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GLOBAL SOLUTIONS TO THE INITIAL-BOUNDARY VALUE PROBLEM FOR THE QUASILINEAR VISCOELASTIC EQUATION WITH A DERIVATIVE NONLINEARITY

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Abstract. We prove the existence and uniqueness of a global decaying solution to the initial boundary value problem for the quasilinear wave equation with Kelvin-Voigt dissipation and a derivative nonlinearity. To derive the required estimates of the solutions we employ a 'loan' method and use a difference inequality on the energy.

Keywords: global solutions, energy decay, quasilinear wave equation, Kelvin-Voigt dissipation, derivative nonlinearity.

Mathematics Subject Classification: 35B35, 35B40, 35L70.

1. INTRODUCTION

We consider the initial-boundary value problem for the quasilinear wave equations with a strong dissipation and a derivative nonlinearity:

$$u_{tt} - \operatorname{div}\{\sigma(|\nabla u|^2)\nabla u\} - \Delta u_t = f(u, \nabla u, u_t) \quad \text{for} \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{for} \quad x \in \Omega, \quad \text{and} \quad u(x,t)|_{\partial\Omega} = 0, t \geq 0, \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth, say C^2-class, boundary $\partial\Omega$ and $\sigma(|\nabla u|^2)$ is a function like $\sigma = 1/\sqrt{1+|\nabla u|^2}$, mean curvature type nonlinearity. The viscosity term $-\Delta u_t$ is often called a Kelvin-Voigt type dissipation or strong dissipation which appears in phenomena of wave propagation in a viscoelastic material (cf. [1, 2, 6, 14]). We make the following assumption on the nonlinear term $f(u, \mathbf{v}, w)$.

Hypothesis A. $f(u, \mathbf{v}, w)$ is a C^1 class function on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ and satisfies:

$$|f(u, \mathbf{v}, w)| \leq k_0 \left(|u|^{\alpha+1} + |u|^{\beta+1}|\mathbf{v}| + |w|^{\gamma+1}\right) \tag{1.3}$$

$$|f_u(u,\mathbf{v},w)| \le k_0\left(1+|u|^\alpha+(1+|u|^\beta)|\mathbf{v}|\right), \tag{1.4}$$
$$|f_{\mathbf{v}}(u,\mathbf{v},w)| \le k_0(1+|u|^{\beta+1}) \tag{1.5}$$

and

$$|f_w(u,\mathbf{v},w)| \le k_0(1+|w|^\gamma) \tag{1.6}$$

with $0<\alpha\le 4/(N-4)^+, \beta\ge 0$, $0<\gamma\le \min\{2/(N-2)^+\}$, and a constant $k_0>0$,

Two typical examples are $f=\nabla\cdot\mathbf{G}(u)$, a nonlinear convection, and $f=|u_t|^\gamma u_t$, a nonlinear perturbation by velocity. In fact some additional restrictions on α,β and γ will be made in our theorem. The conditions (1.4)–(1.6) are made for the uniqueness of solutions. These conditions can be weakened in some way, but we keep the conditions for simplicity of the proof.

Concerning the principal part we make the following assumption.

Hypothesis B. $\sigma(v^2)$ is continuously differentiable in $v^2\ge 0$ and satisfies

$$k_2 \ge \sigma(v^2) \ge k_1 \max\left\{(1+v^2)^{-\nu}, \int_0^{v^2}\sigma(\tau)d\tau/v^2\right\} \tag{1.7}$$

with some ν, $0\le\nu<1$,

$$k_2 \ge \sigma(v^2)+2\sigma'(v^2)v^2 \ge 0, \tag{1.8}$$

and

$$|\sigma'(v^2)v^2| \le k_2, \tag{1.9}$$

where k_1 and $k_2>0$ are some constants.

Let us consider for a moment the typical case $\sigma(|\nabla u|^2)=1/\sqrt{1+|\nabla u|^2}$. In this case Hyp. B is satisfied with $\nu=1/2$. We note that in this case the principal term $-\mathrm{div}\{\sigma(|\nabla u|^2)\nabla u\}$ is not coercive in the sense that

$$\int_\Omega |\nabla u|^2/\sqrt{1+|\nabla u|^2}dx \ge C\|\nabla u\|_2^2, \quad C>0,$$

does not hold, which causes the main difficulty in the existence problem of weak solutions. When $f\equiv 0$, unperturbed problem, and $N=1$ it is not difficult to show the global existence and exponential decay of solutions due to the fact $\|\nabla u\|_\infty \le C\|\Delta u\|_2$ (see [15]). The global existence of smooth solutions for the case $N\ge 2$ is proved by Pecher, Kobayashi and Shibata [5] by a careful use of semi-group theory. But in [5] no decay property of solutions is given. In [9] we assumed that the mean curvature of $\partial\Omega$ with respect to the outward normal is nonnegative and proved that

$$E(t)\equiv \frac{1}{2}\left\{\|u_t(t)\|_2^2+\int_\Omega\int_0^{|\nabla u|^2}\sigma(\tau)d\tau dx\right\} \le C_1(1+t)^{-(1+4/(N-2)^+)}, \tag{1.10}$$

where C_1 is a constant depending on $\|u_0\|_{H_2} + \|u_1\|_{H_1}$. In these papers [5, 8, 15] no smallness condition on the initial data is imposed. When f is a power type nonlinearity of only u like $f = |u|^\alpha u, \alpha > 0$, we can combine an argument of a modified potential well method and the expected decay estimate (1.10) to show the existence and uniqueness of a global solution

$$u(\cdot) \in X_2(\infty) \equiv L^\infty([0,\infty); H_2 \cap H_1^0) \cap W_{loc}^{1,\infty}([0,\infty); H_1^0) \cap W_{loc}^{2,2}([0,\infty); L^2)$$

for each $(u_0, u_1) \in H_2 \cap H_1^0 \times H_1^0$ if $E(0)$ is small (see [3]). But, such a method cannot be applied when f depends on the derivatives of u.

The object of this paper is to show the global existence and uniquness of solutions in $X_2(\infty)$ for the problem (1.1)–(1.2) where f includes derivarives of u. For this we employ a 'loan' method. When f depends on ∇u in an essential manner we must restrict ourselves to $N = 1, 2, 3$ for technical reasons. Otherwise we can also consider the case of more general dimensions. For applications of the 'loan' method in other situations see [8, 10–13].

We make the following assumption.

Hypothesis C. The mean curvature $H(x)$ of $\partial\Omega$ at $x \in \partial\Omega$ is nonnegative.

We note that if we require more regularity on the initial data, say, $(u_0, u_1) \in H_{m+1} \cap H_m^0 \times H_m^0$ with $m > N/2$ and assume that $\|u_0\|_{H_{m+1}} + \|u_1\|_{H_m}$ is sufficiently small, it is not so difficult to prove the existence of corresponding global smooth solutions. Indeed, for such a case we can expect the boundedness of $\|\nabla u(t)\|_\infty$ and the exponential decay of the energy $E(t)$ (cf. [4]). We can expect such a result on global existence of smooth solutions for the quasilinear wave equation with much more weaker dissipation (cf. [8, 10]). But, in the present paper we show the global existence of solutions in $X_2(\infty)$ where we can not expect the boundedness of $\|\nabla u(t)\|_\infty$ except for the case $N = 1$.

