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FRAMES AND FACTORIZATION OF GRAPH LAPLACIANS

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Abstract. Using functions from electrical networks (graphs with resistors assigned to edges), we prove existence (with explicit formulas) of a canonical Parseval frame in the energy Hilbert space $\mathscr{H}_E$ of a prescribed infinite (or finite) network. Outside degenerate cases, our Parseval frame is not an orthonormal basis. We apply our frame to prove a number of explicit results: With our Parseval frame and related closable operators in $\mathscr{H}_E$ we characterize the Friedrichs extension of the $\mathscr{H}_E$-graph Laplacian. We consider infinite connected network-graphs $G = (V, E)$, V for vertices, and E for edges. To every conductance function c on the edges E of G, there is an associated pair $(\mathscr{H}_E, \Delta)$ where $\mathscr{H}_E$ in an energy Hilbert space, and $\Delta\,(= \Delta_c)$ is the c-graph Laplacian; both depending on the choice of conductance function c. When a conductance function is given, there is a current-induced orientation on the set of edges and an associated natural Parseval frame in $\mathscr{H}_E$ consisting of dipoles. Now Δ is a well-defined semibounded Hermitian operator in both of the Hilbert $l^2(V)$ and $\mathscr{H}_E$. It is known to automatically be essentially selfadjoint as an $l^2(V)$-operator, but generally not as an $\mathscr{H}_E$ operator. Hence as an $\mathscr{H}_E$ operator it has a Friedrichs extension. In this paper we offer two results for the Friedrichs extension: a characterization and a factorization. The latter is via $l^2(V)$.

Keywords: unbounded operators, deficiency-indices, Hilbert space, boundary values, weighted graph, reproducing kernel, Dirichlet form, graph Laplacian, resistance network, harmonic analysis, frame, Parseval frame, Friedrichs extension, reversible random walk, resistance distance, energy Hilbert space.

Mathematics Subject Classification: 47L60, 46N30, 46N50, 42C15, 65R10, 05C50, 05C75, 31C20, 46N20, 22E70, 31A15, 58J65, 81S25.

1. INTRODUCTION

We study infinite networks with the use of frames in Hilbert space. While our results apply to finite systems, we will concentrate on the infinite case because of its statistical significance.

By a network we mean a graph G with vertices V and edges E. We assume that each vertex is connected by single edges to a finite number of neighboring vertices, and that resistors are assigned to the edges. From this we define an associated graph-Laplacian Δ, and a resistance metric on V.

The functions on V of interest represent voltage distributions. While there are a number of candidates for associated Hilbert spaces of functions on V, the one we choose here has its norm-square equal to the energy of the voltage function. This Hilbert space is denoted $\mathscr{H}_E$, and it depends on an assigned conductance function (= reciprocal of resistance). We will further study an associated graph Laplacian as a Hermitian semibounded operator with dense domain in $\mathscr{H}_E$.

In our first result we identify a canonical Parseval frame in $\mathscr{H}_E$, and we show that it is not an orthonormal basis except in simple degenerate cases. The frame vectors (for $\mathscr{H}_E$) are indexed by oriented edges e, a dipole vector for each e, and a current through e.

We apply our frame to complete a number of explicit results. We study the Friedrichs extension of the graph Laplacian Δ. And we use our Parseval frame and related closable operators in $\mathscr{H}_E$ to give a factorization of the Friedrichs extension of Δ.

Continuing earlier work [13,17,19–22,28] on analysis and spectral theory of infinite connected network-graphs $G = (V, E)$, V for vertices, and E for edges, we study here a new factorization for the associated graph Laplacians. Our starting point is a fixed conductance function c for G. It is known that, to every prescribed conductance function c on the edges E of G, there is an associated pair $(\mathscr{H}_E, \Delta)$ where $\mathscr{H}_E$ in an energy Hilbert space, and $\Delta\,(= \Delta_c)$ is the c-graph Laplacian; both depending on the choice of conductance function c. For related papers on frames and discrete harmonic analysis, see also [2,5,6,11,18,23,31,32] and the papers cited there.

It is also known that Δ is a well-defined semibounded Hermitian operator in both of the Hilbert $l^2(V)$ and $\mathscr{H}_E$; densely defined in both cases; and in each case with a natural domain, see [19]. As an $l^2(V)$-operator Δ has an $\infty \times \infty$ representation expressed directly in terms of (c, G), and it is further known that Δ is automatically be essentially selfadjoint as an $l^2(V)$-operator, but generally not as an $\mathscr{H}_E$ operator, [17,22]. Hence as an $\mathscr{H}_E$ operator it has a Friedrichs extension. In this paper we offer two results for the $\mathscr{H}_E$ Friedrichs extension: a characterization and a factorization. The latter is via the Hilbert space $l^2(V)$.

We begin with the basic notions needed, and we then turn to our theorem about Parseval frames: In Section 3, we show that, when a conductance function is given, there is a current-induced orientation on the set of edges and an associated natural Parseval frame in the energy Hilbert space $\mathscr{H}_E$ with the frame vectors consisting of dipoles.

2. BASIC SETTING

The graph Laplacian Δ (Definition 2.2) has an easy representation as a densely defined semibounded operator in $l^2(V)$ via its matrix representation, see Remark 2.3. To do this we use implicitly the standard orthonormal (ONB) basis $\{\delta_x\}$ in $l^2(V)$. But in network problems, and in metric geometry, $l^2(V)$ is not useful; rather we need the energy Hilbert space $\mathscr{H}_E$, see Lemma 3.2.

