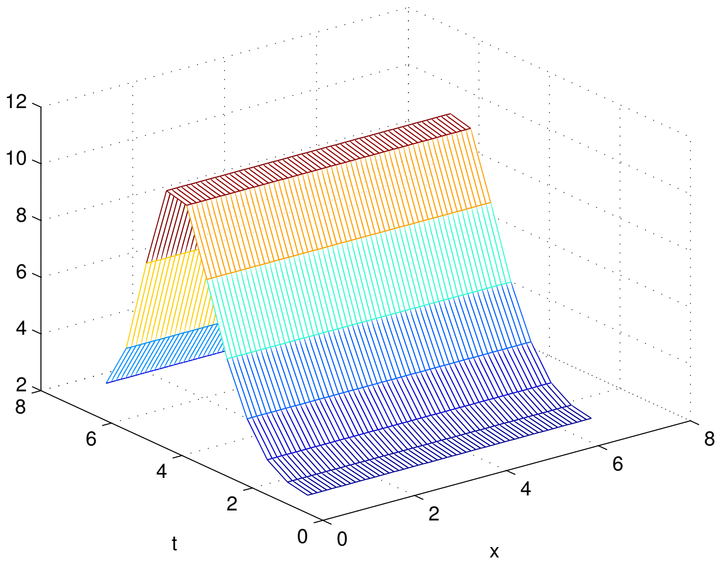

Fig. 1. Numerical solution of Eq. (2.6) after 10 iterations

Example 2.6. Assume $\theta_1, \theta_2 \in (0,1)$, and c, λ, and m are positive constants. Then the equation

$$u_{tt} - u_{xx} + cu_t + \frac{c^2 \sin^2(t+x)}{4} u = \lambda \left(\cos^2 t \cos^2 x + u^{\theta_1} + \frac{1}{(u+m)^{\theta_2}} \right) \tag{2.7}$$

has a unique positive doubly periodic solution for any positive constants c and λ. To see this, let

$$f(t,x,\mathrm{u},\mathrm{v}) = \cos^2 t \cos^2 x + \mathrm{u}^{\theta_1} + \frac{1}{(\mathrm{v}+m)^{\theta_2}}.$$

Clearly (H1)–(H3) hold when $\theta = \max\{\theta_1,\ \theta_2\}$ and $f(t,x,w,w) > 0$ on $\mathbb{T}^2$ for any positive w. Hence, by Theorem 2.3 (1), Eq.(2.7) has a unique positive doubly periodic solution u_λ.

If we further assume $\theta_1, \theta_2 \in (0, 1/2)$, by Theorem 2.3 (3), u_λ is continuous and strictly increasing in λ.

With $c = 10$, $\lambda = 5$, $m = 1$, $\theta_1 = 0.3$, $\theta_2 = 0.4$, Eq. (2.7) is solved using (2.5) with $u_0 \equiv 1$. The maximum absolute errors $E_n = \|u_n - u_{n-1}\|$ between u_n and u_{n-1} for the first 10 iterations are given in Table 2. The graph of u_{10} is given in Figure 2.

Table 2. The maximum absolute error for Eq. (2.7)

n	1	2	3	4	5
E_n	0.7757	0.0310	0.0032	2.3716e-04	1.5449e-05
i	6	7	8	9	10
E_n	1.0135e-06	7.1725e-08	5.3201e-09	3.9626e-10	2.9179e-11

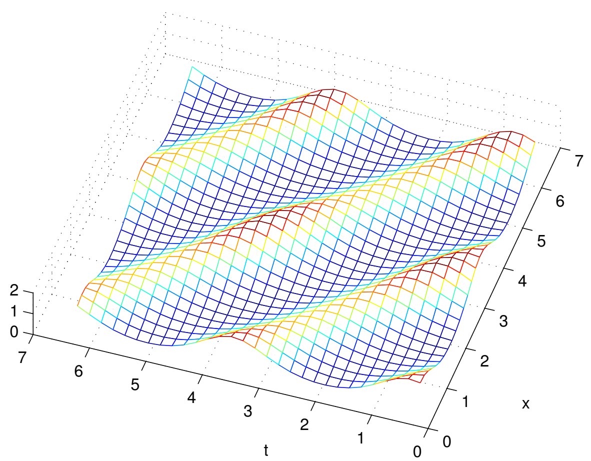

Fig. 2. Numerical solution of Eq. (2.7) after 10 iterations

3. PROOFS

We first consider the linear telegraph equation

$$u_{tt} - u_{xx} + cu_t + a(t,x)u = h(t,x), \tag{3.1}$$

with condition (1.2).