2. PRELIMINARIES AND STATEMENT OF RESULT

We use only familiar function spaces and omit the definition of them. But we note that $\|\cdot\|_p, 1 \le p$, denotes the $L^p(\Omega)$ norm. We write $\|\cdot\|$ for $\|\cdot\|_2$.

Theorem 2.1. *Let $N = 1, 2, 3$ and assume Hyp. A, Hyp. B and Hyp. C. We make the conditions on the exponents α, β and γ such that*

$$\alpha + 2 > \begin{cases} 0 & \text{if } N = 1, 2, \\ 3\nu & \text{if } N = 3, \end{cases}$$

$$\beta + 1 > 3\nu(N-2)^+/(4-N)^+ = \begin{cases} 0 & \text{if } N = 1, 2, \\ 3\nu & \text{if } N = 3, \end{cases}$$

and

$$\frac{2\nu(N-2)^+}{2 + \nu(N-2)^+} < \gamma < \frac{4}{N}.$$

Let $(u_0, u_1) \in H_2 \cap H_1^0 \times H_1^0$ *with*

$$\|\Delta u_0\| < K_2.$$

Then there exists $\delta = \delta(K_2) > 0$ *such that if* $E(0) \le \delta$, *the problem admits a unique solution in the class* $X_2(\infty)$, *satisfying the estimates*

$$E(t) \le C(K_2)(1+t)^{-1-2/\nu(N-2)^+} \quad \text{and} \quad \|\Delta u(t)\| < K_2,$$

where the decay estimate should be replaced if $N = 1, 2$ *as follows:*

$$E(t) \le C(K_2) exp\{-\lambda t\}, \quad \lambda > 0, \quad (N = 1)$$

and

$$E(t) \le C(K_2, m)(1+t)^{-m}, \quad (N = 2)$$

with any $m >> 1$.

Corollary 2.2. *Replace condition* (1.3) *in Hyp. A by*

$$|f(u, \mathbf{v}, w)| \le k_0 \left(|u|^{\alpha+1} + |w|^{\gamma+1}\right) \tag{2.1}$$

and replace (1.4), (1.5) *and* (1.6) *by*

$$|f_u(u, \mathbf{v}, w)| \le k_0(1 + |u|^{\alpha}), \tag{2.2}$$

$$|f_{\mathbf{v}}(u, \mathbf{v}, w)| \le k_0, \tag{2.3}$$

and

$$|f_w(u, \mathbf{v}, w)| \le k_0(1 + |w|^{\gamma}), \tag{2.4}$$

respectively. Assume that

$$(4 - N)\alpha + 2 > 3\nu(N-2)^+$$

and

$$\frac{2\nu(N-2)^+}{2 + \nu(N-2)^+} < \gamma < \frac{4}{N}, \gamma \le 2/(N-2)^+.$$

Then the conclusion of Theorem 2.1 holds for all $N \ge 1$.

To derive the decay estimate of $E(t)$ we use the following lemma.

Lemma 2.3 ([7]). *Let* $\phi(t)$ *be a nonnegative function on* $[0, T]$, $T > 1$, *such that* $\phi(t+1) \le \phi(t)$ *and*

$$\sup_{t \le s \le t+1} \phi(s)^{1+\gamma} \le C_0 \left(\phi(t) - \phi(t+1)\right), \quad 0 \le t \le T - 1,$$

with $C_0 > 0$ *and* $\gamma > 0$. *Then*

$$\phi(t) \le \left((\sup_{0 \le s \le 1} \phi(s))^{-\gamma} + \frac{\gamma}{C_0}(t-1)^+ \right)^{-1/\gamma}, \quad 0 \le t \le T.$$

(*When* $\gamma = 0$ *we have a usual exponential decay of* $\phi(t)$.)

3. LOCAL EXISTENCE AND UNIQUENESS

We begin with the following result concerning the local in time solutions.

Proposition 3.1. *Assume Hyp. A, Hyp. B and Hyp. C, where we make the conditions*

$$(4-N)\alpha + 2 > \nu(N-2), \quad (4-N)(\beta+1) > \nu(N-2)$$

and

$$0 < \gamma < 4/N, \quad \gamma \le 2/(N-2)^+.$$

(*The second condition on β should be dropped under* (2.3).)

Let $(u_0, u_1) \in H_2 \cap H_1^0 \times H_1^0$ and take K_2, K_0 such that

$$K_0 > 1 \quad \text{and} \quad \|\Delta u_0\| < K_2.$$

Then there exists $T = T(K_2, K_0, \|\Delta u_0\| + \|\nabla u_1\|, E(0)) > 0$ such that problem (1.1)–(1.2) *admits a unique solution $u(t)$ in the class*

$$X(T) \equiv L^\infty([0,T); H_2 \cap H_1^0) \cap W^{1,2}([0,T); H_1^0) \cap W^{2,2}([0,T); L^2),$$

satisfying

$$\|\Delta u(t)\| < K_2, \ E(t) < K_0 E(0) \ \text{if}\ E(0) \neq 0$$

and

$$\|\nabla u_t(t)\|^2 + \int_0^T \|u_{tt}(s)\|^2 ds \le C(K_2, T) < \infty, \tag{3.1}$$

where $E(t)$ is the energy defined by (1.10).

Proof. Let $\{\phi_m\}_{m=1}^\infty$ be the basis of $H_2 \cap H_1^0$ consisting of the eigen functions of $-\Delta$ with the Dirichlet boundary condition. Define $u_m(t) = \sum_{i=1}^m C_i(t)\phi_i$ through the solutions $\{C_i(t)\}_{i=1}^m$ of the system of ordinary differential equations

$$(\ddot{u}_m, \phi_i) + (\sigma(|\nabla u_m|^2)\nabla u_m, \nabla\phi_i) + (\nabla u_m, \nabla\phi_i) = f(u_m, \nabla u_m, \dot{u}_m), (i = 1, \dots, m)$$

where initial data are taken as

$$u_m(0) = \sum_{i=1}^m C_i(0)\phi_i \to u_0 \text{ in } H_2 \cap H_1^0 \quad \text{and} \quad \dot{u}_m(0) = \sum_{i=1}^m \dot{C}_i(0)\phi_i \to u_1 \text{ in } H_1^0.$$

The system admits a unique solution $u_m(t)$ on an interval $[0, T_m), T_m > 0$. We derive a priori estimates for $u_m(t)$ independent of sufficiently large m. We define $E_m(t)$ by $E(t)$ with u replaced by u_m. Then we have, by the usual manner,

$$E_m(t) + \int_0^t \|\nabla \dot{u}_m(s)\|^2 ds = E_m(0) + \int_0^t f(u_m, \nabla u_m, \dot{u}_m, \dot{u}_m) ds \tag{3.2}$$

and

$$\frac{1}{2}\|\Delta u_m(t)\|^2 + (\nabla \dot{u}_m(t), \nabla u_m(t)) + \int_0^t \int_\Omega \operatorname{div}\{\sigma(|\nabla u_m|^2)\nabla u\}\Delta u_m dxds =$$
$$= \frac{1}{2}\|\Delta u_m(0)\|^2 + \tag{3.3}$$
$$+ (\nabla \dot{u}_m(0), \nabla u_m(0)) + \int_0^t \|\nabla \dot{u}_m(s)\|^2 ds - \int_0^t (f(u_m, \nabla u_m, \dot{u}_m), \Delta u_m) ds.$$

Assume for a moment that

$$\|\Delta u_m(t)\| \le K_2 \quad \text{and} \quad E_m(t) \le K_0 E_m(0) \tag{3.4}$$

for some interval $0 \le t \le \tilde{T}_m < T_m$. Note that these estimates are certainly valid for sufficiently small $\tilde{T}_m$ and large m by our assumptions $K_2 > \|\Delta u_0\|$ and $K_0 > 1$.

