POSITIVITY PRESERVING RESULTS FOR A BIHARMONIC EQUATION UNDER DIRICHLET BOUNDARY CONDITIONS

Hanen Ben Omrane and Saïma Khenissy

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Abstract. We prove a dichotomy result giving the positivity preserving property for a biharmonic equation with Dirichlet boundary conditions arising in MEMS models. We adapt some ideas in [H.-Ch. Grunau, G. Sweers, *Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions*, Math. Ann. 307 (1997), 589–626].

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1. INTRODUCTION

In this paper, we focus on the following biharmonic equation with Dirichlet boundary conditions

$$\begin{cases} \Delta^2 u - \tau \Delta u = f & \text{in } \Omega, \\ u = \nabla u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\tau > 0$ and Ω is a smooth bounded domain of $\mathbb{R}^n$ $(n \geq 2)$. Equation (1.1) arises for example in the micro-electromechanical systems (MEMS devices) giving the modelization of electrostatic actuation for membranes deflecting on thin plates in the field of nanotechnology detection systems [7,8]. The Dirichlet boundary conditions are also called clamped boundary conditions giving rise to zero vertical displacement and zero slope. The positive parameter τ represents the tension constant rising in the stretching energy sector in the presence of elastic deformation.

Our motivation comes from the fact that the maximum principle or the *positivity preserving property* fails often in the case of Dirichlet boundary conditions with the polyharmonic operators, on the contrary for the case of Navier boundary conditions ($u = \Delta u = 0$ on $\partial\Omega$). More precisely, we are interested in the following question: if the source term f is positive in Ω, do we have the solution u of (1.1) positive in Ω, for any $\tau > 0$?

For the one-dimensional case, Grunau (see [3, Proposition 1]) has given a positive answer to this question. He proved that for $\lambda \leq 0$ and $a \in \mathbb{R}$, for every solution u of

$$\begin{cases} u'''' + au''' + \lambda u'' = f & \text{in } (-1,1), \\ u(-1) = u'(-1) = u(1) = u'(1) = 0, \end{cases} \tag{1.2}$$

we have $0 \not\equiv f \geq 0$ implies $u > 0$ (the solution operator for (1.2) is said to be *strongly positively preserving*). The case $\lambda > 0$ (which corresponds to $\tau < 0$ in our situation) was also well understood in [3].

This result was recently extended by Laurençot and Walker [6] giving a general result on the sign-preserving property in radial symmetry for a ball in $\mathbb{R}^n$, $n \geq 2$. Nevertheless, in Section 2, we give a very simple and direct proof showing that if Ω is a ball in $\mathbb{R}^n$, ($n \geq 2$) and if the source term f is radially symmetric, then for any $\tau > 0$, problem (1.1) is strongly positivity preserving, i.e. $u > 0$ if $0 \not\equiv f \geq 0$ (see Proposition 2.1 below).

Our main concern here is to show a dichotomy result, adapting some ideas and techniques used by Grunau and Sweers [4] in the study of positivity for equations involving $(-\Delta)^m$ and the Dirichlet boundary conditions (see also [2] and the references therein). In fact, we consider the problem

$$\begin{cases} \Delta^2 u - \tau\Delta u = a(x)u + f & \text{in } \Omega, \\ u = \nabla u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Let Λ_1^τ denote the first eigenvalue of the operator $L_\tau = \Delta^2 - \tau\Delta$ under the Dirichlet boundary conditions given by

$$\Lambda_1^\tau = \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_{L^2}^2 + \tau\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2} > 0.$$

Let Φ_1^τ be a corresponding eigenfunction. Define $(L_\tau - a)^{inv}$ the Green operator corresponding to the problem (1.3):

$$u = (L_\tau - a)^{inv} f.$$

It is well known that for $a \in C(\overline{\Omega})$ verifying $\max_{\overline{\Omega}} a < \Lambda_1^\tau$ and $p > 1$, the operators

$$(L_\tau - a)^{inv} \ : \ L^p(\Omega) \to W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$$

are well defined.

We say that an operator $T : L^p(\Omega) \to L^p(\Omega)$ is *positivity preserving* if $0 \not\equiv f \geq 0$ implies $Tf \geq 0$, and T is *strongly positivity preserving* if $0 \not\equiv f \geq 0$ implies $Tf > 0$.

Our main result is the following.