The following lemma is extracted from Li [10, Lemma 2].

Lemma 3.1. *Assume that* (H1) *holds and* $h \in L^1(\mathbb{T}^2)$. *Then Eq.* (3.1) *has a unique solution* $u = Th$, *where* $T:\ L^1(\mathbb{T}^2) \to C(\mathbb{T}^2)$ *is a bounded linear operator with the following properties:*

(a) $T:\ C(\mathbb{T}^2) \to C(\mathbb{T}^2)$ *is a completely continuous operator;*
(b) *if* $h(t,x) \geq 0$ *a.e. in* $\mathbb{T}^2$, *then for any* $(t,x) \in \mathbb{T}^2$,

$$\frac{\|h\|_1}{(e^{3c\pi} - e^{2c\pi})^2} \leq (Th)(t,x) \leq \frac{(1+e^{c\pi})\|h\|_1}{2e^{c\pi}\|a\|_1}. \tag{3.2}$$

Remark 3.2. Lemma 3.1 (b) implies that for any $h_1,\ h_2 \in L^1(\mathbb{T}^2)$ with $h_1(t,x) \leq h_2(t,x)$ a.e. on $\mathbb{T}^2$,

$$(Th_1)(t,x) \leq (Th_2)(t,x) \quad \text{for any } (t,x) \in \mathbb{T}^2.$$

The reader is referred to [10, 12] for more properties of T.

We will use mixed monotone operator theory to prove Theorem 2.2. The following definitions and lemma are needed. The reader is referred to [8] for additional details.

Definition 3.3. Let $(X, \|\cdot\|)$ be a Banach space and $\mathbf{0}$ be the zero element of X.

(a) A nonempty closed convex set $P \subset X$ is said to be a cone if it satisfies (i) $u \in P$ and $\lambda > 0 \Longrightarrow \lambda u \in P$; (ii) $u \in P$ and $-u \in P \Longrightarrow u = \mathbf{0}$.
(b) A cone P is said to be normal if there exists a constant $D > 0$ such that, for all $u, v \in X$, $\mathbf{0} \leq u \leq v \Longrightarrow \|u\| \leq D\|v\|$. The constant D is called the normality constant of P.
(c) The Banach space $(X, \|\cdot\|)$ is partially ordered by a normal cone $P \subset E$, i.e., $u \leq v$ if $v - u \in P$. If $u \leq v$ and $u \neq v$, then we write $u < v$ or $v > u$.
(d) For any $u, v \in X$, we use the notation $u \sim v$ to mean that there exist $\underline{d} > 0$ and $\overline{d} > 0$ such that $\underline{d}v \leq u \leq \overline{d}v$. Given $w > \mathbf{0}$, i.e., $w \geq \mathbf{0}$ and $w \neq \mathbf{0}$, we define $P_w = \{u \in X \mid u \sim w\}$. Clearly, $P_w \subset P$.

Definition 3.4. An operator $\mathcal{A} : P_w \times P_w \to X$ is said to be mixed monotone if $\mathcal{A}(u,v)$ is nondecreasing in u and nonincreasing in v, i.e., for $u_1, u_2, v_1, v_2 \in P_w$, we have

$$u_1 \leq u_2,\ v_1 \geq v_2 \Longrightarrow \mathcal{A}(u_1,v_1) \leq \mathcal{A}(u_2,v_2);$$

Moreover, an element $u \in P_w$ is said to be a fixed point of $\mathcal{A}$ if $\mathcal{A}(u,u) = u$.

Definition 3.5. Assume $\mathcal{B} : P_w \to X$.

