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EXISTENCE RESULTS FOR RANDOM FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, an existence result for a random fractional differential equation is established under a Carathéodory condition. Existence results for extremal random solutions are also proved. Finally, an existence and uniqueness result is given.

Keywords: random fractional differential equations, fractional integral, Caputo fractional derivative, existence.

Mathematics Subject Classification: 26A33, 47H40, 60H25.

1. INTRODUCTION

The nature of a dynamic system in engineering or natural sciences depends on the accuracy of the information we have concerning the parameters that describe that system. If the knowledge about a dynamic system is precise then a deterministic dynamical system arises. Unfortunately in most cases the available data for the description and evaluation of parameters of a dynamic system are inaccurate, imprecise or confusing. In other words, evaluation of parameters of a dynamical system is not without uncertainties. When our knowledge about the parameters of a dynamic system are of statistical nature, that is, the information is probabilistic, the common approach in mathematical modeling of such systems is the use of random differential equations or stochastic differential equations. Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians. We refer the reader to the monographs [3,13,25], the papers [9,22–24] and the references therein. We also refer the reader to recent results [5–7,16–18]. There are real world phenomena with anomalous dynamics such as signals transmissions through strong magnetic fields, atmospheric diffusion of pollution, network traffic, the effect of speculations on the profitability of stocks in financial markets and so on where the classical models are not sufficiently good to describe these features. In this case, the theory of fractional differential equations is a good tool for modeling such phenomena. Therefore, the study of the fractional differential equations with random parameters seem to be a natural one. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs [12, 14, 19, 21] and the paper [15].

In this paper we prove some existence and uniqueness results for initial value problems for fractional differential equations with random parameters (or random fractional differential equations) of the form

$$\begin{cases} D^\alpha x(t,\omega) = f(t, x(t,\omega), \omega), \\ x(0,\omega) = x_0(\omega), \end{cases} \tag{1.1}$$

where x is a random function, x_0 is a random vector, $\omega \in \Omega$ where Ω is the sample space in a probability space $(\Omega, \mathcal{A}, P)$, $D^\alpha x$ is the Caputo fractional derivative of x with respect to the variable $t \in [0,T]$ with $T > 0$, and $f : [0,\infty) \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ is a given function. In [17] the authors established existence and uniqueness result for the solutions of the random fractional differential equations (1.1). This paper continues the study that was initiated in [17]. The paper is organized as follows. In Section 2 we set up the appropriate framework on random processes and on fractional calculus. The main existence result is given in Section 3. In the last section we obtain an existence and uniqueness result.

2. PRELIMINARIES

Let $(\Omega, \mathcal{A})$ be a measurable space; that is, a set Ω with a σ-algebra of subsets of Ω. A probability measure P is a measure on Ω with $P(\Omega) = 1$. Then $(\Omega, \mathcal{A}, P)$ is called a probability space. In the following, assume that $(\Omega, \mathcal{A}, P)$ is a complete probability space. Let $(S, \mathcal{B}(S))$ be a measurable space. If S is a metric space, then the σ-algebra $\mathcal{B}(S)$ will be the σ-algebra of all Borel subsets of S. A measurable function $x : \Omega \to S$ is called a *random element* in S. A random element in S is called a *random variable* when $S = \mathbb{R}$, a *random vector* when $S = \mathbb{R}^m$, and a *random* or *stochastic process* when S is a function space.

The sample path fractional integral. Let $([0,T], \mathcal{L}, \lambda)$ be a Lebesgue-measure space, where $T > 0$ and let $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}^m$ be a product measurable function. We say that $x(\cdot,\cdot)$ *is sample path Lebesgue integrable on* $[0,T]$ if $x(\cdot,\omega) : [0,T] \to \mathbb{R}^m$ is Legesgue integrable on $[0,T]$ for a.e. $\omega \in \Omega$.

Let $\alpha > 0$. If $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}^m$ is sample path Lebesgue integrable on $[0,T]$, then we can consider the fractional integral

$$I^\alpha x(t,\omega) = \int_0^t g_\alpha(t-s) x(s,\omega)\,ds, \tag{2.1}$$

which will be called *the sample path fractional integral of* x. The function $g_\alpha : \mathbb{R} \to [0, \infty)$ is defined by

$$g_\alpha(s) = \begin{cases} \dfrac{s^{\alpha-1}}{\Gamma(\alpha)} & \text{if } s > 0, \\ 0 & \text{if } s \leq 0, \end{cases}$$