By the assumption on σ, we see for $u \in H_2 \cap H_1^0$,

$$J(\nabla u) \equiv \frac{1}{2}\int_\Omega \int_0^{|\nabla u|^2} \sigma(\tau) d\tau dx \ge C \int_\Omega \frac{|\nabla u|^2}{(1+|\nabla u|^2)^\nu} dx$$

for a certain $C > 0$. Hence, setting

$$\mu = \frac{N - \nu(N-2)^+}{N + \nu(N-2)^+}$$

we have

$$\|\nabla u\|_{1+\mu} = \left\{ \int_\Omega \left(\frac{|\nabla u|^2}{(1+|\nabla u|^2)^\nu} \right)^{(1+\mu)/2} (1+|\nabla u|^2)^{(1+\mu)\nu/2} dx \right\}^{1/(1+\mu)} \le$$
$$\le \left(\int_\Omega \frac{|\nabla u|^2}{(1+|\nabla u|^2)^\nu} dx \right)^{1/2} \left(\int_\Omega (1+|\nabla u|^2)^{N/(N-2)^+} dx \right)^{\nu(N-2)^+/2N} \le \tag{3.5}$$
$$\le CJ(|\nabla u|)^{1/2}(1+\|\Delta u\|^\nu) \le C(1+K_2)^\nu \sqrt{E(t)}, \quad 0 \le t \le \tilde{T}_m.$$

Now, by our assumption on f,

$$\left| \int_0^t (f(u_m, \nabla u_m, \dot{u}_m), \dot{u}_m) ds \right| \le$$
$$\le C \int_0^t \int_\Omega (|u_m|^{\alpha+1} + |u_m|^{\beta+1}|\nabla u_m| + |\dot{u}_m|^{\gamma+1})|\dot{u}_m| dxds. \tag{3.6}$$

For simplicity of notation we write $u(t)$ for $u_m(t)$ for a moment. First we note that

$$\|\nabla u\| \le C\|\nabla u\|_{1+\mu}^{1-\theta_0}\|\Delta u\|^{\theta_0} \le CK_2^{\theta_0}(1+K_2)^{\nu(1-\theta_0)}E_m(t)^{(1-\theta_0)/2} \tag{3.7}$$

with

$$\theta_0 = \frac{\nu(N-2)^+}{2+\nu(N-2)^+}.$$

Each term of the right-hand side of (3.6) is estimated as follows:

$$\begin{aligned} I_1 \equiv \int_\Omega |u|^{\alpha+1}|\dot u|dx &\le \|u\|_{2(\alpha+1)}^{\alpha+1}\|\dot u\| \le C\|u\|_p^{(\alpha+1)(1-\theta_1)}\|\Delta u\|^{(\alpha+1)\theta_1}\|\dot u\| \le \\ &\le C\|\nabla u\|_{1+\mu}^{(\alpha+1)(1-\theta_1)}\|\Delta u\|^{(\alpha+1)\theta_1}\|\dot u\|, \end{aligned} \tag{3.8}$$

where $p = N(1+\mu)/(N-1-\mu)^+ = 2N/(N-2)^+(1+\nu))$ and θ_1 is determined by

$$\begin{cases} \theta_1 = 0 & \text{if } \alpha+1 \le \frac{N}{(\nu+1)(N-2)^+}, \\ \theta_1 = \frac{(N-2)^+(\nu+1)-N/(\alpha+1)}{2+\nu(N-2)^+}, & \text{otherwise.} \end{cases}$$

(Note that $\theta \le 1$ since $\alpha \le 4/(N-4)^+$.)

When $1 \le N \le 3$ we see

$$\begin{aligned} I_2 \equiv \int_\Omega |u|^{\beta+1}|\nabla u||\dot u|dx &\le C\|u\|_{(\beta+1)N}^{1+\beta}\|\Delta u\|\|\dot u\| \le \\ &\le C\|\nabla u\|_{1+\mu}^{(\beta+1)(1-\theta_2)}\|\Delta u\|^{(\beta+1)\theta_2}\|\Delta u\|\|\dot u\|, \end{aligned} \tag{3.9}$$

where θ_2 is determined by

$$\begin{cases} \theta_2 = 0 & \text{if } N = 1, 2, \\ \theta_2 = \frac{(N-2)(1+\nu)-2/(\beta+1)}{2+\nu(N-2)} & \text{if } N = 3. \end{cases}$$

(A trivial modification is needed in (3.8) and (3.9) if $N = 2$.)

Finally,

$$I_3 \equiv \int_\Omega |\dot u|^{\gamma+2}dx \le C\|\dot u\|^{(\gamma+2)(1-\theta_3)}\|\nabla \dot u\|^{(\gamma+2)\theta_3} \tag{3.10}$$

with $\theta_3 = N\gamma/2(\gamma+2) < 1$. Note that $(\gamma+2)\theta_3 < 2$ by the assumption on γ.

It follows from (3.2) and (3.5)–(3.10) that

$$
\begin{aligned}
&E_m(t)+\frac{1}{2}\int_0^t\|\nabla\dot u_m(s)\|^2ds\le\\
&\le E_m(0)+C\Big\{K_2^{(\alpha+1)\theta_1}(1+K_2)^{\nu(\alpha+1)(1-\theta_1)}\int_0^t E_m(s)^{(\alpha+1)(1-\theta_1)/2+1/2}ds+\\
&\quad+K_2^{(\beta+1)\theta_2+1}(1+K_2)^{\nu(\beta+1)(1-\theta_2)}\int_0^t E_m(s)^{((\beta+1)(1-\theta_2)/2+1/2}ds+\\
&\quad+\int_0^t E_m(s)^{(4-(N-2)\gamma)/(4-N\gamma)}ds\Big\}\le
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&\le E_m(0)+C\hat T\Big\{K_2^{(\alpha+1)\theta_1}(K_0E_m(0))^{(\alpha+1)(1-\theta_1)/2+1/2}+\\
&\quad+K_2^{(\beta+1)\theta_2+1}(1+K_2)^{\nu(\beta+1)(1-\theta_2}(K_0E_m(0))^{(\beta+1)(1-\theta_2)/2+1/2}+\\
&\quad+(K_0E_m(0))^{(4-(N-2)\gamma)/(4-N\gamma)}\Big\},\quad 0\le t\le\hat T,
\end{aligned}
\tag{3.12}
$$

Since $K_0>1$ and all of the exponents $K_0E(0)$ appearing on the right-hand side of (3.12) are greater than 1 we conclude from (3.12) that there exists $\hat T_1>0$ independent of $\tilde T_m$ such that if $0<t\le\min\{\tilde T_m,\hat T_1\}$, then

$$
E_m(t)+\frac{1}{2}\int_0^t\|\nabla\dot u_m(s)\|^2ds<K_0E_m(0),\tag{3.13}
$$

where we assume $E(0)>0$. (The existence is trivial for the case $E(0)=0$.)

Note that we can choose $\hat T_1$ as large as required if we take $E(0)$ to be sufficiently small.