The problem with this is that there is not an independent characterization of the domain $dom\,(\Delta, \mathscr{H}_E)$ when Δ is viewed as an operator in $\mathscr{H}_E$ (as opposed to in $l^2(V)$); other than what we do in Definition 4.1, i.e., we take for its domain $\mathscr{D}_E =$ finite span of dipoles. This creates an ambiguity with functions on V versus vectors in $\mathscr{H}_E$. Note, vectors in $\mathscr{H}_E$ are equivalence classes of functions on V. In fact we will see that it is not feasible to aim to prove properties about Δ in $\mathscr{H}_E$ without first introducing dipoles; see Lemma 3.4 below. Also the delta-functions $\{\delta_x\}$ from the $l^2(V)$-ONB will typically not be total in $\mathscr{H}_E$. In fact, the $\mathscr{H}_E$ ortho-complement of $\{\delta_x\}$ in $\mathscr{H}_E$ consists of the harmonic functions in $\mathscr{H}_E$.

Let V be a countable discrete set, and let $E \subset V \times V$ be a subset such that:

1. $(x,y) \in E \Longleftrightarrow (y,x) \in E$; $x, y \in V$;
2. $\#\{y \in V \mid (x,y) \in E\}$ is finite, and > 0 for all $x \in V$;
3. $(x,x) \notin E$;
4. there exists $o \in V$ such that for all $y \in V$ there are $x_0, x_1, \ldots, x_n \in V$ with $x_0 = o$, $x_n = y$, $(x_{i-1}, x_i) \in E$ for all $i = 1, \ldots, n$.

The last property is called connectedness. If a conductance function c is given we require $c_{x_{i-1}x_i} > 0$.

Definition 2.1. A function $c : E \to \mathbb{R}_+ \cup \{0\}$ is called *conductance function* if $c\,(e) \geq 0$, for all $e \in E$, and if for all $x \in V$, and $(x,y) \in E$, $c_{xy} > 0$, and $c_{xy} = c_{yx}$.

If $x \in V$, we set

$$c(x) := \sum_{y} c_{xy}, \quad \text{sum over } \big\{y \in V \,\big|\, (x,y) \in E\big\} := E(x). \tag{2.1}$$

The summation in (2.1) is denoted $x \sim y$. We say that $x \sim y$ if $(x,y) \in E$.

Definition 2.2. When c is a conductance function (see Definition 2.1) we set $\Delta = \Delta_c$ (the corresponding graph Laplacian)

$$(\Delta u)\,(x) = \sum_{y \sim x} c_{xy}\,(u(x) - u(y)) = c(x)u(x) - \sum_{y \sim x} c_{xy} u(y). \tag{2.2}$$

Remark 2.3. Given (V, E, c) as above, and let $\Delta = \Delta_c$ be the corresponding graph Laplacian. With a suitable ordering on V, we obtain the following banded $\infty \times \infty$

matrix-representation for Δ:

$$\begin{bmatrix} c\left(x_{1}\right) & -c_{x_{1}x_{2}} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \cdots \\ -c_{x_{2}x_{1}} & c\left(x_{2}\right) & -c_{x_{2}x_{3}} & 0 & \cdots & \cdots & \cdots & \vdots & \cdots \\ 0 & -c_{x_{3}x_{2}} & c\left(x_{3}\right) & -c_{x_{3}x_{4}} & 0 & \cdots & \cdots & 0 & \cdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \cdots \\ \vdots & 0 & \cdots & 0 & -c_{x_{n}x_{n-1}} & c\left(x_{n}\right) & -c_{x_{n}x_{n+1}} & 0 & \cdots \\ \vdots & \vdots & \cdots & \cdots & 0 & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{2.3}$$

(We refer to [14] for a number of applications of infinite banded matrices.)

If $\#E(x) = 2$ for all $x \in V$, where $E(x) := \{y \in V \mid (x, y) \in E\}$, we say that the corresponding (V, E, c) is nearest neighbor, and in this case, the matrix representation takes the following form relative the an ordering in V:

$$o \underset{c_{ox_1}}{\longleftrightarrow} x_1 \underset{c_{x_1x_2}}{\longleftrightarrow} x_2 \quad \cdots \quad x_{n-1} \underset{c_{x_{n-1}x_n}}{\longleftrightarrow} x_n \underset{c_{x_nx_{n+1}}}{\longleftrightarrow} x_{n+1} \cdots \tag{2.4}$$

$$\begin{bmatrix} c\left(o\right) & -c_{ox_{1}} & 0 & \cdots & \cdots & \cdots & 0 \\ -c_{ox_{1}} & c\left(x_{1}\right) & -c_{x_{1}x_{2}} & 0 & \cdots & \cdots & \vdots \\ 0 & -c_{x_{1}x_{2}} & c\left(x_{2}\right) & c_{x_{2}x_{3}} & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & -c_{x_{n}x_{n-1}} & c\left(x_{n}\right) & -c_{x_{n}x_{n+1}} & 0 \\ \vdots & \cdots & & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \cdots & 0 & & \end{bmatrix} \tag{2.5}$$

Remark 2.4 (Random walk)**.** If (V, E, c) is given as in Definition 2.2, then for $(x, y) \in E$, set

$$p_{xy} := \frac{c_{xy}}{c(x)} \tag{2.6}$$

and note then $\{p_{xy}\}$ in (2.6) is a system of transition probabilities, i.e., $\sum_y p_{xy} = 1$ for all $x \in V$ (see Figure 2.1 below).

A Markov-random walk on V with transition probabilities (p_{xy}) is said to be *reversible* iff there exists a positive function $\widetilde{c}$ on V such that

$$\widetilde{c}(x)p_{xy} = \widetilde{c}(y)p_{yx} \text{ for all } (x, y) \in E. \tag{2.7}$$

Lemma 2.5. *There is a bijective correspondence between reversible Markov-walks on the one hand, and conductance functions on the other.*

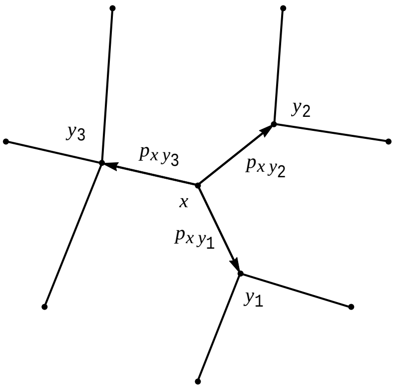


Fig. 2.1. Transition probabilities p_{xy} at a vertex x (in V)

Proof. If c is a conductance function on E (see Definition 2.1), then (p_{xy}), defined in (2.6), is a reversible walk. This follows from $c_{xy} = c_{yx}$.