where Γ is the Euler's Gamma function. In fact, $I^\alpha x(t,\omega) = (g_\alpha * x)(t,\omega)$ for a.e. $\omega \in \Omega$, where $g_\alpha * x$ denotes the convolution product (see [1]). By the properties of convolution product (see [1]), it follows that, if $x(\cdot,\omega) : [0,T] \to \mathbb{R}^m$ is Lebesgue integrable on $[0,T]$ for a.e. $\omega \in \Omega$, then $t \mapsto I^\alpha x(t,\omega)$ is also Lebesgue integrable on $[0,T]$ for a.e. $\omega \in \Omega$. A function $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}^m$ is said to be a *Carathéodory function* if $t \mapsto x(t,\omega)$ is continuous for a.e. $\omega \in \Omega$, and $\omega \mapsto x(t,\omega)$ is measurable for each $t \in [0,T]$. We recall that a Carathéodory function is a product measurable function (see [10]). If $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}^m$ is a Carathéodory function, then the function $(t,\omega) \mapsto I^\alpha x(t,\omega)$ is also a Carathéodory function (see [17]).

The sample path fractional derivative. A function $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}^m$ is said to have *a sample path derivative at* $t \in (0,T)$ if the function $t \mapsto x(t,\omega)$ has a derivative at t for a.e. $\omega \in \Omega$. We will denote by $\frac{d}{dt}x(t,\omega)$ or by $x'(t,\omega)$ the sample path derivative of $x(\cdot,\omega)$ at t. We say that $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}^m$ is sample path differentiable on $[0,T]$ if $x(\cdot,\cdot)$ has a sample path derivative for each $t \in (0,T)$ and possesses a one-sided sample path derivative at the end points 0 and T.
If $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}^m$ is a sample path absolutely continuous on $[0,T]$ (that is, $t \mapsto x(t,\omega)$ is absolutely continuous on $[0,T]$ for a.e. $\omega \in \Omega$), then the sample path derivative $x'(t,\omega)$ exists for λ-a.e. $t \in [0,T]$. Let $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}^m$ be a sample path absolutely continuous on $[0,T]$ and let $\alpha \in (0,1]$. Then, for λ-a.e. $t \in [0,T]$ and for a.e. $\omega \in \Omega$, we define the *Caputo sample path fractional derivative* of x by

$$D^\alpha x(t,\omega) = I^{1-\alpha} x'(t,\omega) = \int_0^t g_{1-\alpha}(t-s) x'(s,\omega) ds. \tag{2.2}$$

Obviously, if $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is sample path differentiable on $[0,T]$ and $t \mapsto x'(t,\omega)$ is continuous on $[0,T]$, then $D^\alpha x(t,\omega)$ exists for every $t \in [0,T]$ and $t \mapsto D^\alpha x(t,\omega)$ is continuous on $[0,T]$.
The following properties are well known (see [12, 21]). If $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}^m$ is a Carathéodory function then ([12, Lemma 2.21])

$$D^\alpha I^\alpha x(t,\omega) = x(t,\omega) \tag{2.3}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. If $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is sample path absolutely continuous on $[0,T]$ then ([12, Lemma 2.22])

$$I^\alpha D^\alpha x(t,\omega) = x(t,\omega) - x(0,\omega) \tag{2.4}$$

for λ-a.e. $t \in [0,T]$ and a.e. $\omega \in \Omega$. In particular, if $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}^m$ is sample path differentiable on $[0,T]$ and $t \mapsto x'(t,\omega)$ is continuous on $[0,T]$, then (2.4) holds for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. If $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is sample path absolutely continuous on $[0,T]$ then ([12, Lemma 2.22])

$$t \mapsto y(t,\omega) := \int_0^t g_{1-\alpha}(t-s)x(s,\omega)ds$$

is also sample path absolutely continuous on $[0,T]$. Moreover, for λ-a.e. $t \in [0,T]$ and a.e. $\omega \in \Omega$, we have that

$$y'(t,\omega) = \frac{d}{dt}\int_0^t g_{1-\alpha}(t-s)x(s,\omega)ds = D^\alpha x(t,\omega) + \frac{x(0,\omega)}{\Gamma(1-\alpha)}t^{-\alpha}. \tag{2.5}$$

Let $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}^m$ be a sample path Lebesgue integrable on $[0,T]$ and let $\alpha \in (0,1]$. For a.e. $\omega \in \Omega$ we define the *Riemann-Liouville sample path fractional derivative* of x by

$$D^\alpha_{RL}x(t,\omega) = \frac{d}{dt}I^{1-\alpha}x(t,\omega) = \frac{d}{dt}\int_0^t g_{1-\alpha}(t-s)x(s,\omega)ds. \tag{2.6}$$

If $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is sample path absolutely continuous on $[0,T]$ then, by (2.5), it follows that

$$D^\alpha_{RL}x(t,\omega) = D^\alpha x(t,\omega) + \frac{x(0,\omega)}{\Gamma(1-\alpha)}t^{-\alpha}, \tag{2.7}$$

for λ-a.e. $t \in [0,T]$ and a.e. $\omega \in \Omega$.