Theorem 1.1. *Let $p > 1$. Suppose that there holds $\Phi_1^\tau > 0$ in Ω. Then there exists $-\infty < \lambda_c \le \Lambda_1^\tau$ such that for all $a \in C(\overline{\Omega})$, we have, for u and $f \in L^p(\Omega)$ verifying* (1.3)*:*

1) *If $\lambda_c \le a < \Lambda_1^\tau$ in $\overline{\Omega}$, then $(L_\tau - a)^{inv}$ is positivity preserving, i.e. $f \ge 0 \Longrightarrow u \ge 0$.*
2) *If $a < \lambda_c$ in $\overline{\Omega}$, then $(L_\tau - a)^{inv}$ is not positivity preserving. Indeed, we have in this case:*
   i) $0 \not\equiv f \ge 0 \Longrightarrow u \not\le 0$,
   ii) $\exists\, 0 \not\equiv f \ge 0$ *with* $u \ge 0$,
   iii) $\exists\, 0 \not\equiv f \ge 0$ *with* $u \not\ge 0$.

This result is analogous to Corollary 6.4 in [4] stated for general bounded domains Ω in $\mathbb{R}^n$, $n \ge 2$, with $\partial\Omega \in C^{2m,\gamma}$ for some $\gamma > 0$, $m \ge 2$:

$$\begin{cases} (-\Delta)^m u = a(x)u + f & \text{in } \Omega, \\ \mathcal{D}_m u := (D^k u)_{k \in \mathbb{N}^n, |k| \le m-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The problem (1.4) with $a(x) \equiv 0$ and Ω is a unit ball $B \subset \mathbb{R}^n$ for any n, was studied by Boggio [1] who showed the positivity preserving result by proving the positivity of the Green function. However, it is well known that the *positivity preserving property* could no longer hold true even for some convex two-dimensional domains with Δ^2 and the Dirichlet boundary conditions. On the other hand, Grunau and Sweers proved in [5] that for domains with small C^k deformation from the ball, the Green function remains positive, while the positivity preserving breaks down when the deformation becomes important. For more discussions, we refer to the book [2].

Here we consider the case $m = 2$ but with $\tau > 0$ in (1.3). That means that we consider a lower order perturbation of the biharmonic operator Δ^2 by adding the term $-\tau\Delta u$. We should remark that the coercivity of the operator L_τ cannot imply the positivity preserving property for L_τ. We know also by Theorem 5.1 in [4] that when τ and $a(x)$ are small and Ω is a ball, problem (1.3) is strongly positivity preserving. Our contribution consists then in considering $\Delta^2 - \tau\Delta$ instead of just the biharmonic operator Δ^2 (in view of Corollary 6.4) and a general bounded smooth domain Ω instead of a ball.

Following the ideas in [4], a key point to prove in Theorem 1.1 above is some kind of *continuity result* for the positivity preserving property (see Proposition 3.2) and a crucial result asserting that when $\lambda << 0$, the operator

$$(L_\tau - \lambda)^{inv} \;:\; L^p(\Omega) \to W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega) \subset L^p(\Omega)$$

is not positivity preserving (see Proposition 3.4). Such a result for $\lambda << 0$ is proved using the continuity result together with the construction of some changing-sign solution. We adapted techniques in [4] and used a simple but very helpful result (see Lemma 3.3) in order to overcome the difficulty introduced by the harmonic perturbation. Actually, what we will need is a pointwise estimate for the infinity norm of a solution u to problem (1.1) with a constant souce term $f \equiv 1$ on a ball B_R, and this is provided by the technical Lemma 3.3.

2. POSITIVITY PRESERVING FOR RADIAL SOLUTIONS IN DIMENSION n

Consider the problem

$$\begin{cases} \Delta^2 u - \tau\Delta u = f(x) & \text{in } B, \\ u = \nabla u = 0 & \text{on } \partial B, \end{cases} \tag{2.1}$$

where $B := B(0,1)$ is the unit ball in $\mathbb{R}^n$, $n \geq 2$.

Proposition 2.1. *Suppose that the source term f is radially symmetric, then for any $\tau > 0$, problem (2.1) is strongly positivity preserving, i.e. $u > 0$ if $0 \not\equiv f \geq 0$.*

Proof. Let f be nonnegative and radially symmetric, i.e. $f(x) = f(r)$, with $0 \leq r = |x| \leq 1$. Then the solution u to problem (2.1) is radially symmetric. Let $v = -\Delta u$ in B then v is radially symmetric and satisfies

$$(-\Delta + \tau)v = f \geq 0 \quad \text{in } B. \tag{2.2}$$

If $v(1) \geq 0$, by the strong maximum principle for $L_\tau := -\Delta + \tau$, we know that $v > 0$ in B. This is just impossible, since using the Dirichlet boundary condition, there holds

$$\int_B v dx = \int_{\partial B} \frac{\partial u}{\partial \nu} d\sigma = 0.$$

Therefore, $v(1) < 0$.