We proceed to the estimation of $\|\Delta u_m(t)\|$. We again write $u(t)$ for $u_m(t)$. Under the Hyp. C the second term on the left-hand side of (3.3) is treated by integration by parts as follows.

$$\int_\Omega \operatorname{div}\{\sigma(|\nabla u|^2)\nabla u\}\Delta u dx =$$

$$= \int_\Omega \left\{\sigma(|\nabla u|^2)\sum_{i,j=1}^N \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + 2\sigma'(|\nabla u|^2)\sum_{i=1}^N \left(\sum_{j=1}^N \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2\right\} dx +$$

$$+ (N-1)\int_{\partial\Omega}\left(\frac{\partial u}{\partial n}\right)^2 H(x) dS \geq 0,$$

where $H(x)$ is the mean curvature of $\partial\Omega$ at $x \in \partial\Omega$ with respect to the outward normal and we have used the assumption $\sigma(v^2) + 2\sigma'(v^2)v^2 \geq 0$. Therefore, we see from (3.3) that

$$\begin{aligned} \frac{1}{2}\|\Delta u(t)\|^2 - (\dot u(t), \Delta u(t)) \leq \frac{1}{2}\|\Delta u(0)\|^2 - (\dot u(0), \Delta u(0)) + \\ + \int_0^t \int_\Omega |f(u,\nabla u, \dot u)||\Delta u| dx ds. \end{aligned} \tag{3.14}$$

The last term of (3.14) is treated as in (3.7)–(3.10) and we have

$$\begin{aligned} &\int_0^t |f(u,\nabla u,\dot u)||\Delta u| ds \leq \\ &\leq C\int_0^t \left(\int_\Omega (|u|^{2(\alpha+1)} + |u|^{2(\beta+1)}|\nabla u|^2 + |\dot u|^{2(\gamma+1)}) dx\right)^{1/2} \|\Delta u(s)\| ds \leq \\ &\leq C K_2^{(\alpha+1)\theta_1+1}(1+K_2)^{(\alpha+1)\nu((1-\theta_1)}\int_0^t E_m(s)^{(\alpha+1)(1-\theta_1)/2} ds + \\ &\quad + C K_2^{(\beta+1)\theta_2+1}(1+K_2)^{\nu(\beta+1)(1-\theta_2)}\int_0^t E_m(s)^{(\beta+1)(1-\theta_2)/2} ds + \\ &\quad + C K_2\left(\int_0^t E(s)^{(2-(N-2)\gamma)/(4-N\gamma)} ds\right)^{1-N\gamma/4}\left(\int_0^t \|\nabla \dot u(s)\|^2 ds\right)^{N\gamma/4}, \end{aligned} \tag{3.15}$$

where we have used the assumption $0 < \gamma < 4/N$ and $0 < \gamma \leq 2/(N-2)^+$ in the treatment of the last term. It follows from (3.14) and (3.15) that

$$
\begin{aligned}
\|\Delta u(t)\|^2 \leq & \|\Delta u(0)\| + 4K_2\sqrt{K_0E_m(0)}+ \\
& + CK_2^{(\alpha+1)\theta_1+1}(1+K_2)^{\nu(\alpha+1)((1-\theta_1)}\int_0^t E_m(s)^{(\alpha+1)(1-\theta_1)/2}ds+ \\
& + CK_2^{(\beta+1)\theta_2+1}(1+K_2)^{\nu(\beta+1)(1-\theta_2)}\int_0^t E_m(s)^{(\beta+1)(1-\theta_2)/2}ds+ \\
& + CK_2\left(K_0E_m(0)\right)^{N\gamma/4}\left(\int_0^t E_m(s)^{(2-(N-2)\gamma)/(4-N\gamma)}ds\right)^{1-N\gamma/4} \leq \\
& \leq \|\Delta u(0)\| + 4K_2\sqrt{K_0E_m(0)}+ \\
& + CtK_2^{'\alpha+1)\theta_1+1}(1+K_2)^{\nu(\alpha+1)(1-\theta_1)}\left(K_0E_m(0)\right)^{(\alpha+1)(1-\theta_1)/2}+ \\
& + CtK_2^{(\beta+1)\theta_2+1}(1+K_2)^{\nu(\beta+1)(1-\theta_2)}\left(K_0E_m(0)\right)^{(\beta+1)(1-\theta_2)/2}+ \\
& + CtK_2\left(K_0E_m(0)\right)^{N\gamma/4}\left(K_0E_m(0)\right)^{(2-(N-2)\gamma)/4}, \quad 0 \leq t < \min\{\tilde{T}_m, \hat{T}_1\}.
\end{aligned}
$$

Since $\|\Delta u(0)\| < K_2$, we see from (3) that there exists $\hat{T}_2$ $(\leq \hat{T}_1)$ independent of $\tilde{T}_m$ and large m such that

$$\sup_{m>>1}\|\Delta u_m(t)\| < K_2 \quad \text{for} \quad 0 \leq \min\{\tilde{T}_m, \hat{T}_2\}. \tag{3.16}$$

Note that $\hat{T}_2$ can be chosen as large as we want if we take $E(0)$ to be sufficiently small. We conclude from (3.13) and (3.16) that the solutions $u_m(t), m >> 1$, in fact exists on $[0, \hat{T}_2]$, that is, we may assume $T_m > \tilde{T}_m > \hat{T}_2$, and they satisfy the estimates

$$\sup_{m>>1} E_m(t) < K_0E(0) \quad \text{and} \quad \sup_{m>>1}\|\Delta u_m(t)\| < K_2, \quad 0 \leq t \leq \hat{T}_2. \tag{3.17}$$

We write $\hat{T} = \hat{T}_2$. Recall that

$$K_0 > 1 \quad \text{and} \quad \|\Delta u(0)\| < K_2.$$

We fix K_0 and K_2 arbitrarily. Further we take arbitrary $\hat{T} > 1$. Then we can conclude that there exists $\delta_0 = \delta_0(K_2, K_0, \hat{T}) > 0$ such that if $E(0) < \delta_0$, the solutions u_m, m large, exist on $[0, \hat{T}]$ and the estimates in (3.17) hold on $[0, \hat{T}]$.

Further, multiplying the equation by $u_{tt}(t)$ and integrating we have

$$
\begin{aligned}
&\|u_{tt}\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_t(t)\|^2 \leq \\
&\leq \int_\Omega |\mathrm{div}\{\sigma(|\nabla u\|^2)\nabla u\}||u_{tt}|dx+ \\
&\quad + \int_\Omega |f(u,\nabla u,u_t)||u_{tt}|dx \leq \\
&\leq C\int_\Omega \left\{2|\sigma'(|\nabla u|^2)||\nabla u|^2|D^2u| + \sigma(|\nabla u|^2)|\Delta u|\right\}|u_{tt}|dx + \int_\Omega |f(u,\nabla u,u_t)||u_{tt}|dx
\end{aligned}
\tag{3.18}
$$

which implies

$$\int_0^t \|u_{tt}(s)\|^2 ds + \|\nabla u_t(t)\|^2 \le \|\nabla u_t(0)\|^2 + C\int_0^t \|\Delta u(t)\|^2 + \int_\Omega |f(u,\nabla u,u_t)|^2 dx \le$$
$$\le \|\nabla u_t(0)\| + q_1(K_2)\hat{T} < \infty, \quad 0 \le t \le \hat{T}. \tag{3.19}$$

with some quantity $q(K_2)$. Now, it is a standard argument to show that the limit of $u_m(t)$ as $m \to \infty$ becomes the desired solution in $X_2(\hat{T})$. Now we change the notation $\hat{T}$ by T. The proof of the existence part of local in time solutions is complete.