Conversely, if (2.7) holds for a system of transition probabilities $(p_{xy} = = \mathrm{Prob}(x \mapsto y))$, then $c_{xy} := \widetilde{c}(x)p_{xy}$ is a conductance function, where

$$\widetilde{c}(x) = \sum_{y \sim x} c_{xy}.$$

□

For results on reversible Markov chains, see e.g., [27].

3. ELECTRICAL CURRENT AS FRAME COEFFICIENTS

The role of the graph-network setting $(V, E, c, \mathscr{H}_E)$ introduced above is, in part, to model a family of electrical networks; one application among others. Here G is a graph with vertices V, and edges E. Since we study large networks, it is helpful to take V infinite, but countable. Think of a network of resistors placed on the edges in G. In this setting, the functions $v_{(x,y)}$ in $\mathscr{H}_E$, indexed by pairs of vertices, represent dipoles. They measure voltage drop in the network through all possible paths between the two vertices x and y. Now the conduction function c is given, and so (electrical) current equals the product of conductance and voltage drop; in this case voltage drop is computed over the paths made up of edges from x to y. For infinite systems (V, E, c) the corresponding dipoles v_{xy} are *not* in $l^2(V)$, but they are always in $\mathscr{H}_E$; see Lemma 3.4 below.

For a fixed function u in $\mathscr{H}_E$ (voltage) and e in E we calculate the current $I(u, e)$, and we show that these numbers yield *frame coefficients* in a natural Parseval frame (for $\mathscr{H}_E$) where the frame vectors making up the Parseval frame are v_e, $e = (x, y)$ in E.

This result will be proved below. For general background references on frames in Hilbert space, we refer to [8, 12, 16, 24, 25, 30], and for electrical networks, see [3,4,9,15,29,33,35]. The facts on electrical networks we need are the laws of Kirchhoff and Ohm, and our computation of the frame coefficients as electrical currents is based on this, in part.

Definition 3.1. Let $\mathscr{H}$ be a Hilbert space with inner product denoted $\langle\cdot,\cdot\rangle$, or $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ when there is more than one possibility to consider. Let J be a countable index set, and let $\{w_j\}_{j\in J}$ be an indexed family of non-zero vectors in $\mathscr{H}$. We say that $\{w_j\}_{j\in J}$ is a *frame* for $\mathscr{H}$ iff there are two finite positive constants b_1 and b_2 such that

$$b_1 \left\|u\right\|_{\mathscr{H}}^2 \le \sum_{j\in J} \left|\langle w_j, u\rangle_{\mathscr{H}}\right|^2 \le b_2 \left\|u\right\|_{\mathscr{H}}^2 \tag{3.1}$$

holds for all $u \in \mathscr{H}$. We say that it is a *Parseval* frame if $b_1 = b_2 = 1$.

Lemma 3.2. *If $\{w_j\}_{j\in J}$ is a Parseval frame in $\mathscr{H}$, then the (analysis) operator $A = A_{\mathscr{H}} : \mathscr{H} \longrightarrow l^2(J)$,*

$$Au = \left(\langle w_j, u\rangle_{\mathscr{H}}\right)_{j\in J} \tag{3.2}$$

is well-defined and isometric. Its adjoint $A^* : l^2(J) \longrightarrow \mathscr{H}$ is given by

$$A^*\left(\left(\gamma_j\right)_{j\in J}\right) := \sum_{j\in J} \gamma_j w_j \tag{3.3}$$

and the following hold:

1. *The sum on the RHS in* (3.3) *is norm-convergent;*
2. *$A^* : l^2(J) \longrightarrow \mathscr{H}$ is co-isometric; and for all $u \in \mathscr{H}$, we have*

$$u = A^*Au = \sum_{j\in J} \langle w_j, u\rangle\, w_j, \tag{3.4}$$

where the RHS in (3.4) *is norm-convergent.*

Proof. The details are standard in the theory of frames; see the cited papers above. Note that (3.1) for $b_1 = b_2 = 1$ simply states that A in (3.2) is isometric, and so $A^*A = I_{\mathscr{H}}$ = the identity operator in $\mathscr{H}$, and AA^* = the projection onto the range of A. □

When a conductance function $c : E \to \mathbb{R}_+ \cup \{0\}$ is given, we consider the energy Hilbert space $\mathscr{H}_E$ (depending on c) with inner product and norm:

$$\langle u, v\rangle_{\mathscr{H}_E} := \frac{1}{2}\sum\sum_{(x,y)\in E} c_{xy}\left(\overline{u(x)} - \overline{u(y)}\right)(v(x) - v(y)) \tag{3.5}$$

and

$$\left\|u\right\|_{\mathscr{H}_E}^2 = \langle u, u\rangle_{\mathscr{H}_E} = \frac{1}{2}\sum\sum_{(x,y)\in E} c_{xy}\left|u(x) - u(y)\right|^2 < \infty. \tag{3.6}$$

We shall assume that (V, E, c) is *connected* (see Definition 2.1). It is shown that then (3.4)–(3.5) define $\mathscr{H}_E$ as a Hilbert space of functions on V; functions defined modulo constants; see also 4 below.

Further, for any pair of vertices $x, y \in V$, there is a unique dipole vector $v_{xy} \in \mathscr{H}_E$ such that

$$\langle v_{xy}, u \rangle_{\mathscr{H}_E} = u(x) - u(y) \tag{3.7}$$

holds for all $u \in \mathscr{H}_E$, see Lemma 3.4 below.