Using again the maximum principle for L_τ, we cannot have a subdomain $\Omega \subset B$ such that $v < 0$ in Ω and $v = 0$ on $\partial\Omega$. So there exists $r_0 \in (0,1)$ such that $v \geq 0$ in $(0, r_0)$ and $v < 0$ in $(r_0, 1]$. Then the function

$$r \mapsto h(r) := \int_{B_r} v dx = -u'(r) r^{n-1} |\partial B|$$

is nondecreasing in $(0, r_0)$ and decreasing in $(r_0, 1]$, which means $h(r) \geq 0$ for $0 < r < 1$ as $h(0) = h(1) = 0$. So u is nonincreasing w.r.t. r, which yields finally $u > 0$ in B since $u|_{\partial B} = 0$ and u is decreasing near 1. $\square$

3. A DICHOTOMY RESULT FOR POSITIVITY PRESERVING PROPERTY

In what follows, Ω is a general smooth domain and we prove some results inspired by [4].

Proposition 3.1. *Let $p > 1$. Let $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ and $f \in L^p(\Omega)$ satisfy (1.3). Suppose that we have $\Phi_1^\tau > 0$ in Ω. If $0 \not\equiv a - \Lambda_1^\tau \geq 0$, then*

$$0 \not\equiv f \geq 0 \Longrightarrow u \not\geq 0 \quad (\textit{positivity killing}).$$

If $a \equiv \Lambda_1^\tau$, then for $0 \not\equiv f \geq 0$ there is no solution.

Proof. Suppose that we have $0 \not\equiv f \geq 0$ and $0 \not\equiv u \geq 0$. Then

$$0 < \int\limits_{\Omega} f\Phi_1^\tau dx = \int\limits_{\Omega} (L_\tau u - au)\Phi_1^\tau dx = \int\limits_{\Omega} u(\Lambda_1^\tau - a)\Phi_1^\tau dx \leq 0, \tag{3.1}$$

which is a contradiction if $0 \not\equiv a - \Lambda_1^\tau \geq 0$.

If $a \equiv \Lambda_1^\tau$, again from (3.1), we see that no solution could exist. □

Proposition 3.2. *Let $\tau > 0$, $a \in C(\overline{\Omega})$ with $a < \Lambda_1^\tau$. Suppose that*

$$(L_\tau - a)^{inv} \ : \ L^p(\Omega) \to W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega) \subset L^p(\Omega)$$

is positivity preserving (resp. strongly positivity preserving) for $p > 1$. Then for all $b \in C(\overline{\Omega})$ with $a \leq b < \Lambda_1^\tau$ in $\overline{\Omega}$, we have $(L_\tau - b)^{inv}$ is also positivity preserving (resp. strongly positivity preserving).

Proof. We will prove only for the positivity preserving case, since the other one is completely similar.

First assume that $p \geq 2$. Suppose that $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ satisfies $L_\tau u - bu = f$ in Ω with $b - a \geq 0$. Then u solves $L_\tau u - au = f + (b-a)u$ in Ω.

Consider now the nonlinear problem

$$\begin{cases} (L_\tau - a)v = f + (b-a)|v| & \text{in } \Omega, \\ \qquad\quad v = \nabla v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

and the corresponding energy functional F on $H_0^2(\Omega)$ defined by

$$F(v) = \int\limits_{\Omega} \Big[(\Delta v)^2 + \tau|\nabla v|^2 - av^2 - 2fv - (b-a)|v|v\Big]dx.$$

Set

$$\varepsilon = \min\left\{2\Lambda_1^\tau \ , \ \min_{x\in\overline{\Omega}}(\Lambda_1^\tau - b(x))\right\} > 0.$$

Then we have

$$\begin{aligned} F(v) &\geq \int\limits_{\Omega} \Big[(\Delta v)^2 + \tau|\nabla v|^2 - 2fv - bv^2\Big]dx \geq \\ &\geq \frac{\varepsilon}{2\Lambda_1^\tau}\int\limits_{\Omega} \Big[(\Delta v)^2 + \tau|\nabla v|^2\Big]dx + \int\limits_{\Omega} \left[\left(\Lambda_1^\tau - \frac{1}{2}\varepsilon - b\right)v^2 - 2fv\right]dx \geq \\ &\geq \frac{\varepsilon}{2\Lambda_1^\tau}\int\limits_{\Omega} (\Delta v)^2 dx + \int\limits_{\Omega} \left(\frac{1}{2}\varepsilon v^2 - 2fv\right)dx \geq \\ &\geq \frac{\varepsilon}{2\Lambda_1^\tau}\int\limits_{\Omega} (\Delta v)^2 dx - \frac{1}{\varepsilon}\int\limits_{\Omega} f^2 dx. \end{aligned}$$