(a) $\mathcal{B}$ is said to be sub-homogeneous if it satisfies

$$\mathcal{B}(\kappa u) \geq \kappa \mathcal{B}(u) \quad \text{for all } u \in P_w \text{ and } \kappa \in (0,1);$$

(b) let $\theta \in [0,1)$, $\mathcal{B}$ is said to be θ-concave if it satisfies

$$\mathcal{B}(\kappa u) \geq \kappa^\theta \mathcal{B}u \text{ for all } u \in P_w \quad \text{and} \quad \kappa \in (0,1).$$

Lemma 3.6 ([8, Lemma 2.1]). *Let $\theta \in (0,1)$ and $\mathcal{A} : P_w \times P_w \to X$ be a mixed monotone operator satisfying*

$$\mathcal{A}(\kappa u, \kappa^{-1} v) \geq \kappa^\theta \mathcal{A}(u,v) \quad \text{for all } u, v \in P_w \text{ and } \kappa \in (0,1).$$

(A) *Assume that $\mathcal{B} : P_w \to X$ is an increasing sub-homogeneous operator and the following conditions hold:*

(i) *$\mathcal{A}(w,w) \in P_w$ and $\mathcal{B}(w) \in P_w$;*
(ii) *there exists a constant $\eta > 0$ such that $\mathcal{A}(u,v) \geq \eta \mathcal{B}(u)$ for all $u, v \in P_w$.*

Then, we have:

1. *for any $\lambda > 0$, the equation $\lambda(\mathcal{A}(u,u) + \mathcal{B}(u)) = u$ has a unique solution u_λ in P_w;*
2. *for any initial values $u_0, v_0 \in P_w$, the sequences $\{u_n\}$ and $\{v_n\}$ defined by*

$$\begin{aligned} u_n &= \lambda(\mathcal{A}(u_{n-1}, v_{n-1}) + \mathcal{B}(u_{n-1})), \\ v_n &= \lambda(\mathcal{A}(v_{n-1}, u_{n-1}) + \mathcal{B}(v_{n-1})), \end{aligned} \qquad n = 1, 2, \ldots,$$

   satisfy $\|u_n - u_\lambda\| \to 0$ and $\|v_n - u_\lambda\| \to 0$ as $n \to \infty$;
3. *if we further assume that $\theta \in (0, 1/2)$ and $\mathcal{B}$ is θ-concave, then the unique solution u_λ satisfies the following properties:*
   (a) *u_λ is strictly increasing in λ, i.e., $u_{\lambda_1} > u_{\lambda_2}$ if $\lambda_1 > \lambda_2 > 0$;*
   (b) *$\lim_{\lambda \to 0+} \|u_\lambda\| = 0$ and $\lim_{\lambda \to \infty} \|u_\lambda\| = \infty$;*
   (c) *u_λ is continuous in λ, i.e., $\|u_\lambda - u_{\lambda_0}\| \to 0$ when $\lambda \to \lambda_0 > 0$.*

(B) *Assume $\mathcal{A}(w,w) \in P_w$. Then:*

1. *for any $\lambda > 0$, the equation $\lambda \mathcal{A}(u,u) = u$ has a unique solution u_λ in P_w;*
2. *for any initial values $u_0, v_0 \in P_w$, the sequences $\{u_n\}$ and $\{v_n\}$ defined by*

$$u_n = \lambda \mathcal{A}(u_{n-1}, v_{n-1}), \quad v_n = \lambda \mathcal{A}(v_{n-1}, u_{n-1}), \quad n = 1, 2, \ldots,$$

   satisfy $\|u_n - u_\lambda\| \to 0$ and $\|v_n - u_\lambda\| \to 0$ as $n \to \infty$;
3. *if we further assume that $\theta \in (0, 1/2)$, then the unique solution u_λ satisfies all the properties in* (3) *of Part* (A).

In the sequel, we let $X = C(\mathbb{T}^2)$ with the standard maximum norm $\|\cdot\|$ and

$$P = \{u \in X \mid u(t,x) \geq 0 \text{ on } \mathbb{T}^2\}.$$

Clearly, P is a normal cone with normality constant $D = 1$. For w given in (H4), it is easy to see that

$$P_w = \{u \in P \mid \min_{(t,x)\in\mathbb{T}^2} u(t,x) > 0\}$$

since for any $u \in P_w$, $0 < \min_{(t,x)\in\mathbb{T}^2} u(t,x) \le u(t,x) \le \|u\|$ on $\mathbb{T}^2$. We define $\mathcal{A}: P_w \times P_w \to X$ and $\mathcal{B}: P_w \to X$ by

$$\mathcal{A}(u,v)(t,x) = (Tf[u,v])(t,x) \tag{3.3}$$

and

$$\mathcal{B}(u)(t,x) = (Tr[u])(t,x), \tag{3.4}$$

where $f[u,v](t,x) = f(t,x,u(t,x),v(t,x))$, $r[u](t,x) = r(t,x,u(t,x))$, and T is defined in Lemma 3.1.