The metric space of random elements. In the following, consider the space $C := C([0,T],\mathbb{R}^m)$ of all continuous functions from $[0,T]$ into $\mathbb{R}^m$, endowed with the uniform metric $d(x,y) := \sup_{0\le t\le T}\|x(t)-y(t)\|$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^m$. Let $\mathcal{B}(C)$ be the σ-algebra of all Borel subsets of C. Then, by [11, Lemma 14.1], $\mathcal{B}(C)$ coincides with the σ-algebra generated in C by all evaluation maps $\pi_t : C \to \mathbb{R}^m$, $t \in [0,T]$, given by $\pi_t(x) = x(t)$. If $x : \Omega \to C$, then $x_t := \pi_t \circ x$ maps Ω into $\mathbb{R}^m$. Therefore, x may also be regarded as a function $x(t,\omega) := x_t(\omega)$ from $[0,T] \times \Omega$ into $\mathbb{R}^m$. Then, by [11, Lemma 2.1], $x : \Omega \to C$ is a random element in C (that is, a random process) if and only if $x_t : \Omega \to \mathbb{R}^m$ is measurable for all $t \in [0,T]$. In fact, it is easy to see that if $x : \Omega \to C$ is a random element, then the function $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}^m$ is a Carathéodory function. If $x : \Omega \to C$ is a random element, then the function $x(\cdot,\omega) : [0,T] \to \mathbb{R}^m$ is said to be a realization or a trajectory of the random element x, corresponding to the outcome $\omega \in \Omega$. Let $\mathcal{M}(C)$ be the space of all probability measures on $\mathcal{B}(C)$. If $x : \Omega \to C$ is a random element in C, then the probability measure μ_x, defined by $\mu_x(B) := P(x^{-1}(B)) = P(\{\omega \in \Omega; x(\omega) \in B\})$, $B \in \mathcal{B}(C)$, is called the *distribution* of x. Since C is a complete and separable metric space, then it is well known that $\mathcal{M}(C)$ is a complete and separable metric space with respect to the Prohorov metric $D : \mathcal{M}(C) \to [0,\infty)$, defined by ([20]):

$$D(\mu,\nu) = \inf\{\varepsilon > 0 : \mu(B) \le \nu(B^\varepsilon) + \varepsilon, B \in \mathcal{B}(C)\},$$

where $B^\varepsilon := \{x \in C : \inf_{y\in B} d(x,y) < \varepsilon\}$. If we identify random elements in C that have the same distribution, then $\rho(x,y) := D(\mu_x, \mu_y)$ is a metric on the set of random elements in C (see [8]). Denote by $\mathcal{R}(\Omega, C)$ the metric space of all random elements in C. A sequence of random variables $\{x_n\} \subset \mathcal{R}(\Omega, C)$ is said to *converge almost everywhere* (a.e.) to $x \in \mathcal{R}(\Omega, C)$ if there exists $N \subset \Omega$ such that $P(N) = 0$, and $\lim_{n\to\infty} d(x_n(\omega), x(\omega)) = 0$ for every $\omega \in \Omega \setminus N$. We use the notation $x_n \to x$ a.e. for almost everywhere convergence. If $\{x_n\} \subset \mathcal{R}(\Omega, C)$ is a ρ-convergent sequence, then it is known that $\{x_n\}$ is a ρ-Cauchy sequence. The converse is also true in the following sense.

Theorem 2.1 (Skorokhod [2]). *IF $\{x_n\} \subset \mathcal{R}(\Omega, C)$ is a ρ-Cauchy sequence, then one can construct a sequence $\{y_n, y\} \subset \mathcal{R}(\Omega, C)$ such that $\rho(x_n, y_n) = 0$ and $y_n \to y$ a.e.*

A sequence $\{x_n\} \subset \mathcal{R}(\Omega, C)$ is called *ρ-relatively compact* if every subsequence has a subsequence that converges with respect to the metric ρ. Since C is a complete and separable metric space, then ρ-relative compactness and tightness are equivalent.