Finally, we prove the uniqueness of the local solutions. Let u, v be two solutions with the same initial data and set $w = u - v$. We may assume that both of solutions satisfy the estimates which have been proved for u in the above. Then multiplying the difference of two equations by w we easily see,

$$\begin{aligned}
&\frac{d}{dt}\left((w_t,w) + \frac{1}{2}\|\nabla w(t)\|^2\right) + \int_\Omega (\sigma(|\nabla u|^2)\nabla u - \sigma(|\nabla v|^2)\nabla v)\nabla w dx = \\
&= \|w_t(t)\|^2 + \int_\Omega \left(f(u,\nabla u,u_t) - f(v,\nabla v,v_t)\right) w dx \le \\
&\le \|w_t(t)\|^2 + C\int_\Omega \big((1+|u|^\alpha+|v|^\alpha)|w|^2 + (1+|u|^\beta+|v|^\beta)(|\nabla u|+|\nabla v|)|w|^2 + \\
&\quad +(1+|u|^{\beta+1}+|v|^{\beta+1})|\nabla w||w| + (1+|u_t|^\gamma+|v_t|^\gamma)|w_t||w|\big)\, dx \le \\
&\le \|w_t(t)\|^2 + C(K_2)\|\nabla w\|^2 + C(1+\|\nabla u_t\|^\gamma + \|\nabla v_t\|^\gamma)\|\nabla w_t\|\|w\|.
\end{aligned} \tag{3.20}$$

Note that the second term of the left-hand side of (3.20) is nonnegative and hence, integrating (3.20) and using (3.19) we have

$$\begin{aligned}
&(w_t(t),w(t)) + \frac{1}{2}\|\nabla w(t)\|^2 \le \\
&\le \int_0^t \|w_t(s)\|^2 ds + C(K_2,T)\int_0^t (\|\nabla w(s)\|^2 + \|\nabla w_t(s)\|\|\nabla w(s)\|)ds
\end{aligned}$$

and hence

$$\begin{aligned}
&\|w(t)\|^2 + \int_0^t \|\nabla w(s)\|^2 ds \le \\
&\le 2t\int_0^t \|w_t(s)\|^2 ds + C(K_2,T)\int_0^t (\|\nabla w(s)\|^2 + \|\nabla w_t(s)\|^2)ds, \quad 0 \le t \le T.
\end{aligned} \tag{3.21}$$

Next, multiplying the difference of two equations by $w_t(t)$ and integrating we have easily,

$$\frac{1}{2}\|w_t(t)\|^2 + \int_0^t \|\nabla w_t(s)\|^2 ds \le \int_0^t \left\{ \int_\Omega |(\sigma(|\nabla u|^2)\nabla u - \sigma(|\nabla v|^2)\nabla v)\nabla w_t| dx \right\} ds +$$

$$+ \int_0^t \left\{ \int_\Omega \left(f(u,\nabla u, u_t) - f(v,\nabla v, v_t)\right) w_t dx \right\} ds \le$$

$$\le C \int_0^t \|\nabla w(s)\| \|\nabla w_t(s)\| ds +$$

$$+ C(K_2) \int_0^t \left(\|\nabla w(s)\| \|w_t(s)\| + \|\nabla w_t(s)\| \|w_t(s)\|\right) ds$$

and hence

$$\|w_t(t)\|^2 + \int_0^t \|\nabla w_t(s)\|^2 ds \le C(K_2) \int_0^t (\|\nabla w(s)\|^2 + \|w_t(s)\|^2) ds. \tag{3.22}$$

It follows from (3.21) and (3.22) that for any $k > 0$,

$$\|w(t)\|^2 + k\|w_t(t)\|^2 + \int_0^t (\|\nabla w(s)\|^2 + k\|\nabla w_t(s)\|^2) ds \le$$

$$\le 2t \int_0^t \|w_t(s)\|^2 ds + C(K_2, T) \int_0^t (\|\nabla w(s)\|^2 + \|\nabla w_t(s)\|^2) ds + \tag{3.23}$$

$$+ kC(K_2) \int_0^t (\|\nabla w(s)\|^2 + \|w_t(s)\|^2) ds.$$

We take $k = 1/(C(K_2) + 1)$ and $T_2 = \min\{T, k/C(K_2, T)\}$. Then, by (3.23), we deduce that

$$\|w(t)\|^2 + k\|w_t(t)\|^2 \le (2T_2 + 1) \int_0^t \|w_t(s)\|^2 ds, \quad 0 \le t \le T_2, \tag{3.24}$$

which implies $w_t(t) = 0$ and hence $w(t) = 0, 0 \le t \le T_2$. Repeating this argument we conclude $w(t) = 0$, $0 \le t \le T$. Thus uniqueness is proved. □

Remark 3.2. Without Hyp. C we can prove a similar local existence and uniqueness result as in Proposition 3.1. But, in this case, in order to take the existence time T to be large we must assume that both of K_2 and $E(0)$ are sufficiently small.

4. A DIFFERENCE INEQUALITY

We take K_2 such that $\|\Delta u(0)\| < K_2$. The solution $u(t)$ exists on $[0,T)$ for some $T>1$ under a smallness condition $E(0) < \delta_0 = \delta_0(K_2)$. Further we know $E(t) < K_0 E(0)$ on $[0,1]$ if $E(0) \neq 0$. We may assume that

$$\|\Delta u(t)\| \le K_2 \quad \text{and} \quad E(t) \le K_0 E(0) \tag{4.1}$$

on some interval $[0,\tilde{T}], 1 < \tilde{T} < T$. If we can derive the estimates

$$\|\Delta u(t)\| < K_2 \quad \text{and} \quad E(t) < K_0 E(0), \quad 0 \le t \le \tilde{T}, \tag{4.2}$$

we can conclude that estimates (4.1) in fact hold on the interval $[0,T)$, and consequently the solution in fact exists on the whole interval $[0,\infty)$. We call such an argument a 'loan' method.

Multiplying the equation by u_t and integrating we have

$$\int_t^{t+1} \|\nabla u_t(s)\|^2 ds = E(t) - E(t+1) + \int_t^{t+1}\int_\Omega F u_t dx ds \equiv D(t)^2, \quad 0 \le t \le \tilde{T}-1, \tag{4.3}$$

where we set $F = f(u, \nabla u, u_t)$. We derive the following inequality.

Proposition 4.1.

$$\begin{aligned} &\sup_{t \le s \le t+1} E(s) + \int_t^{t+1} \|\nabla u_t(s)\|^2 ds \le \\ &\le q(K_2) D(t)^{2/(1+\theta_0)} + \\ &\quad + C D(t)^2 + C \int_t^{t+1} \left(\int_\Omega |F|(|u| + |u_t|) dx \right) ds, \quad 0 \le t \le \tilde{T} - 1, \end{aligned} \tag{4.4}$$

where $q(K_2)$ is a certain positive constant depending on K_2.

