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A NOTE ON M_2-EDGE COLORINGS OF GRAPHS

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Abstract. An edge coloring φ of a graph G is called an M_2-edge coloring if $|\varphi(v)| \leq 2$ for every vertex v of G, where $\varphi(v)$ is the set of colors of edges incident with v. Let $K_2(G)$ denote the maximum number of colors used in an M_2-edge coloring of G. Let G_1, G_2 and G_3 be graphs such that $G_1 \subseteq G_2 \subseteq G_3$. In this paper we deal with the following question: Assuming that $K_2(G_1) = K_2(G_3)$, does it hold $K_2(G_1) = K_2(G_2) = K_2(G_3)$?

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1. INTRODUCTION

Let $G = (V, E)$ be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. An edge coloring φ of a graph G is an assignment of colors to the edges of G. The coloring φ is proper if no two adjacent edges have the same color.

Many generalizations of the proper edge coloring problem have been introduced and studied in the research of graph properties. These generalizations are usually obtained by a relaxation of the condition that no two edges of the same color are adjacent. For instance, in an f-coloring it is allowed that at each vertex v the same color may occur at most $f(v)$ times, where f is a function which assigns a positive integer $f(v)$ to each vertex v (see [5,7–10]). This type of coloring has many important applications in scheduling problems, e.g. the file transfer problem in a computer network (see [2,3,6]). If the requirement "the number of edges of the same color incident with a vertex v is at most $f(v)$" is replaced by "the number of colors of edges incident with a vertex v is at most $f(v)$", then we obtain a definition of a generalized M_i-edge coloring, which was introduced in [4]. In this paper we deal with M_2-edge colorings.

An edge coloring of a graph G is an M_2-edge coloring if at most two colors appear at any vertex of G. The problem is to determine the maximum number of colors $K_2(G)$ used in an M_2-edge coloring of G.

An M_2-edge coloring of G is called optimal if it uses $K_2(G)$ colors.

Observe that this graph invariant is not monotone. For example, the maximum number of colors used in an M_2-edge coloring of the complete graph on four vertices is three, but it has subgraphs - a cycle and a star on four vertices - which use four and two colors in an optimal coloring.

In the following the fact that H is a subgraph of G will be denoted by $H \subseteq G$.

In this paper we deal with the following question.

Question 1.1. *Let G_1, G_2 and G_3 be three graphs such that $G_1 \subseteq G_2 \subseteq G_3$ and $K_2(G_1) = K_2(G_3)$. Is it true that $K_2(G_1) = K_2(G_2) = K_2(G_3)$?*

We prove that the answer to this question is negative. Moreover, we show that the gap between $K_2(G_2)$ and $K_2(G_3)$ can be arbitrarily large.

2. RESULTS

First we prove that any optimal coloring of a complete graph on $n \geq 4$ vertices uses $\lfloor \frac{n}{2} \rfloor + 1$ colors.

A matching in a graph is a set of pairwise nonadjacent edges. A maximum matching is a matching that contains the largest possible number of edges. The number of edges in a maximum matching of a graph G is denoted by $\alpha(G)$.

Lemma 2.1. *If G is a complete graph on $n \geq 4$ vertices, then $K_2(G) = \alpha(G) + 1$.*

Proof. First observe that $\alpha(G) = \lfloor \frac{n}{2} \rfloor$. In [1], it is proved that $K_2(\widetilde{G}) \geq \alpha(\widetilde{G}) + 1$ for any graph $\widetilde{G}$ with maximum degree at least two. Therefore, it suffices to show that any M_2-edge coloring of G uses at most $\alpha(G) + 1$ colors.

For $|V(G)| = 4$ the result follows from [4]. Let $|V(G)| = 5$ and let ψ be an optimal coloring of G. Choose one edge from each color class. These edges induce a graph G' of maximum degree two. The graph G does not contain any multicolored (no two edges have the same color) path or cycle on four edges, because no such coloring can be extended to an M_2-edge coloring of G. Moreover, G does not contain any multicolored cycle on three edges and an other edge whose color does not appear on that 3-cycle, since otherwise it also contains a multicolored path on four edges. Therefore G' has at most three edges. Consequently, ψ uses at most three colors.

Let G be a complete graph on at least six vertices and let φ be an optimal coloring of G. The coloring φ uses at least four colors since $\alpha(G) \geq 3$. Let u_1u_2 and u_2u_3 be two adjacent edges which have different colors, say a, b, respectively. First we show that $\varphi(u_1u_3) \in \{a, b\}$. Assume that $\varphi(u_1u_3) = c \notin \{a, b\}$ and let u_4u_5 be an edge of color $d \notin \{a, b, c\}$. The color of the edge u_1u_4 is a or c since it is incident with the vertex u_1. If $\varphi(u_1u_4) = a$, then $\varphi(u_3u_4) \in \{b, c\} \cap \{a, d\} = \emptyset$. If $\varphi(u_1u_4) = c$, then $\varphi(u_2u_4) \in \{a, b\} \cap \{c, d\} = \emptyset$. This means that the color of u_1u_3 is a or b. Without loss of generality assume that $\varphi(u_1u_3) = a$.

Let v_1v_2 and v_3v_4 be edges in G of different colors c and d, where $\{a, b\} \cap \{c, d\} = \emptyset$. The edges v_1v_2 and v_3v_4 cannot be adjacent. Otherwise, without loss of generality assume that $v_2 = v_3$. Then the color of the edge u_2v_2 must be in $\{a, b\} \cap \{c, d\}$, a contradiction.