Remark 3.3. We illustrate this Parseval frame in Section 7.4 with a finite--dimensional example.

3.1. DIPOLES

Let $(V, E, c, \mathscr{H}_E)$ be as described above, and assume that (V, E, c) is connected. Note that "vectors" in $\mathscr{H}_E$ are equivalence classes of functions on V (= the vertex set in the graph $G = (V, E)$).

Lemma 3.4 ([19])**.** *For every pair of vertices $x, y \in V$, there is a unique vector $v_{xy} \in \mathscr{H}_E$ satisfying*

$$\langle u, v_{xy} \rangle_{\mathscr{H}_E} = u(x) - u(y) \tag{3.8}$$

for all $u \in \mathscr{H}_E$.

Proof. Fix a pair of vertices x, y as above, and pick a finite path of edges $(x_i, x_{i+1}) \in E$ such that $c_{x_i, x_{i+1}} > 0$, and $x_0 = y$, $x_n = x$. Then

$$\begin{aligned} u(x) - u(y) &= \sum_{i=0}^{n-1} u(x_{i+1}) - u(x_i) \\ &= \sum_{i=0}^{n-1} \frac{1}{\sqrt{c_{x_i x_{i+1}}}} \sqrt{c_{x_i x_{i+1}}} \left(u(x_{i+1}) - u(x_i) \right) \end{aligned} \tag{3.9}$$

and, by the Schwarz inequality, we have the following estimate:

$$\begin{aligned} |u(x) - u(y)|^2 &\leq \left(\sum_{i=0}^{n-1} \frac{1}{c_{x_i x_{i+1}}} \right) \sum_{j=0}^{n-1} c_{x_j x_{j+1}} \left| u(x_{j+1}) - u(x_j) \right|^2 \\ &\leq (\mathrm{Const}_{xy}) \left\| u \right\|^2_{\mathscr{H}_E}, \end{aligned}$$

valid for all $u \in \mathscr{H}_E$, where we used (3.6) in the last step of this *a priori* estimate. But this states that the linear functional:

$$L_{xy} : \mathscr{H}_E \ni u \longmapsto u(x) - u(y) \tag{3.10}$$

is continuous on $\mathscr{H}_E$ w.r.t. the norm $\|\cdot\|_{\mathscr{H}_E}$. Hence existence and uniqueness for $v_{xy} \in \mathscr{H}_E$ follows from Riesz' theorem. We get a unique $v_{xy} \in \mathscr{H}_E$ such that

$$L_{xy}(u) = \langle v_{xy}, u \rangle_{\mathscr{H}_E} \text{ for all } u \in \mathscr{H}_E.$$

□

Remark 3.5. Let $x, y \in V$ be as above, and let $v_{xy} \in \mathscr{H}_E$ be the dipole. One checks, using (2.1)–(2.2) that

$$\Delta v_{xy} = \delta_x - \delta_y. \tag{3.11}$$

But this equation (3.11) does not determine v_{xy} uniquely. Indeed, if w is a function on V satisfying $\Delta w = 0$, i.e., w is harmonic, then $v_{xy} + w$ also satisfies eq. (3.11). (In [20, 21] we studied when (V, E, c) has non-constant harmonic functions in $\mathscr{H}_E$.)

The system of vectors v_{xy} in (3.7) indexed by pairs of vertices carry a host of information about the given system $(V, E, c, \mathscr{H}_E)$, for example the computation of *resistance metric*.

Lemma 3.6. *When c (conductance) is given and assume (V, c) is connected, set*

$$d_c(x, y) := \sup \left\{ \frac{1}{\|u\|^2_{\mathscr{H}_E}} \;\Big|\; u \in \mathscr{H}_E, u(x) = 1, u(y) = 0 \right\}. \tag{3.12}$$

Then $d_c(x, y)$ is a metric on V, and

$$d_c(x, y) = \|v_{xy}\|^2_{\mathscr{H}_E}, \tag{3.13}$$

where the dipole vectors are specified as in (3.7).

Proof. Consider $u \in \mathscr{H}_E$ as in (3.12), i.e., $u(x) = 1$, $u(y) = 0$. Using (3.7) and the Schwarz inequality, we then get

$$\begin{aligned} 1 = u(x) - u(y) &= \left| \langle v_{xy}, u \rangle_{\mathscr{H}_E} \right|^2 \\ &\leq \|v_{xy}\|^2_{\mathscr{H}_E} \|u\|^2_{\mathscr{H}_E}. \end{aligned}$$

Since we know the optimizing vectors in the Schwarz inequality, the desired formula (3.13) now follows from (3.12) and (3.7). But (3.12) is known to yield a metric and the lemma follows. □

The next lemma offers a lower bound for the resistance metric between any two vertices when (V, E, c) is given. Given any two vertices x and y, we prove the following estimate: $\mathrm{dist}_c(x, y) \geq$ sum of dissipation along any path of edges from x to y.

Lemma 3.7. *Let $G = (V, E, c)$ be as before. For all finite paths*

$$x_0 := x \to (e_i) \to x_n := y,$$

we have

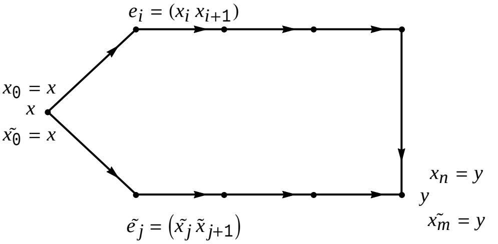
$$dist_c(x, y) \geq \underbrace{\sum_{i=0}^{n-1} Res_{x_i x_{i+1}} \left| I(v_{xy})_{i,i+1} \right|^2}_{dissipation}, \tag{3.14}$$

where $Res = \frac{1}{c}$ denotes the resistance.