Theorem 2.2 (Prohorov [20]). *A sequence $\{x_n\} \subset \mathcal{R}(\Omega, C)$ is ρ-relatively compact if and only if $\{x_n\}$ is a tight, that is, for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset C$ such that $P(\{\omega \in \Omega; x_n(\omega) \in K_\varepsilon\}) > 1 - \varepsilon$ for all $n \geq 1$.*

3. LOCAL EXISTENCE

In the following, we consider the initial value problem

$$\begin{cases} D^\alpha x(t,\omega) = f(t, x(t,\omega), \omega), \\ x(0,\omega) = x_0(\omega), \end{cases} \tag{3.1}$$

where x_0 is a random vector, $D^\alpha x$ is the Caputo fractional derivative of x with respect to the variable $t \in [0,T]$ with $T > 0$, and $f : [0,T] \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ is a given function. In the following, assume that:

(H1) the function $(t,x) \mapsto f(t,x,\omega)$ is continuous for a.e. $\omega \in \Omega$,
(H2) the function $\omega \mapsto f(t,x,\omega)$ is measurable for each $(t,x) \in [0,T] \times \mathbb{R}^m$,
(H3) there exists $y_0 \in \mathbb{R}^m$ such that $x_0(\omega) \in \overline{B}(y_0, \rho)$ for a.e. $\omega \in \Omega$, where

$$\overline{B}(y_0, \rho) := \{x \in \mathbb{R}^m; \|x - x_0\| \leq \rho\}.$$

(H4) There exist $K > 0$, $\rho > 0$ and $x_0 \in \mathbb{R}^m$ such that

$$\|f(t,x,\omega)\| \leq K$$

for every $(t,x) \in [0,T] \times \overline{B}(y_0, \rho)$ and for a.e. $\omega \in \Omega$.

A product measurable function $x : [0,T] \times \Omega \to \mathbb{R}^m$ is said to be a *sample solution* for problem (3.1) if

(a) $x(\cdot,\omega)$ is continuous on $[0,T]$ for a.e. $\omega \in \Omega$,
(b) $x(t,\omega)$ satisfies (3.1) for a.e. $t \in [0,T]$ and for a.e. $\omega \in \Omega$.

If $(t,\omega) \mapsto f(t,x(t,\omega),\omega)$ is product measurable and $t \mapsto f(t,x(t,\omega),\omega)$ is Lebesgue integrable on $[0,T]$ for a.e. $\omega \in \Omega$ then, by (2.3) and (2.4), we observe that $x : [0,T] \times \Omega \to \mathbb{R}^m$ is a solution for (3.1) if and only if

$$x(t,\omega) = x_0(\omega) + \int_0^t g_\alpha(t-s) f(s,x(s,\omega),\omega)ds \tag{3.2}$$

for all $t \in [0,T]$ and for a.e. $\omega \in \Omega$.

Remark 3.1. We can consider the random differential equation (3.1) as a family (with respect to parameter ω) of deterministic differential equations, namely

$$\begin{cases} D^\alpha x(t,\omega) = f(t,x(t,\omega),\omega), & t \in [0,T], \\ x(0,\omega) = x_0(\omega). \end{cases} \tag{3.3}$$

Generally, it is not correct to solve each problem (3.3) to obtain the solutions of (3.1). Let us give two examples.

Example 3.2. Let $(\Omega, \mathcal{A}, P)$ be a complete probability measure space. Consider an initial value problem of the form

$$\begin{cases} D^{1/2} x(t,\omega) = \sqrt{\pi} x(t,\omega), & t \in [0,\infty), \\ x(0,\omega) = K(\omega), \end{cases} \tag{3.4}$$

where $K : \Omega \to (0,\infty)$ is a random variable. It is easy to show that, for each $\omega \in \Omega$,

$$x(t,\omega) = \frac{K(\omega)}{\sqrt{1-(K(\omega))^2 t}}$$

is a solution of (3.4) on the interval $[0, 1/(K(\omega))^2)$. Let us choose an $a > 0$ such that $\Omega \diagdown K^{-1}\left((0,1/\sqrt{a}\right) \neq \emptyset$. Since

$$\Omega = K^{-1}((0,\infty)) = K^{-1}\left((0,1/\sqrt{a}\right) \cup K^{-1}([1/\sqrt{a},\infty)),$$

it follows that

$$P(1/(K(\omega))^2 > a) = 1 - P(1/(K(\omega))^2 \le a) < 1.$$

Therefore, not all solutions $x(\cdot,\omega)$ are well defined on some common interval $[0,a)$.

Example 3.3. Let $(\Omega, \mathcal{A}, P)$ be a complete probability measure space and let $\Omega_0 \notin \mathcal{A}$. It is easy to check that, for each $\omega \in \Omega$, the function $x(\cdot,\cdot) : [0,1] \times \Omega \to \mathbb{R}$, given by

$$x(t,\omega) = \begin{cases} 0 & \text{if } \omega \in \Omega_0, \\ t^{1/2} & \text{if } \omega \in \Omega \setminus \Omega_0, \end{cases}$$

is a solution of the initial value problem

$$\begin{cases} D^{1/2}x(t,\omega) = \frac{\sqrt{\pi}}{2}x(t,\omega), & t \in [0,1], \\ x(0,\omega) = 0. \end{cases}$$

However $x(\cdot,\cdot)$ is not a stochastic process. Indeed, we have that

$$\Big\{\omega \in \Omega : x(1,\omega) \in \Big[-\frac{1}{2},\frac{1}{2}\Big]\Big\} = \Omega_0 \notin \mathcal{A},$$

that is, $\omega \mapsto x(1,\omega)$ is not a measurable function.