Let H be a graph obtained from G by removing all edges of colors a, b and also removing all isolated vertices. By the fact described above, every component of H is monochromatic. Therefore, the number of components of H increased by two is equal to the number of colors in any optimal coloring of G. Each component of H has at least one edge, thus at least two vertices. Therefore, the number of components of H is at most $\left\lfloor \frac{|V(H)|}{2} \right\rfloor \leq \left\lfloor \frac{|V(G)|-2}{2} \right\rfloor$ (we removed at least two vertices from G, namely u_2, u_3). This means that

$$K_2(G) \leq \left\lfloor \frac{|V(G)|-2}{2} \right\rfloor + 2.$$

If $|V(G)|$ is even, then

$$\left\lfloor \frac{|V(G)|-2}{2} \right\rfloor + 2 = \frac{|V(G)|-2}{2} + 2 = \frac{|V(G)|}{2} + 1 = \alpha(G) + 1.$$

If $|V(G)|$ is odd, then

$$\left\lfloor \frac{|V(G)|-2}{2} \right\rfloor + 2 = \frac{|V(G)|-3}{2} + 2 = \frac{|V(G)|-1}{2} + 1 = \alpha(G) + 1. \qquad \square$$

Lemma 2.2 ([4]). *If G is a graph such that each of its vertices has degree 1 or 3, then $K_2(G) = \frac{|V(G)|}{2} + t$, where t is the maximum number of disjoint cycles in G.*

Theorem 2.3. *For any positive integer ℓ there are graphs G_1, G_2 and G_3 such that $G_1 \subseteq G_2 \subseteq G_3$, $K_2(G_1) = K_2(G_3)$ and $K_2(G_2) - K_2(G_3) \geq \ell$.*

Proof. Let $k \geq 2$ be a fixed positive integer and let $n = 2k$. Let G_2 be an n-sided prism, with vertex set V and edge set E, where

$$V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\},$$

$$E = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_i \mid i = 1, \ldots, n\}$$

(see Figure 1 for illustration).

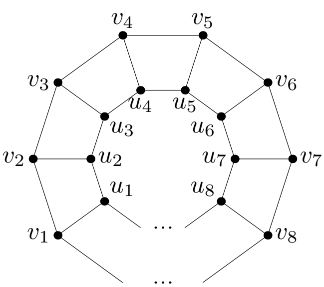

Fig. 1. An n-sided prism

Observe that the length of a shortest cycle in G_2 is four. Hence, the number of disjoint cycles in G_2 is at most $\frac{|V(G_2)|}{4} = \frac{n}{2}$. On the other hand, the cycles $u_i u_{i+1} v_{i+1} v_i u_i$ are disjoint for $i = 1, 3, \ldots, n-1$. Consequently, the number of disjoint cycles in G_2 equals $\frac{n}{2}$. Lemma 2.2 implies that $K_2(G_2) = n + \frac{n}{2}$.

Let G_1 be a spanning subgraph of G_2 with edge set

$$E = \{u_i u_{i+1}, u_i v_i \,|\, i = 1, \ldots, n\}.$$

From Lemma 2.2 it follows that $K_2(G_1) = n+1$, since G_1 is a unicylic subcubic graph without vertices of degree two.

Let G_3 be a complete graph on $2n$ vertices. From Lemma 2.1 we have $K_2(G_3) = n + 1$.

Clearly, $G_1 \subseteq G_2 \subseteq G_3$. Moreover, $K_2(G_1) = K_2(G_3) = n+1$ and $K_2(G_2) = n+\frac{n}{2}$. Therefore,

$$K_2(G_2) - K_2(G_3) = \frac{n}{2} - 1 = k - 1.$$

Consequently, it is sufficient to take $k = \ell + 1$. □

REFERENCES

[1] K. Budajová, J. Czap, *M_2-edge coloring and maximum matching of graphs*, Int. J. Pure Appl. Math. **88** (2013), 161–167.

[2] H. Choi, S.L. Hakimi, *Scheduling file transfers for trees and odd cycles*, SIAM J. Comput. **16** (1987), 162–168.

[3] E.G. Coffman Jr., M.R. Garey, D.S. Johnson, A.S. LaPaugn, *Scheduling file transfers*, SIAM J. Comput. **14** (1985), 744–780.

[4] J. Czap, *M_i-edge colorings of graphs*, Appl. Math. Sciences **5** (2011), 2437–2442.

[5] S.L. Hakimi, O. Kariv, *A generalization of edge-coloring in graphs*, J. Graph Theory **10** (1986), 139–154.

[6] H. Krawczyk, M. Kubale, *An approximation algorithm for diagnostic test scheduling in multicomputer systems*, IEEE Trans. Comput. **C-34** (1985), 869–872.

[7] S. Nakano, T. Nishizeki, N. Saito, *On the f-coloring of multigraphs*, IEEE Trans. Circuits Syst. **35** (1988), 345–353.

[8] X. Zhang, G. Liu, *Some sufficient conditions for a graph to be of C_f 1*, Appl. Math. Lett. **19** (2006), 38–44.

[9] X. Zhang, G. Liu, *f-colorings of some graphs of f-class 1*, Acta Math. Sin., Engl. Ser. **24** (2008), 743–748.

[10] X. Zhang, G. Liu, *Some graphs of class 1 for f-colorings*, Appl. Math. Lett. **21** (2008), 23–29.

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