Proof. In general there are many paths from x to y when x and y are fixed vertices (see Figure 3.1).

$$\begin{aligned}
\|v_{xy}\|^2_{\mathscr{H}_E} &= \mathrm{dist}_c(x,y) \\
&= \sum_{e\in E^{(dir)}} \left|\langle w_e, v_{xy}\rangle_{\mathscr{H}_E}\right|^2 \quad (w_e := \sqrt{c_e}v_e) \\
&= \sum_{e\in E^{(dir)}} c_c \left|\langle v_e, v_{xy}\rangle_{\mathscr{H}_E}\right|^2 \\
&\geq \sum_{i=0}^{n-1} c_{i,i+1} \left|v_{xy}(x_i) - v_{xy}(x_{i+1})\right|^2 \\
&= \sum_{i=0}^{n-1} \mathrm{Res}_{x_i x_{i+1}} \left|I(v_{xy})_{x_i x_{i+1}}\right|^2 .
\end{aligned}$$

□



$$v_{xy} = \sum_{i=0}^{n-1} v_{i,i+1} = \sum_{j=0}^{m-1} v_{j,j+1}$$

Fig. 3.1. Two finite paths connecting x and y, where $e_i = (x_i x_{i+1}), \widetilde{e_j} = (\widetilde{x}_j \widetilde{x}_{j+1}) \in E$

Now pick an orientation for each edge, and denote by $E^{(ori)}$ the set of oriented edges.

Theorem 3.8. *Let* $\left(V, E, c, E^{(ori)}\right)$ *and* $\mathscr{H}_E$ *be as above. Then the system of vectors*

$$w_{xy} := \sqrt{c_{xy}}v_{xy}, \text{ indexed by } (xy) \in E^{(ori)}, \tag{3.15}$$

is a Parseval frame for $\mathscr{H}_E$*.*

Proof. We will show that (3.1) holds for constants $b_1 = b_2 = 1$ for the vectors $\left(w_{(xy)}\right)_{(xy)\in E^{(ori)}}$, see (3.7)–(3.15). Indeed, we have for $u \in \mathscr{H}_E$:

$$\begin{aligned}
\|u\|^2_{\mathscr{H}_E} &\underset{\text{(by (3.6))}}{=} \sum_{(xy)\in E^{(ori)}} c_{xy} \left|u(x) - u(y)\right|^2 \\
&\underset{\text{(by (3.7))}}{=} \sum_{(xy)\in E^{(ori)}} c_{xy} \left|\langle v_{xy}, u\rangle_{\mathscr{H}_E}\right|^2 \\
&= \sum_{(xy)\in E^{(ori)}} \left|\left\langle \sqrt{c_{xy}} v_{xy}, u\right\rangle_{\mathscr{H}_E}\right|^2 \\
&\underset{\text{(by (3.15))}}{=} \sum_{(xy)\in E^{(ori)}} \left|\langle w_{xy}, u\rangle_{\mathscr{H}_E}\right|^2
\end{aligned}$$

which is the desired conclusion. □

Remark 3.9. While the vectors $w_{xy} := \sqrt{c_{xy}} v_{xy}$, $(xy) \in E^{(ori)}$, form a Parseval frame in $\mathscr{H}_E$ in the general case, typically this frame is not an orthogonal basis (ONB) in $\mathscr{H}_E$, although it is in Example 7.1 below.

To see when our Parseval frames are in fact ONBs, we use the following lemma.

Lemma 3.10. *Let* $\{w_j\}_{j\in J}$ *be a Parseval frame in a Hilbert space* $\mathscr{H}$*, then* $\|w_j\|_{\mathscr{H}_E} \leq 1$*, and it is an ONB in* $\mathscr{H}$ *if and only if* $\|w_j\|_{\mathscr{H}} = 1$ *for all* $j \in J$*.*

Proof. Follows from an easy application of

$$\|u\|^2_{\mathscr{H}} = \sum_{j\in J} \left|\langle w_j, u\rangle_{\mathscr{H}}\right|^2, \quad u \in \mathscr{H}. \tag{3.16}$$

Plug in w_{j_0} for u in (3.16). □

Remark 3.11. Frames in $\mathscr{H}_E$ consisting of our system (3.15) are not ONBs when resisters are configured in non-linear systems of vertices, for example, resisters in parallel. See Figure 3.2 and Example 7.4.

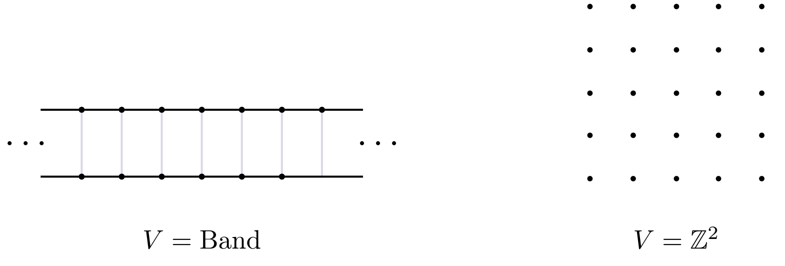


Fig. 3.2. Non-linear system of vertices

In these examples one checks that

$$1 > \|w_{xy}\|^2_{\mathscr{H}_E} = c_{xy} \|v_{xy}\|^2_{\mathscr{H}_E} = c_{xy} \left(v_{xy}(x) - v_{xy}(y)\right). \tag{3.17}$$

That is the current flowing through each edge $e = (x, y) \in E$ is < 1, or equivalently the voltage-drop across e is $<$ resistance

$$v_{xy}(x) - v_{xy}(y) < \frac{1}{c_{cy}} = \text{resistance}.$$

Definition 3.12. Let $V, E, c, \mathscr{H}_E$ be as above and for $u \in \mathscr{H}_E$, set

$$I(u)_{(xy)} := c_{xy} \left(u(x) - u(y)\right) \text{ for } (x, y) \in E. \tag{3.18}$$

By Ohm's law, the function $I(u)_{(xy)}$ in (3.18) represents the current in a network.