Theorem 3.4. *Let $f : [0,T]\times\mathbb{R}^m\times\Omega \to \mathbb{R}^m$ be a given function, and let $x_0 : \Omega \to \mathbb{R}^m$ be a random vector. If* (H1)–(H4) *hold, then the problem* (3.1) *has at least one solution on $[0,a]$ for a suitable $a \le T$.*

Proof. Let us define the sequence $\{x_n(t,\omega)\}$ by $x_1(t,\omega) = x_0(\omega)$, and

$$x_n(t,\omega) = x_0(\omega) + \int_0^t g_\alpha(t-s)f(s,x_{n-1}(s,\omega),\omega)ds, \quad n \ge 2,$$

for all $t \in [0,T]$ and for a.e. $\omega \in \Omega$. Since, by (H1) and (H2), $(t,\omega) \mapsto I^\alpha x(t,\omega)$ is a Carathéodory function (see [17]), it follows that, for each $n \ge 1$, $(t,\omega) \mapsto x_n(t,\omega)$ is also a Carathéodory function. Choose $0 < a \le T$ such that $\frac{Ka^\alpha}{\Gamma(\alpha+1)} \le \frac{\rho}{2}$. Then we have that

$$\begin{aligned} \|x_n(t,\omega) - x_0(\omega)\| &\le \int_0^t g_\alpha(t-s)\,\|f(s,x_{n-1}(s,\omega),\omega)\|\,ds \\ &\le \frac{K}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \le \frac{Ka^\alpha}{\Gamma(\alpha+1)} \le \frac{\rho}{2}, \end{aligned}$$

for all $t \in [0,a]$ and for a.e. $\omega \in \Omega$. Thus, $x_n(t,\omega) \in \overline{B}(y_0,\rho)$ for all $t \in [0,a]$ and for a.e. $\omega \in \Omega$. Let $\varepsilon > 0$. If we choose $\delta > 0$ such that $\delta^\alpha \le \frac{\varepsilon\Gamma(\alpha+1)}{2K}$, then for each $s,t \in [0,a]$ with $0 < t-s < \delta$ we have that

$$\begin{aligned} \|x_n(t,\omega) - x_n(s,\omega)\| &\le \frac{K}{\Gamma(\alpha)}\int_0^s [(s-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1}]d\tau \\ &\quad + \frac{K}{\Gamma(\alpha)}\int_s^t (t-\tau)^{\alpha-1}d\tau \le \frac{2K}{\Gamma(\alpha+1)}(t-s)^\alpha < \varepsilon, \end{aligned}$$

for a.e. $\omega \in \Omega$. It follows that the family $\{x_n(\cdot,\omega)\}$ is sample equicontinuous and sample uniformly bounded, and hence by the Ascoli-Arzela's theorem, $\{x_n(\cdot,\omega)\}$ is a relatively compact subset of $C_a := C([0,a],\mathbb{R}^m)$ for a.e. $\omega \in \Omega$. Now by Theorem 2.2, $\{x_n\} \subset \mathcal{R}(\Omega, C_a)$ is ρ-relatively compact. Thus, $\{x_n\}$ has a ρ-Cauchy subsequence, again denoted by $\{x_n\}$. Further, by Theorem 2.1, we can construct a sequence $\{y_n\} \subset \mathcal{R}(\Omega, C_a)$ and $x \in \mathcal{R}(\Omega, C_a)$ such that $\rho(x_n, y_n) = 0$ and $y_n(\cdot,\omega) \to x(\cdot,\omega)$ for a.e. $\omega \in \Omega$. Next, we show that $x(\cdot,\cdot)$ is a solution of (3.1). Since, for fixed $t \in [0,a]$, the function $x \mapsto f(t,x,\omega)$ is continuous for a.e. $\omega \in \Omega$, then it follows that $f(\cdot, y_n(\cdot,\omega),\omega) \to f(\cdot, x(\cdot,\omega),\omega)$ for a.e. $\omega \in \Omega$. Therefore, for every $\varepsilon > 0$ there exists a $n_0 \geq 1$ such that

$$d(f(\cdot, y_n(\cdot,\omega),\omega), f(\cdot, x(\cdot,\omega),\omega)) \leq \frac{\varepsilon\Gamma(\alpha+1)}{a^\alpha},$$