Remark 3.7. It is easy to verify that u is a solution of Eq. (1.1) if and only if u is a solution of the equation $\lambda(A(u,u) + B(u)) = u$.

Proof of Theorem 2.2. (1). By (H2), (H4), and Remark 3.2, $\mathcal{A}$ is mixed monotone and $\mathcal{B}$ is increasing. For $u, v \in P_w$ and $\kappa \in (0,1)$, (H3) implies that for any $(t,x) \in \mathbb{T}^2$ and $\kappa \in (0,1)$, $f[\kappa u, \kappa^{-1}v](t,x) \ge \kappa^\theta f[u,v](t,x)$. Then by (3.3) and Remark 3.2,

$$\mathcal{A}(\kappa u, \kappa^{-1}v)(t,x) = Tf[\kappa u, \kappa^{-1}v](t,x) \ge \kappa^\theta Tf[u,v](t,x) = \kappa^\theta \mathcal{A}(u,v)(t,x).$$

Similarly, by (H5) and (3.4), for any $(t,x) \in \mathbb{T}^2$ and $\kappa \in (0,1)$,

$$\mathcal{B}(\kappa u)(t,x) = Tr[\kappa u](t,x) \ge \kappa Tr[u](t,x) = \kappa\mathcal{B}(u)(t,x),$$

i.e., $\mathcal{B}$ is sub-homogeneous. By (H6), (3.3), and (3.4), for any $u, v \in P_w$,

$$\mathcal{A}(u,v)(t,x) = Tf[u,v](t,x) \ge \eta Tr[u](t,x) = \eta\mathcal{B}(u)(t,x).$$

In particular, we have

$$\mathcal{A}(w,w)(t,x) = Tf[w,w](t,x) \ge \eta Tr[w](t,x) = \eta\mathcal{B}(w)(t,x).$$

By (H4), (3.2), and (3.4), for any $(t,x) \in \mathbb{T}^2$,

$$\mathcal{B}(w)(t,x) = Tr[w](t,x) \ge \frac{\|r[w]\|_1}{(e^{3c\pi} - e^{2c\pi})^2} > 0.$$

Hence, both $\mathcal{A}(w,w)$ and $\mathcal{B}(w) \in P_w$.

Therefore, by Lemma 3.6 (A) (1), for any $\lambda > 0$, Eq. (1.1) has a unique solution $u_\lambda \in P_w$.

By the definition of X and P_w, it is easy to see that if $\hat{u} \in X$ is a positive solution of Eq. (1.1), then $\hat{u} \in P_w$. Thus, u_λ is the unique positive doubly periodic solution of Eq. (1.1).

(2). By Lemma 3.1, for $n = 1, 2, \ldots$, $u_n = T(\lambda(f[u_{n-1}, v_{n-1}] + r[u_{n-1}]))$ is the solution of Eq. (2.3). Then by (3.3) and (3.4),

$$u_n = \lambda T(f[u_{n-1}, v_{n-1}] + r[u_{n-1}]) = \lambda(\mathcal{A}(u_{n-1}, v_{n-1}) + \mathcal{B}(u_{n-1})).$$

Similarly, we can show that for $n = 1, 2, \dots$, $v_n = \lambda(\mathcal{A}(v_{n-1}, u_{n-1}) + \mathcal{B}(v_{n-1}))$ is the solution of Eq. (2.4). The the conclusion follows from Lemma 3.6 (A) (2).

(3). If (H7) holds, then $\theta \in (0, 1/2)$ and

$$\mathcal{B}(\kappa u)(t,x) = Tr[\kappa u](t,x) \geq \kappa^\theta Tr[u](t,x) = \kappa^\theta \mathcal{B}(u)(t,x),$$

i.e., $\mathcal{B}$ is θ-concave. Then Theorem 2.2 (3) follows from Lemma 3.6 (A) (3). □

The proof of Theorem 2.3 proceeds in the same way by using Lemma 3.6 (B); we omit the details.

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Received: September 3, 2013.
Revised: November 12, 2013.
Accepted: November 12, 2013.