A choice of *orientation* may be assigned as follows (three different ways):

1. The orientation of every $(xy) \in E$ may be chosen arbitrarily.
2. The orientation may be suggested by geometry; for example in a binary tree, as shown in Figure 3.3 below.
3. Or the orientation may be assigned by the experiment of inserting one Amp at a vertex, say $o \in V$, and extracting one Amp at a distinct vertex, say $x_{dist} \in V$. We then say that an edge $(xy) \in E$ is positively oriented if $I(u)_{xy} > 0$ where $I(u)_{xy}$ is the induced current; see (3.18). See Figure 3.4 below.

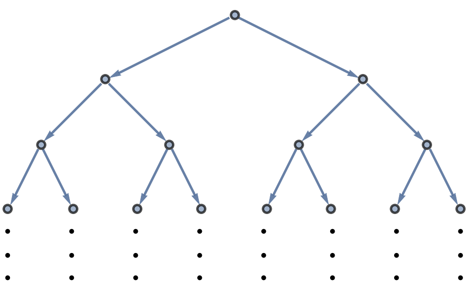


Fig. 3.3. Geometric orientation (see also Figure 7.4 below)

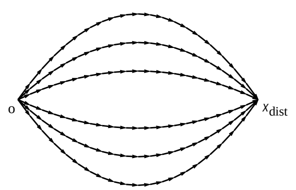


Fig. 3.4. Electrically induced orientation

Corollary 3.13 (Figure 3.4)**.** *Let $(V, E, c, \mathscr{H}_E)$ be as above, and let $E^{(ori)}$ be assigned as in* (3) *of Definition 3.12. Then every $u \in \mathscr{H}_E$ has a norm-convergent representation*

$$u = \sum_{(xy)\in E^{(ori)}} I\left(u\right)_{xy} v_{xy}. \tag{3.19}$$

Proof. By Theorem 3.8 and Lemma 3.2, we have the following norm-convergent representation

$$u = \sum_{(xy)\in E^{(ori)}} \left\langle w_{xy}, u\right\rangle_{\mathscr{H}_E} w_{xy}, \tag{3.20}$$

see (3.4). Now, by statements (3.15) and (3.18), we get

$$\left\langle w_{xy}, u\right\rangle_{\mathscr{H}_E} w_{xy} = I\left(u\right)_{xy} v_{xy}. \tag{3.21}$$

Considering (3.20) and (3.21), the desired conclusion (3.19) then follows. □

4. LEMMAS

Starting with a given network (V, E, c), we introduce functions on the vertices V, voltage, dipoles, and point-masses; and on the edges E, conductance, and current. We introduce a system of operators which will be needed throughout, the graph-Laplacian Δ, and the transition operator P. We show that there are two Hilbert spaces serving different purposes, $l^2(V)$, and the energy Hilbert space $\mathscr{H}_E$; the latter depending on choice of conductance function c.

Lemma 4.2 below summarizes the key properties of Δ as an operator, both in $l^2(V)$ and in $\mathscr{H}_E$. The metric properties of networks (V, E, c) depend on $\mathscr{H}_E$ (Lemma 4.3), and not on $l^2(V)$.

Recall that the graph-Laplacian Δ is automatically essentially selfadjoint as a densely defined operator in $l^2(V)$, but not as a $\mathscr{H}_E$ operator [17, 21]. In Section 7 we compute examples where $(\Delta, \mathscr{H}_E)$ has deficiency indices (m, m), $m > 0$. These results make use of an associated reversible random walk, as well as the transition operator P.

Let (V, E, c) be as above; note we are assuming that $G = (V, E)$ is connected; so there is a point o in V such that every $x \in V$ is connected to o via a finite path of edges. We will set $V' := V\setminus\{o\}$, and consider $l^2(V)$ and $l^2(V')$. If $x \in V$, we set

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases} \tag{4.1}$$

Set $\mathscr{H}_E :=$ the set of all functions $u : V \to \mathbb{C}$ such that

$$\|u\|^2_{\mathscr{H}_E} := \frac{1}{2}\sum\sum_{(x,y)\in E} c_{xy} \left|u(x) - u(y)\right|^2 < \infty, \tag{4.2}$$

and we note ([19]) that $\mathscr{H}_E$ is a Hilbert space. Moreover, for all $x, y \in V$, there is a real-valued solution $v_{xy} \in \mathscr{H}_E$ to the equation

$$\Delta v_{x,y} = \delta_x - \delta_y. \tag{4.3}$$

If $y = o$, we set $v_x := v_{x,o}$, and note

$$\Delta v_x = \delta_x - \delta_o. \tag{4.4}$$

In this case, we assume that v_x is defined only for $x \in V'$.

Definition 4.1. Let $\left(V, E, c, o, \Delta, \{v_x\}_{x\in V'}\right)$ be as above, and set

$$\mathscr{D}_2 := span\left\{\delta_x \;\middle|\; x \in V\right\} \tag{4.5}$$

and

$$\mathscr{D}_E := span\left\{v_x \;\middle|\; x \in V'\right\}, \tag{4.6}$$

where by "span" we mean of all finite linear combinations.