for all $n \geq n_0$ and for a.e. $\omega \in \Omega$. Then we have that

$$\begin{aligned}
&\left\| \int_0^t g_\alpha(t-s) f(s, y_n(s,\omega),\omega)ds - \int_0^t g_\alpha(t-s) f(s, x(s,\omega),\omega)ds \right\| \\
&\leq \int_0^t g_\alpha(t-s) \left\| f(s, y_n(s,\omega),\omega)ds - f(s, x(s,\omega),\omega)ds \right\| \\
&\leq \int_0^t g_\alpha(t-s) d(f(\cdot, y_n(\cdot,\omega),\omega), f(\cdot, x(\cdot,\omega),\omega))ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{\varepsilon\Gamma(\alpha+1)}{a^\alpha} ds \leq \varepsilon,
\end{aligned}$$

that is,

$$\lim_{n\to\infty} \int_0^t g_\alpha(t-s) f(s, y_n(s,\omega),\omega)ds = \int_0^t g_\alpha(t-s) f(s, x(s,\omega),\omega)ds,$$

for all $t \in [0,a]$ and for a.e. $\omega \in \Omega$. It follows that

$$\begin{aligned}
x(t,\omega) = \lim_{n\to\infty} y_n(t,\omega) &= \lim_{n\to\infty} \left[x_0(\omega) + \int_0^t g_\alpha(t-s) f(s, y_{n-1}(s,\omega),\omega)ds \right] \\
&= x_0(\omega) + \int_0^t g_\alpha(t-s) f(s, x(s,\omega),\omega)ds,
\end{aligned}$$

for all $t \in [0,a]$ and for a.e. $\omega \in \Omega$. Therefore, $x(\cdot,\cdot)$ is a solution of (3.2), and so a solution of (3.1). □

4. MAXIMAL SOLUTIONS AND GLOBAL EXISTENCE

First, we prove a result relative to random fractional differential inequalities. A product measurable random process $x : [0,T) \times \Omega \to \mathbb{R}$ is said to be sample locally absolutely continuous continuous if the function $t \mapsto x(t,\omega)$ is locally absolutely continuous for a.e. $\omega \in \Omega$.

Lemma 4.1. *Let $v, w : [0,T) \times \Omega \to \mathbb{R}$ be sample locally absolutely continuous random processes, and assume that the function $g : [0,T) \times \mathbb{R} \times \Omega \to \mathbb{R}$ satisfies* (H1) *and* (H2)*. If*

(i) $D^\alpha v(t,\omega) \le g(t, v(t,\omega), \omega)$,
(ii) $D^\alpha w(t,\omega) \ge g(t, w(t,\omega), \omega)$,

for a.e. $t \in [0,T)$ and a.e. $\omega \in \Omega$, and one of the inequalities is strict, then $v(0) < w(0)$ implies that

$$v(t,\omega) < w(t,\omega), \tag{4.1}$$

for a.e. $t \in [0,T)$ and a.e. $\omega \in \Omega$.

Proof. Let $J_0 \subset [0,T)$ be such that $\lambda(J_0) = 0$ and (i) and (ii) hold for all $t \in [0,T) \setminus J_0$ and a.e. $\omega \in \Omega$. Suppose that the conclusion (4.1) is not true. Also, let us suppose that the inequality (ii) is strict. Then there exists a $\tau \in [0,T) \setminus J_0$ such that $v(\tau,\omega) = w(\tau,\omega)$ for a.e. $\omega \in \Omega$, and $v(t,\omega) \ge w(t,\omega)$, $0 \le t \le \tau$, $\omega \in \Omega$. Then, for $h > 0$, we have

$$\frac{v(\tau,\omega) - v(\tau - h,\omega)}{h} \ge \frac{w(\tau,\omega) - w(\tau - h,\omega)}{h} \text{ for a.e. } \omega \in \Omega.$$

It follows that $v'(\tau,\omega) \ge w'(\tau,\omega)$ for a.e. $\omega \in \Omega$, and hence $D^\alpha v(\tau,\omega) \ge D^\alpha w(\tau,\omega)$ for a.e. $\omega \in \Omega$. Using (i) and (ii), we obtain that

$$g(\tau, v(\tau,\omega), \omega) \ge D^\alpha v(\tau,\omega) \ge D^\alpha w(\tau,\omega) > g(\tau, w(\tau,\omega), \omega),$$

for a.e. $\omega \in \Omega$. This is a contradiction since $v(\tau,\omega) = w(\tau,\omega)$ for a.e. $\omega \in \Omega$. Hence $v(t,\omega) < w(t,\omega)$ for a.e. $t \in [0,T)$ and a.e. $\omega \in \Omega$. □

Let $r(\cdot,\cdot) : [0,T) \times \Omega \to \mathbb{R}$ be a sample solution process on $[0,T)$ of the random differential equation

$$\begin{cases} D^\alpha u(t,\omega) = g(t, u(t,\omega), \omega), \\ u(0,\omega) = u_0(\omega). \end{cases} \tag{4.2}$$

Then $r(\cdot,\cdot)$ is said to be a *sample maximal solution process* of (4.2) on $[0,T)$ if, for every sample solution of (4.2) on $[0,T)$, the inequality $u(t,\omega) \le r(t,\omega)$ holds.