Since $F(\cdot)$ is weakly lower semicontinuous and coercive in $H_0^2(\Omega)$, there exists a minimizer $u_* \in H_0^2(\Omega)$ which verifies the Euler-Lagrange equation $(L_\tau - a)u_* = f + (b-a)|u_*|$. Using regularity theory and appropriate embeddings, a boot-strapping argument leads to $u_* \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$.

Coming back to the positivity preserving hypothesis of the operator $(L_\tau - a)^{inv}$, as $f + (b-a)|u_*| \geq 0$, we have then the solution u_* of (3.2) is nonnegative:

$$u_* = (L_\tau - a)^{inv}(f + (b-a)|u_*|) \geq 0.$$

Hence u_* solves (1.3) with b instead of a, that is $u = u_* \geq 0$.

Finally, for $p \in (1,2)$, the claim follows by approximating $f \in L^p(\Omega)$ by a sequence $f_k \in L^2(\Omega)$. □

Lemma 3.3. *Let $\tau > 0$ and u be the solution to*

$$\begin{cases} \Delta^2 u - \tau \Delta u = 1 & \text{in } B_R := B(0,R) \subset \mathbb{R}^n, \\ u = \Delta u = 0 & \text{on } \partial B_R. \end{cases} \tag{3.3}$$

Then $\|u\|_\infty < \dfrac{R^2}{2\tau n}$.

Proof. Let $v = -\Delta u$. Then $-\Delta v + \tau v = 1$ and $v = 0$ on ∂B_R. So immediately $0 \leq v < \tau^{-1}$ by the maximum principle, or equivalently, as u is radial,

$$0 \leq -\frac{1}{r^{n-1}}(r^{n-1}u')' < \frac{1}{\tau}, \quad 0 < r \leq R.$$

Integrating from 0 to $r > 0$, we obtain $0 \leq -r^{n-1}u' < \frac{r^n}{\tau n}$, that is $0 \leq -u' < \frac{r}{\tau n}$ for $r \in (0,R]$. Then integrating from 0 to R, as u is decreasing and $u = 0$ on ∂B_R, we obtain $u(0) = \|u\|_\infty < \frac{R^2}{2\tau n}$. □

Applying Proposition 3.2 and Lemma 3.3, we get the following crucial result.

Proposition 3.4. *For $\lambda << 0$, the operator*

$$(L_\tau - \lambda)^{inv} \ : \ L^p(\Omega) \to W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega) \subset L^p(\Omega),$$

$p > 1$, is not positivity preserving for any $\tau > 0$.

Proof. Using first Proposition 3.2, if $(L_\tau)^{inv}$ is not positivity preserving, we are done as $\Lambda_1^\tau > 0$.

Hence we may assume that $(L_\tau)^{inv}$ preserves the positivity. We will construct as in [4] a nonpositive nontrivial test function $a \in C(\overline{\Omega})$ such that $(L_\tau - a)^{inv}$ is not positivity preserving. Therefore we would have $(L_\tau - \lambda)^{inv}$ is not positivity preserving for $\lambda \leq \min a < 0$, since otherwise we get a contradiction, again by Proposition 3.2.

The construction of such a function a is related to the construction of a sign-changing function $\tilde{u}$ satisfying $(L_\tau - a)^{inv}\tilde{u} \geq 0$ in Ω and $\tilde{u} = \nabla \tilde{u} = 0$ on $\partial\Omega$. Set $v_1 = (L_\tau)^{inv}1$, which is then positive by our assumption. By scaling, we may assume that Ω contains the unit ball B and let u_0 be the solution to (3.3) on B. Consider $\varepsilon > 0$ small and the function

$$u_\varepsilon(x) = 16r^2 - \varepsilon - 32\tau n u_0(0) + 32\tau n u_0(x),$$

where $r = |x|$. As $u_0(0) < \frac{1}{2\tau n}$ by Lemma 3.3, we can choose ε small enough such that $u_\varepsilon(1) = 16 - \varepsilon - 32\tau n u_0(0) > 0$. Hence there exists an interval $[r_1, r_2] \subset (0,1)$ verifying $u_\varepsilon > 0$ in $[r_1, r_2]$.