Lemma 4.2. *The following hold:*

1. $\langle \Delta u, v\rangle_{l^2} = \langle u, \Delta v\rangle_{l^2}$ *for all* $u, v \in \mathscr{D}_2$*;*
2. $\langle \Delta u, v\rangle_{\mathscr{H}_E} = \langle u, \Delta v\rangle_{\mathscr{H}_E}$ *for all* $u, v \in \mathscr{D}_E$*;*
3. $\langle u, \Delta u\rangle_{l^2} \geq 0$ *for all* $u \in \mathscr{D}_2$*;*
4. $\langle u, \Delta u\rangle_{\mathscr{H}_E} \geq 0$ *for all* $u \in \mathscr{D}_E$*, where for* $u, v \in \mathscr{H}_E$ *we set*

$$\langle u, v\rangle_{\mathscr{H}_E} = \frac{1}{2}\sum\sum_{(x,y)\in E} c_{xy}\overline{(u(x) - u(y))}\,(v(x) - v(y))\,. \tag{4.7}$$

Moreover, we have

5. $\langle v_{x,y}, u\rangle_{\mathscr{H}_E} = u(x) - u(y)$ *for all* $x, y \in V$*.*

Finally,

6.
$$\delta_x(\cdot) = c(x)v_x(\cdot) - \sum_{y\sim x} c_{xy}v_y(\cdot) \;\; \textit{for all } x \in V'.$$

Proof. (2) We have $\langle \Delta u, v\rangle_{\mathscr{H}_E} = \langle u, \Delta v\rangle_{\mathscr{H}_E}$, for all $u, v \in \mathscr{D}_E$. Set $v_x := v_{xo}$, where o is a fixed base-point in V, $V' := V\setminus\{o\}$, so $\Delta v_x = \delta_x - \delta_o$, $x \in V'$. Set $u = \sum_{x\in V'}\xi_x v_x$, $v = \sum_{x\in V'}\eta_x v_x$, where the summations are finite by convention. Then

$$\begin{aligned}
\langle \Delta u, v\rangle_{\mathscr{H}_E} &= \sum_{V'}\sum_{V'}\overline{\xi_x}\eta_y\,\langle \delta_x - \delta_o, v_y\rangle_{\mathscr{H}_E}\\
&= \sum_{V'}\sum_{V'}\overline{\xi_x}\eta_y\Big((\delta_x(y) - \underbrace{\delta_x(o)}_{=0}) - (\underbrace{\delta_o(y)}_{=0} - \underbrace{\delta_o(o)}_{=1})\Big)\\
&= \sum_{V'}\sum_{V'}\overline{\xi_x}\eta_y\,(\delta_{xy} + 1)\\
&= \sum_{V'}\overline{\xi_x}\eta_y + \Big(\overline{\sum_{V'}\xi_x}\Big)\Big(\sum_{V'}\eta_y\Big)\\
&= \langle u, \Delta v\rangle_{\mathscr{H}_E} \text{ (by symmetry).}
\end{aligned}$$

For the remaining, see [19, 20]. ∎

Lemma 4.3. *Let (V, E, c, o) be as above. Then the function*

$$N_c(x,y) := \|v_x - v_y\|^2_{\mathscr{H}_E} \tag{4.8}$$

is conditionally negative definite, i.e., for all finite system $\{\xi_x\} \subseteq \mathbb{C}$ such that $\sum_{x\in V'} \xi_x = 0$, we have

$$\sum_x \sum_y \overline{\xi_x}\xi_y N_c(x,y) \le 0. \tag{4.9}$$

Proof. Compute the LHS in (4.9) as follows. If $\sum \xi_x = 0$, we have

$$\begin{aligned}
&\sum \overline{\xi_x}\xi_y N_c(x,y)\\
&= -\sum\sum \overline{\xi_x}\xi_y \langle v_x, v_y\rangle_{\mathscr{H}_E} - \sum\sum \overline{\xi_x}\xi_y \langle v_y, v_x\rangle_{\mathscr{H}_E}\\
&= -2\Big\| \sum_x \xi_x v_x \Big\|^2_{\mathscr{H}_E}.
\end{aligned}$$

□

We show also the following

Lemma 4.4 ([19])**.**

$$\left\{u \in \mathscr{H}_E \;\middle|\; \langle u, \delta_x\rangle_{\mathscr{H}_E} = 0 \text{ for all } x \in V\right\} = \left\{u \in \mathscr{H}_E \;\middle|\; \Delta u = 0\right\}. \tag{4.10}$$

When N is a fixed negative definite function, we get an associated Hilbert space $\mathscr{H}_N$ by completing finitely supported functions ξ on V subject to the condition $\sum_{x\in V} \xi_x = 0$, under the inner product

$$\|\xi\|^2_{\mathscr{H}_N} := -\sum_x \sum_y \overline{\xi_x}\xi_y N(x,y)$$

and quotienting out with

$$\sum_x \sum_y \overline{\xi_x}\xi_y N(x,y) = 0.$$

Lemma 4.5. *Assume (V, E, c) is connected. If a negative definite function N on $V \times V$ satisfies $N = N_c$, then*

$$\mathscr{H}_{N_c} = \mathscr{H}_E, \tag{4.11}$$

where $\mathscr{H}_E$ is the energy Hilbert space from 3 defined on the prescribed functions.

Proof. By Lemma 4.3, we have

$$\|\xi\|^2_{\mathscr{H}_{N_c}} = \Big\| \sum_{x\in V'} \xi_x v_x \Big\|^2_{\mathscr{H}_E}, \tag{4.12}$$

where $v_x = v_{ox}$ is a system of dipoles corresponding to a fixed base point $o \in V$, and $V' = V\setminus\{o\}$, and

$$\langle v_x, u\rangle_{\mathscr{H}_E} = u(x) - u(o) \text{ for all } u \in \mathscr{H}_E. \tag{4.13}$$

Hence, we need only prove that all the finite summations $\sum_{x\in V'}\xi_x v_x$ subjecting to $\sum_{x\in V'}\xi_x = 0$ are dense in $\mathscr{H}_E$. But if $u \in \mathscr{H}_E$, $u \in \left\{\sum_{x\in V'}\xi_x v_x \mid \sum_{x\in V'}\xi_x = 0\right\}^{\perp}$ (the orthogonal-complement), then

$$\langle v_x, u\rangle_{\mathscr{H}_E} - \langle v_y, u\rangle_{\mathscr{H}_E} = 0$$

for all $x, y \in V'$. Hence by (4.13) we get $u(x) = u(y)$ for all pairs $x, y \in V'$; and $u(x) = u(o)$, $x \in V'$. Since (V, E, c) is connected, it follows that u is constant. But with the normalization $v_x(o) = 0$ $(x \in V')$, we conclude that u must be zero. □