Proof. We use the argument as in [9]. We know from (4.3) that there exist $t_1 \in [t, t+1/4], t_2 \in [t+3/4, t+1]$ such that

$$\|u_t(t_i)\| \le C \|\nabla u_t(t_i)\| \le 4CD(t)^2, \quad i = 1,2. \tag{4.5}$$

Next, multiplying the equation by u and integrating we have

$$\begin{aligned} &\int_{t_1}^{t_2} \left(\int_\Omega \sigma(|\nabla u|^2) |\nabla u|^2 dx \right) ds = \\ &= \sum_{i=1,2} \pm (u_t(t_i), u(t_i)) + D(t)^2 + \int_{t_1}^{t_2} (\nabla u_t, \nabla u) ds + \int_{t_1}^{t_2} \int_\Omega F u dx ds. \end{aligned} \tag{4.6}$$

Recall that

$$J(\nabla u) \equiv \frac{1}{2}\int\limits_\Omega \Bigg(\int\limits_0^{|\nabla u|^2} \sigma(\eta)d\eta\Bigg)dx \le C\int\limits_\Omega \sigma(|\nabla u|^2)|\nabla u|^2 dx$$

and

$$E(t) = \frac{1}{2}\|u_t(t)\|^2 + J(\nabla u).$$

We already know that

$$J(\nabla u) \ge C(1+K_2)^{-2\nu}\|\nabla u(t)\|_{1+\mu}^2$$

and

$$\|\nabla u(t)\| \le C\|\nabla u(t)\|_{1+\mu}^{1-\theta_0}\|\Delta u(t)\|^{\theta_0} \le \\ \le CK_2^{\theta_0}(1+K_2)^{(1-\theta_0)\nu}\sup_{t\le s\le t+1} E(s)^{(1-\theta_0)/2},$$

where

$$\mu = \frac{N-\nu(N-2)^+}{N+\nu(N-2)+} \quad \text{and} \quad \theta_0 = \frac{\nu(N-2)^+}{2+\nu(N-2)^+}.$$

Thus,

$$|(u_t(t_i), u(t_i)| \le C\|u_t(t_i)\|\|\nabla u(t_i)\| \le \\ \le CK_2^{\theta_0}(1+K_2)^{(1-\theta_0)\nu}D(t)\sup_{t\le s\le t+1} E(s)^{(1-\theta_0)/2}$$

and

$$\Big|\int\limits_{t_1}^{t_2}(\nabla u_t, \nabla u)dx\Big| \le C\Bigg(\int\limits_{t_1}^{t_2}\|\nabla u_t(s)\|^2 ds\Bigg)^{1/2}\Bigg(\int\limits_{t_1}^{t_2}\|\nabla u(s)\|^2 ds\Bigg)^{1/2} \le \\ \le CK_2^{\theta_0}(1+K_2)^{(1-\theta_0)\nu}D(t)\sup_{t\le s\le t+1} E(s)^{(1-\theta_0)/2}.$$

Therefore we see from (4.3), (4.5) and (4.6) that

$$\int\limits_{t_1}^{t_2} E(s)ds \le q_1(K_2)D(t)\sup_{t\le s\le t+1} E(s)^{(1-\theta_0)/2} + \\ + D(t)^2 + \int\limits_t^{t+1}\Bigg(\int\limits_\Omega |F|(|u|+|u_t|)dx\Bigg)ds \equiv A(t)^2$$

with a certain constant $q_1(K_2)$, which implies

$$E(t^*) \le 2A(t)^2$$

for some $t^* \in [t_1, t_2]$, and hence, by energy identity (see (3.2)),

$$
\begin{aligned}
\sup_{t \le s \le t+1} E(s) \le E(t^*) + \int_t^{t+1} \|\nabla u_t(s)\|^2 ds + \\
+ \int_t^{t+1} \int_\Omega |F u_t| dx ds \le \\
\le 2 q_1(K_2) D(t) \sup_{t \le s \le t+1} E(s)^{(1-\theta_0)/2} + D(t)^2 + \\
+ 2 \int_t^{t+1} \left(\int_\Omega |F|(|u| + |u_t|) dx \right) ds.
\end{aligned} \tag{4.7}
$$

Inequality (4.7) easily yields the desired inequality (4.4). □

5. BOUNDEDNESS AND DECAY OF $E(t)$ ON $[0, \tilde{T}]$

From difference inequality (4.4) we first derive the boundedness of $E(t)$, $0 \le t \le \tilde{T}$.

Assume that $E(t) \le E(t+1)$ for some t, $0 \le t \le \tilde{T} - 1$. Then, inequality (4.4) implies

$$
\begin{aligned}
\sup_{t \le s \le t+1} E(s) + \int_t^{t+1} \|\nabla u_t(s)\|^2 ds \le \\
\le q(K_2) \left(\int_t^{t+1} \int_\Omega |F||u_t| dx ds \right)^{1/(1+\theta_0)} + C \int_t^{t+1} \left(\int_\Omega |F|(|u| + |u_t|) dx \right) ds.
\end{aligned} \tag{5.1}
$$

By the argument in the proof of Proposition 3.1 (see (3.10), (3.11)), we have

$$
\begin{aligned}
\int_t^{t+1} \int_\Omega |F u_t| dx ds \le C K_2^{\theta_1(\alpha+1)} \sup_{t \le s \le t+1} E(s)^{(\alpha+1)(1-\theta_1)/2 + 1/2} + \\
+ C K_2^{\theta_2(\beta+1)+1} \sup_{t \le s \le t+1} E(s)^{(\beta+1)(1-\theta_2)/2 + 1/2} + \\
+ C \sup_{t \le s \le t+1} E(s)^{(\gamma+2)(1-\theta_3)/2} ds \left(\int_t^{t+1} \|\nabla u_t(s)\|^2 \right)^{(\gamma+2)\theta_3} \equiv \\
\equiv I_1 + I_2 + I_3.
\end{aligned} \tag{5.2}
$$

Further,

$$\int_t^{t+1}\int_\Omega |Fu|dxds \le C \sup_{t\le s\le t+1}\{\int_\Omega (|u|^{\alpha+2}+|u|^{\beta+2}|\nabla u|+|u_t|^{\gamma+1}|u|)dx\} \equiv \\ \equiv \tilde{I}_1+\tilde{I}_2+\tilde{I}_3. \tag{5.3}$$

Here we see

$$\tilde{I}_1 \le C \sup_{t\le s\le t+1}\|\nabla u(s)\|_{1+\mu}^{(\alpha+2)(1-\tilde{\theta}_1)}\|\Delta u(s)\|^{(\alpha+2)\tilde{\theta}_1} \le \\ \le CK_2^{(\alpha+2)\tilde{\theta}_1}(1+K_2)^{\nu(\alpha+2)(1-\tilde{\theta}_1)}\sup_{t\le s\le t+1}E(s)^{(\alpha+2)(1-\tilde{\theta}_1)/2} \tag{5.4}$$

with

$$\tilde{\theta}_1 = \begin{cases} 0 & \text{if } N=1,2, \\ \frac{((N-2)(\nu+1)-2N/(\alpha+1)}{2+\nu(N-2))^+} & \text{if } N\ge 3, \end{cases}$$

and

$$\tilde{I}_2 \le C \sup_{t\le s\le t+1}\|u(s)\|_{2(\beta+2)}^{\beta+2}\|\nabla u(s)\| \le \\ \le C \sup_{t\le s\le t+1}\|\nabla u\|^{(\beta+2)(1-\tilde{\theta}_2)+1}\|\Delta u\|^{\tilde{\theta}_2(\beta+2)} \le \\ \le CK_2^{\tilde{\theta}_2(\beta+2)+((\beta+2)(1-\tilde{\theta}_2)+1)\theta_0}(1+K_2)^{\nu(1-\theta_0)(\beta+2)(1-\tilde{\theta}_2+1)}\times \\ \times \sup_{t\le s\le t+1}E(s)^{((\beta+2)(1-\tilde{\theta}_2)+1)(1-\theta_0)/2} \tag{5.5}$$

with

$$\tilde{\theta}_2 = \begin{cases} 0 & \text{if } N=1,2, \\ (\beta-1)^+/3(\beta+2) & \text{if } N=3. \end{cases}$$