Now fix $r_3 \in (r_1, r_2)$ and consider the cut-off function $\chi \in C_c^\infty(\Omega)$ such that $\mathrm{supp}(\chi) \subset \{|x| \le r_3\}$ and $\chi \equiv 1$ on $\{|x| \le r_0\}$ for $r_1 < r_0 < r_3$. Define finally

$$\tilde{u}(x) = \chi(r) u_\varepsilon(x) + [1 - \chi(r)]\, v_1(x).$$

For $|x| \ge r_3$, $\chi \equiv 0$, we have then $\tilde{u}(x) = v_1(x) \ge 0$ and $L_\tau \tilde{u} = L_\tau v_1 = 1$.

We have also $\tilde{u}(x) = u_\varepsilon(x)$ in B_{r_0}, hence $L_\tau \tilde{u} = L_\tau u_\varepsilon = 0$ on B_{r_0} and $\tilde{u}(0) = u_\varepsilon(0) = -\varepsilon < 0$. Moreover, in $B_{r_3} \setminus B_{r_0}$, $\tilde{u} = \chi(r) u_\varepsilon(x) + [1 - \chi(r)]\, v_1(x) > C_\varepsilon > 0$ and $L_\tau \tilde{u} \ge -C$ for some positive constants C_ε and C. We can conclude then

$$L_\tau \tilde{u} = \begin{cases} 0 & \text{in } B_{r_0}, \\ 1 & \text{in } \Omega \setminus B_{r_3}, \end{cases} \qquad \text{and} \qquad \begin{cases} \tilde{u}(0) < 0, & \\ \tilde{u}(x) > 0 & \text{in } \Omega \setminus B_{r_1}, \\ \tilde{u} = \nabla \tilde{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

The Dirichlet boundary condition comes from tha fact $\tilde{u} \equiv v_1$ for $|x| \ge r_3$.

Finally, set $a(x) = -M\chi_1(r) \le 0$ with χ_1 a cut-off function $\chi_1(r) \equiv 1$ on $\{r_0 \le |x| \le r_3\}$ and $\mathrm{supp}(\chi_1) \subset \{r_1 < |x| < r_2\}$. By choosing the constant $M > 0$ sufficiently large, we can claim that the sign-changing function $\tilde{u}$ verifies $(L_\tau - a)\tilde{u} \ge 0$ in Ω. Hence Proposition 3.2 yields that $(L_\tau - \lambda)^{inv}$ is not positivity preserving for $\lambda \le \min a < 0$. $\square$

Using Propositions 3.1, 3.2 and 3.4, we are now able to prove the dichotomy result.

Proof of Theorem 1.1. From Propositions 3.1 and 3.2, one finds that the set of $\lambda \in \mathbb{R}$ for which the operator $(L_\tau - a)^{inv} : L^p(\Omega) \to W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega) \subset L^p(\Omega)$ is positivity preserving is an interval $[\lambda_c, \Lambda_1^\tau)$ or $(\lambda_c, \Lambda_1^\tau)$. Proposition 3.4 yields that $\lambda_c > -\infty$. Since $\lambda \mapsto (L_\tau - \lambda)^{inv}$ is continuous except at the eigenvalues, we find that the interval is left closed.

The previous argument by Proposition 3.2 implies clearly claim 2 iii). Moreover, for claim 2 ii), we can just take $u = \Phi_1^\tau$ which verifies $\Phi_1^\tau > 0$ and $(L_\tau - a)\Phi_1^\tau = (\Lambda_1^\tau - a)\Phi_1^\tau > 0$.

Let $a < \lambda_c \le \Lambda_1^\tau$ in $\overline{\Omega}$ and suppose that for some $0 \not\equiv f \ge 0$, $u = (L_\tau - a)^{inv} f \le 0$. Then

$$0 < \int\limits_\Omega f \Phi_1^\tau dx = \int\limits_\Omega (L_\tau u - au)\Phi_1^\tau dx = \int\limits_\Omega u(\Lambda_1^\tau - a)\Phi_1^\tau dx \le 0,$$

which is impossible, hence we get claim 2 i). $\square$

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Hanen Ben Omrane
benomranehanen@yahoo.fr

Institut Préparatoire des Études d'Ingénierie de Tunis
Mont-Fleury, Tunisie

Saïma Khenissy
saima.khenissy@gmail.com

Département de Mathématiques Appliquées
Institut Supérieur d'Informatique, Ariana, Tunisie

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