Theorem 4.2. *Assume that $g : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ satisfies* (H1)–(H4) *and that the function $u \mapsto g(t,u,\omega)$ is sample nondecreasing for fixed $t \in [0,T]$ and a.e. $\omega \in \Omega$. Then the problem* (4.2) *has a maximal solution on $[0,a]$ for a suitable $a < T$.*

Proof. Let $0 < \varepsilon < \frac{\rho}{2}$. Consider the following random fractional differential equation

$$\begin{cases} D^\alpha u(t,\omega) = g_\varepsilon(t, u(t,\omega), \omega), \\ u(0,\omega) = u_0(\omega) + \varepsilon, \end{cases} \tag{4.3}$$

where $g_\varepsilon(t, u(t,\omega), \omega) = g(t, u(t,\omega), \omega) + \varepsilon$ and $u_0(\omega) \in \overline{B}(x_0, \frac{\rho}{2})$. It is easy to see that g_ε satisfies (H_1)-(H_4) with K replaced by $K + \frac{\rho}{2}$. Then by Theorem 3.4, the random fractional differential equation (4.3) has a sample solution $u_\varepsilon(\cdot,\omega)$ on $[0,a]$ for some $a \in (0,T]$. Let $\{\varepsilon_n\}$ be a strictly decreasing sequence such that $0 < \varepsilon_n \le \varepsilon$ for all $n \ge 1$ and $\lim\limits_{n\to\infty} \varepsilon_n = 0$. Also, for each $n \ge 1$, let $u_{\varepsilon_n}(\cdot,\cdot)$ be solution of

$$\begin{cases} D^\alpha u_{\varepsilon_n}(t,\omega) = g_{\varepsilon_n}(t, u_{\varepsilon_n}(t,\omega), \omega), \\ u_{\varepsilon_n}(0,\omega) = u_0(\omega) + \varepsilon_n. \end{cases}$$

This implies that $u_{\varepsilon_n}(0,\omega) > u_{\varepsilon_{n+1}}(0,\omega)$ and

$$\begin{aligned} D^\alpha u_{\varepsilon_{n+1}}(t,\omega) &= g_{\varepsilon_{n+1}}(t, u_{\varepsilon_{n+1}}(t,\omega), \omega) = g(t, u_{\varepsilon_{n+1}}(t,\omega), \omega) + \varepsilon_{n+1} \\ &< g(t, u_{\varepsilon_{n+1}}(t,\omega), \omega) + \varepsilon_n = g_{\varepsilon_n}(t, u_{\varepsilon_{n+1}}(t,\omega), \omega). \end{aligned}$$

Then, from Lemma 4.1, it follows that $u_{\varepsilon_{n+1}}(t,\omega) < u_{\varepsilon_n}(t,\omega)$ for all $t \in [0,a]$ and a.e. $\omega \in \Omega$. Since the family of functions $\{u_{\varepsilon_n}(\cdot,\cdot)\}$ is equicontinuous and uniformly bounded, then the uniform limit $r(t,\omega) = \lim_{n\to\infty} u_{\varepsilon_n}(t,\omega)$ exists on $[0,a]$ for a.e. $\omega \in \Omega$. Next, we show that $r(\cdot,\cdot)$ is a solution of (4.2). It is obvious that $r(0,\omega) = u_0(\omega)$. Since $u \mapsto g(t,u,\omega)$ is a sample continuous function for fixed $t \in [0,a]$, then $\lim_{n\to\infty} g_{\varepsilon_n}(t, u_{\varepsilon_n}(t,\omega), \omega) = g(t, r(t,\omega), \omega)$ a.e. on $[0,a]$. As in the proof of Theorem 3.4, we can show that

$$\lim_{n\to\infty} \int_0^t g_\alpha(t-s) g(s, u_{\varepsilon_n}(t,\omega), \omega) ds = \int_0^t g_\alpha(t-s) g(s, r(s,\omega), \omega) ds,$$