Further,

$$\tilde{I}_3 \le C\int_t^{t+1}\|u_t\|_{2N(\gamma+1)/(N+2)}^{\gamma+1}\|\nabla u\|ds \le \\ \le C\int_t^{t+1}\|u_t\|^{(\gamma+1)(1-\tilde{\theta}_3)}\|\nabla u_t\|^{(\gamma+1)\tilde{\theta}_3}\|\nabla u\|ds \le \\ \le CK_2^{\theta_0}(1+K_2)^{\nu(1-\theta_0)}\sup_{t\le s\le t+1}E(s)^{(\gamma+1)(1-\tilde{\theta}_3)/2+(1-\theta_0)/2}\times \\ \times\Big\{\int_t^{t+1}\|\nabla u_t(s)\|^2ds\Big\}^{(\gamma+1)\tilde{\theta}_3/2} \tag{5.6}$$

with $\tilde{\theta}_3 = (N\gamma-2)^+/2(\gamma+1)$.

It follows from (5.1) and (5.2)–(5.6) that

$$
\begin{aligned}
&\sup_{t\le s\le t+1} E(s) + \int_t^{t+1} \|\nabla u_t(s)\|^2 ds \le \\
&\le Cq(K_2)(I_1+I_2+I_3)^{1/(1+\theta_0)} + C(I_1+I_2+I_3+\tilde I_1+\tilde I_2+\tilde I_3) \le \\
&\le \tilde q(K_2)\{\sup_{t\le s\le t+1} E(s)^{(\alpha+1)(1-\theta_1)+1)/2(1+\theta_0)} + \sup_{t\le s\le t+1} E(s)^{((\beta+1)(1-\theta_2)+1)/2(1+\theta_0)} + \\
&\quad + \sup_{t\le s\le t+1} E(s)^{(\gamma+2)(1-\theta_3)/(2(1+\theta_0)-(\gamma+2)\theta_3)} + \sup_{t\le s\le t+1} E(s)^{((\alpha+1)(1-\theta_1)+1)/2}\} + \\
&\quad + \sup_{t\le s\le t+1} E(s)^{((\beta+1)(1-\theta_2)+1)/2} + \sup_{t\le s\le t+1} E(s)^{(\gamma+2)(1-\theta_3)/(2-\theta_3(\gamma+2))} + \\
&\quad + C\{\sup_{t\le s\le t+1} E(s)^{((\alpha+2)(1-\tilde\theta_1)/2} + \sup_{t\le s\le t+1} E(s)^{((\beta+2)(1-\tilde\theta_2)+1)(1-\theta_0)/2} + \\
&\quad + \sup_{t\le s\le t+1} E(s)^{((\gamma+1)(1-\tilde\theta_3)+1-\theta_0)/(2-(\gamma+1)\tilde\theta_3)}\} + \frac{1}{2}\int_t^{t+1} \|\nabla u_t(s)\|^2 ds
\end{aligned} \tag{5.7}
$$

with a certain constant $\tilde q(K_2)$. We note that by our assumption on α, β and γ, all of the exponents of $E(s)$ appearing in the right-hand side of (5.7) are greater than 1. Hence, using assumption $E(t) \le K_0 E(0), 0 \le t \le \tilde T$, we obtain

$$
\begin{aligned}
&\sup_{t\le s\le t+1} E(s) + \int_t^{t+1} \|\nabla u_t(s)\|^2 ds \le \\
&\le Q_1(K_2, K_0E(0)) \sup_{t\le s\le t+1} E(s), \quad 0 \le t \le \tilde T - 1
\end{aligned} \tag{5.8}
$$

with a certain quantity $Q_1(K_2, K_0E(0))$ which depends on K_2 and $K_0E(0)$ continuously and $Q(K_2, 0) = 0$. Therefore, there exists $\delta_1(K_2) > 0$ such that if $K_0E(0) < \delta_1(K_2)$, then $Q_1(K_2, K_0E(0)) < 1$ and consequently,

$$
\sup_{t\le s\le t+1} E(s) \le 0, \quad \text{i.e.} \quad E(s) = 0, \quad t \le s \le t+1. \tag{5.9}
$$

Recall that (5.9) is deduced under the assumption $E(t) \le E(t+1)$ for some t, $0 \le t \le \tilde T - 1$. Thus we conclude that $E(t+1) \le E(t)$ for all t, $0 \le t \le \tilde T - 1$. In particular we see

$$
E(t) \le \sup_{0\le s\le 1} E(s) < K_0E(0) \quad \text{for all } t,\ 0 \le t \le \tilde T. \tag{5.10}
$$

Returning to the difference inequality (4.4) and using the above fact we obtain

$$
\sup_{t\le s\le t+1} E(s) + \int_t^{t+1} \|\nabla u_t(s)\|^2 ds \le q(K_2) D_0(t)^{2/(1+\theta_0)} + C D_0(t)^2 +
$$

$$+CQ_1(K_2, K_0E(0)) \sup_{t\le s\le t+1} E(s), \quad 0 \le t \le \tilde{T}-1,$$

where we set

$$D_0(t)^2 = E(t) - E(t+1).$$

There exists $\delta_2(K_2) > 0$ such that if $K_0E(0) < \delta_2(K_2)$, we have

$$CQ_1(K_2, K_0E(0)) \le \frac{1}{2}. \tag{5.11}$$

(We may assume $\delta_2 < \delta_1$.)

Then we conclude that if $K_0E(0) < \delta_2(K_2)$,

$$\sup_{t\le s\le t+1} E(s) + \int_t^{t+1} \|\nabla u_t(s)\|^2 ds \le C(K_2)\{D_0(t)^{2/(1+\theta_0)} + D_0(t)^2\}, \quad 0 \le t \le \tilde{T}-1. \tag{5.12}$$

For simplicity we may assume $K_0E(0) \le 1$, and hence (5.12) implies

$$\sup_{t\le s\le t+1} E(s)^{1+\theta_0} \le C(K_2)(E(t) - E(t+1)). \tag{5.13}$$

Applying Lemma 1.1 to (5.12) we arrive at the decay estimate of $E(t)$,

$$E(t) \le \left((K_0E(0))^{-\theta_0} + C(K_2)^{-1}\theta_0(t-1)^+\right)^{-1/\theta_0}, \quad 0 \le t \le \tilde{T}. \tag{5.14}$$

When $N = 1$ (5.14) should be changed to the exponential decay

$$E(t) \le C(K_2)E(0)\mathrm{e}^{-\lambda t}$$

for some $\lambda > 0$ independent of $E(0)$, and when $N = 2$ we have

$$E(t) \le \left((K_0E(0))^{-1/m} + C(K_2)^{-1}m(t-1)^+\right)^{-m}, \quad 0 \le t \le \tilde{T}$$

for arbitrarily large $m >> 1$.