for all $t \in [0,a]$ and for a.e. $\omega \in \Omega$. It follows that

$$\begin{aligned} r(t,\omega) &= \lim_{n\to\infty} u_{\varepsilon_n}(t,\omega) = \lim_{n\to\infty} \left[u_0(\omega) + \int_0^t g_\alpha(t-s) g(s, u_{\varepsilon_n}(s,\omega), \omega) ds \right] \\ &= u_0(\omega) + \int_0^t g_\alpha(t-s) g(s, r(s,\omega), \omega) ds, \end{aligned}$$

for all $t \in [0,a]$ and for a.e. $\omega \in \Omega$. Therefore, $r(\cdot,\cdot)$ is a solution of (4.2). Now we show that $r(\cdot,\cdot)$ is the desired maximal solution of (4.2). For this, let $u(\cdot,\cdot)$ be any solution of (4.2). As above, by Lemma 4.1, we have that $u(t,\omega) < u_{\varepsilon_n}(t,\omega)$ for all $t \in [0,a]$ and a.e. $\omega \in \Omega$. It follows that $u(t,\omega) \le \lim_{n\to\infty} u_{\varepsilon_n}(t,\omega) = r(t,\omega)$ uniformly on $[0,a]$, and the theorem is proved. □

Corollary 4.3. *Assume that* $g : [0,T) \times \mathbb{R} \times \Omega \to \mathbb{R}$ *satisfies the assumptions of Theorem 4.2, and that* $r(\cdot,\cdot)$ *is a sample maximal solution of* (4.2) *on* $[0,T)$. *Let* $v \in C([0,T), \mathcal{R}(\Omega,\mathbb{R}))$ *be such that* $v(0,\omega) \le u_0(\omega)$ *for a.e.* $\omega \in \Omega$, *and* $D^\alpha v(t,\omega) = g(t, v(t,\omega),\omega)$ *for a.e.* $t \in [0,a]$ *and a.e.* $\omega \in \Omega$. *Then* $v(t,\omega) \le r(t,\omega)$ *for a.e.* $t \in [0,a]$ *and a.e.* $\omega \in \Omega$.

Proof. If $u_\varepsilon(\cdot,\cdot)$ is a sample solution of (4.3) for $\varepsilon > 0$ sufficiently small, then by Lemma 4.1, we have $v(t,\omega) \le u_\varepsilon(t,\omega)$ for a.e. $t \in [0,a]$ and a.e. $\omega \in \Omega$. Since $\lim_{\varepsilon\to 0} u_\varepsilon(t,\omega) = r(t,\omega)$ on every closed interval contained in $[0,T)$, the proof is complete. □

Theorem 4.4. *Assume that:*

(i) $f : [0,T) \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ *satisfies* (H1)–(H4),
(ii) $g : [0,T) \times \mathbb{R} \times \Omega \to \mathbb{R}$*satisfies the assumptions of Theorem 4.2,*
(iii) $u(t,\omega) \equiv 0$, *for a.e.* $\omega \in \Omega$, *is the unique solution of the random fractional differential equation*

$$D^\alpha u(t,\omega) = g(t, u(t,\omega),\omega), \quad u(0,\omega) = 0, \tag{4.4}$$

existing on $[0,a]$ *for a suitable* $a < T$,
(iv) $\|f(t,x,\omega) - f(t,y,\omega)\| \le g(t, \|x-y\|,\omega)$ *for all* $(t,x),(t,y) \in [0,a] \times \overline{B}(x_0,\rho)$ *and a.e.* $\omega \in \Omega$.

Then the random fractional differential equation (3.1) *has a unique solution on* $[0,a]$.

Proof. From (i) and Theorem 3.4, it follows that (3.1) has a solution. Let $x(\cdot,\cdot)$ and $y(\cdot,\cdot)$ be two solution of (3.1) on $[0,a]$, and let $v(t,\omega) := \|x(t,\omega) - y(t,\omega)\|$ for $t \in [0,a]$ and a.e. $\omega \in \Omega$. Then $v(t,\omega)$ is sample absolutely continuous and, using (iv), we have

$$\begin{aligned} D^\alpha v(t,\omega) &\le \|D^\alpha x(t,\omega) - D^\alpha y(t,\omega)\| \\ &= \|f(t,x(t,\omega),\omega) - f(t,y(t,\omega),\omega)\| \le g(t, v(t,\omega),\omega), \end{aligned}$$

for a.e. $t \in [0,a]$ and a.e. $\omega \in \Omega$. Since $v(0,\omega) = 0$ for a.e. $\omega \in \Omega$, then by Corollary 4.3, we have $v(t,\omega) \le r(t,\omega)$ for a.e. $t \in [0,a]$ and a.e. $\omega \in \Omega$, where $r(t,\omega)$ is the maximal solution process of (4.4). Therefore, using (iii), it follows that $v(t,\omega) \equiv 0$ on $[0,a]$, for a.e. $\omega \in \Omega$. Thus the proof of the theorem is complete. □

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