6. ESTIMATION OF $\|\Delta u(t)\|$ ON $[0, \tilde{T}]$ AND COMPLETION OF THE PROOF OF THEOREM 2.1

We proceed to the estimation of $\|\Delta u(t)\|$ under the assumption (4.1).

Multiplying the equation by $-\Delta u(t)$ and integrating we know (see (3.14)) that

$$\|\Delta u(t)\|^2 \le \|\Delta u(0)\|^2 + 2\sqrt{K_0E(0)}K_2 + \\ + C\int_0^t \int_\Omega (|u|^{\alpha+1} + |u|^{\beta+1}|\nabla u| + |u_t|^{\gamma+1}|)|\Delta u| dxds. \tag{6.1}$$

We know also (see (3.15))

$$
\begin{aligned}
&\int_0^t \int_\Omega |u|^{\alpha+1}|\Delta u| dxds \leq \\
&\leq C K_2^{\theta_1(\alpha+1)+1}(1+K_2)^{2\nu(1-\theta_1)(\alpha+1)} \int_0^t E(s)^{(1-\theta_1)(\alpha+1)/2} ds \leq \\
&\leq C K_2^{(\alpha+1)\theta_1+1}(1+K_2)^{\nu(\alpha+1)(1-\theta_1)} \int_0^t \big((K_0 E(0))^{-\theta_0} + \\
&\qquad\qquad + C(s-1)^+\big)^{-(\alpha+1)(1-\theta_1)/2\theta_0} ds \leq \\
&\leq C K_2^{(\alpha+1)\theta_1+1}(1+K_2)^{(\alpha+1)\nu(1-\theta_1)} \big((K_0 E(0))^{(\alpha+1)(1-\theta_1)/2} + \\
&\qquad\qquad + (K_0 E(0))^{(\alpha+1)(1-\theta_1)/2-\theta_0}\big),
\end{aligned} \tag{6.2}
$$

where we have used the fact

$$(\alpha+1)(1-\theta_1)/2 - \theta_0 > 1/2 > 0.$$

Similarly,

$$
\begin{aligned}
&\int_0^t \int_\Omega |u|^{\beta+1}|\nabla u||\Delta u| dxds \leq \\
&\leq C K_2^{(\beta+1)\theta_2}(1+K_2)^{\nu(\beta+1)(1-\theta_2)} \int_0^t E(s)^{(\beta+1)(1-\theta_2)/2} ds \leq \\
&\leq C K_2^{(\beta+1)\theta_2+1}(1+K_2)^{\nu(\beta+1)(1-\theta_2)} (K_0 E(0))^{(\beta+1)(1-\theta_2)/2} + \\
&\quad + (K_0 E(0))^{(\beta+1)(1-\theta_2)/2-\theta_0}.
\end{aligned} \tag{6.3}
$$

The treatment of the last term of (6.1) is also similar. We have

$$
\begin{aligned}
&\int_0^t \int_\Omega |u_t|^{\gamma+1}|\Delta u| dxds \leq C K_2 \int_0^t \|u_t(s)\|_{2(\gamma+1)}^{\gamma+1} ds \leq \\
&\leq C K_2 \left(\int_0^t E(s)^{(2\gamma+2-N\gamma)/(4-N\gamma)} ds\right)^{1-N\gamma/4} \left(\int_0^t \|\nabla u_t(s)\|^2 ds\right)^{N\gamma/4}.
\end{aligned} \tag{6.4}
$$

Here, using the fact $(2\gamma+2-N\gamma)/(4-N\gamma)\theta_0 > 1$, we have

$$
\begin{aligned}
&\int_0^t E(s)^{(2\gamma+2-N\gamma)/(4-N\gamma)} ds \leq \\
&\leq (K_0 E(0))^{(2\gamma+2-N\gamma)} + C(K_0 E(0))^{(2+2\gamma-N\gamma)/(4-N\gamma)-\theta_0}.
\end{aligned} \tag{6.5}
$$

Further (see (3.11)),

$$\begin{aligned}
&\int_0^t \|\nabla u_t(s)\|^2 ds \leq \\
&\leq 2E(0)+ \\
&\quad +C\Bigg\{ K_2^{(\alpha+1)\theta_1} \int_0^t E(s)^{(\alpha+1)(1-\theta_1)/2+1/2} ds + \\
&\qquad + CK_2^{(\beta+1)\theta_2+\theta_0}(1+K_2)^{\nu(\beta+1)(1-\theta_0)} \int_0^t E(s)^{((\beta+1)(1-\theta_2)+1)/2} ds + \\
&\qquad + CK_2 \int_0^t E(s)^{(4-(N-2)\gamma)/(4-N\gamma)} ds \Bigg\} \leq \qquad (6.6)\\
&\leq 2E(0) + CK_2^{(\alpha+1)\theta_1}\Big((K_0E(0))^{((\alpha+1)((1-\theta_1)+1)/2\theta_0} + \\
&\qquad\qquad + (K_0E(0))^{((\alpha+1)(1-\theta_1)+1)/2-\theta_0}\Big) + \\
&\quad + CK_2^{(\beta+1)\theta_2+\theta_0}(1+K_2)^{\nu(\beta+1)(1-\theta_0} \Big((K_0E(0))^{((\beta+1)(1-\theta_2)+1)/2\theta_0} + \\
&\quad + C(K_0E(0))^{((\beta+1)(1-\theta_2)+1)/2-\theta_0}\Big) + \\
&\quad + C\left((K_0E(0))^{(4-(N-2)\gamma)/(4-N\gamma)\theta_0} + (K_0E(0))^{(4-(N-2)\gamma)/(4-N\gamma)-\theta_0}\right).
\end{aligned}$$

Note that all of the exponents of $K_0E(0)$ appearing in (6.2), (6.5) and (6.6) are all positive. Thus we conclude from (6.1) that

$$\|\Delta u(t)\|^2 \leq \|\Delta u(0)\|^2 + Q_2(K_2, K_0E(0)), \quad 0 \leq t \leq \tilde{T}, \tag{6.7}$$

where $Q_2(K_2, K_0E(0))$ is a certain quantity depending on $K_0E(0)$ and K_2 continuouesly and satisfying $Q_2(K_2, 0) = 0$. Therefore, under the assumption $\|\Delta u(0)\| < K_2$, there exists $\delta_3 = \delta_3(K_2) > 0$ such that if $K_0E(0) < \delta_3$, then

$$\|\Delta u(t)\| < K_2, \quad 0 \leq t \leq \tilde{T}. \tag{6.8}$$

(We may assume that $\delta_3(K_2) < \delta_2(K_2) < \delta_1(K_2)$.)

Now under the asumptions $E(t) \leq K_0E(0)$ and $\|\Delta u(t)\| \leq K_2$ on $[0, \tilde{T}]$ we have derived the estimates

$$E(t) < K_0E(0) \quad \text{and} \quad \|\Delta u(t)\| < K_2, \quad 0 \leq t \leq \tilde{T},$$

provided that $0 < E(0) < \delta_3(K_2)/K_0$.

We fix $K_0 > 1$. Then, if $E(0) < \delta_3(K_2)/K_0 \equiv \delta(K_2)$ we can conclude that the solution in fact exists on the whole interval $[0, \infty)$ and all of the estimates derived on $[0, \tilde{T}]$ so far are valid on $[0, \infty)$. The proof of Theorem 2.1 is now complete. The proof of Corollary 2.2 is also included in the above argument.

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