Lemma 4.6. *Let (V, E, c) be a connected network, and let $\mathscr{H}_E$ be the energy Hilbert space. Then, for all $f \in \mathscr{H}_E$ and $x \in V$, we have*

$$\langle \delta_x, f\rangle_{\mathscr{H}_E} = (\Delta f)(x). \tag{4.14}$$

Proof. We compute $\text{LHS}_{(4.14)}$ with the use of eq. (4.7) in Lemma 4.2. Indeed,

$$\begin{aligned}\langle \delta_x, f\rangle_{\mathscr{H}_E} &\underset{(4.7)}{=} \frac{1}{2}\sum\sum_{(st)\in E} c_{st}\left(\delta_x(s) - \delta_x(t)\right)\left(f(s) - f(t)\right)\\ &= \sum_{t\sim x} c_{xt}\left(f(x) - f(t)\right) = (\Delta f)(x),\end{aligned}$$

where we used (2.2) in Definition 2.2 in the last step. □

Corollary 4.7. *Let (V, E, c) be as above. Then*

$$\langle \delta_x, \delta_y\rangle_{\mathscr{H}_E} = \begin{cases}\widetilde{c}(x) = \sum_{t\sim x} c_{xt} & \text{if } y = x,\\ -c_{xy} & \text{if } (xy) \in E,\\ 0 & \text{if } (xy) \in E \text{ and } x \neq y.\end{cases} \tag{4.15}$$

Proof. Immediate from the lemma. □

Corollary 4.8. *Let (V, E, c, Δ) be as above. Then*

$$\mathscr{H}_E \ominus \{\delta_x \mid x \in V\} = \{u \in \mathscr{H}_E \mid \Delta u = 0\}. \tag{4.16}$$

Proof. This is immediate from (4.14) in Lemma 4.6. (Note that "$\ominus$" in (4.16) means ortho-complement.) □

Corollary 4.9. *Let (V, E, c, Δ) be as above. Then, for every $x \in V$, we have*

$$\sum_{y\sim x} c_{xy} v_{xy} = \delta_x. \tag{4.17}$$

Proof. Immediate from (4.14) in Lemma 4.6. □

5. THE HILBERT SPACES $\mathscr{H}_E$ AND $l^2(V')$, AND OPERATORS BETWEEN THEM

The purpose of this section is to prepare for the two results (Sections 5 and 6) on factorization to follow.

Definition 5.1. Let $\left(V, E, c, o, \{v_x\}_{x \in V'}\right)$ be specified as in Sections 2–4, where c is a fixed conductance function. Set

$$\mathscr{D}'_{l^2} := \text{all finitely supported functions } \xi \text{ on } V' \text{such that } \sum_x \xi_x = 0. \tag{5.1}$$

Then assuming that V is infinite, we conclude that $\mathscr{D}'_{l^2}$ is dense in $l^2(V')$.

Lemma 5.2. *For* $(\xi_x) \in \mathscr{D}'_{l^2}$*, set*

$$K(\xi) := \sum_{x \in V'} \xi_x v_x \in \mathscr{H}_E. \tag{5.2}$$

Then $K (= K_c)$ *is a densely defined, and closable operator*

$$K : l^2(V') \longrightarrow \mathscr{H}_E$$

with domain $\mathscr{D}'_{l^2}$.

Proof. We must prove that the norm-closure of the graph of K in $l^2 \times \mathscr{H}_E$ is again the graph of a linear operator; equivalently, if $\lim_{n\to\infty} \left\|\xi^{(n)}\right\|_{l^2} = 0$, $\xi^{(n)} \in \mathscr{D}_{l^2}$; and if there exists $u \in \mathscr{H}_E$ such that

$$\lim_{n\to\infty} \left\|K\left(\xi^{(n)}\right) - u\right\|_{\mathscr{H}_E} = 0, \tag{5.3}$$

then $u = 0$ in $\mathscr{H}_E$.

We prove this by establishing a formula for an adjoint operator,

$$K^* : \mathscr{H}_E \longrightarrow l^2(V')$$

having as its domain

$$\left\{\sum_x \xi_x v_x \;\middle|\; \text{finite sums, } \xi_x \in \mathbb{C} \text{ such that } \textstyle\sum_x \xi_x = 0\right\}. \tag{5.4}$$

Setting

$$K^*\left(\sum_x \xi_x v_x\right) = (\zeta_x),$$

where

$$\zeta_x = \sum_y \langle v_x, v_y \rangle_{\mathscr{H}_E} \xi_y \tag{5.5}$$

on the space in (5.4), we show that this is a well-defined, densely defined, linear operator, and that

$$\left\langle K^*\Big(\sum_x \xi_x v_x\Big), \eta\right\rangle_{l^2} = \left\langle \sum_x \xi_x v_x, K\eta\right\rangle_{\mathscr{H}_E} \tag{5.6}$$

holds for all $\eta \in \mathscr{D}'_{l^2}$. This shows that a well-defined adjoint K^* operator exists (by (5.5)), and that therefore the implication in (5.3) is valid.

Now let $\xi, \eta \in \mathscr{D}'_{l^2}$ as in (5.1). Then

$$\begin{aligned}
(\mathrm{LHS})_{(5.6)} &= \left\langle \sum_y \langle v_x, v_y\rangle_{\mathscr{H}_E} \xi_y, \eta \right\rangle_{l^2} \\
&= \sum_x \sum_y \overline{\xi_y} \langle v_y, v_x\rangle_{\mathscr{H}_E} \eta_x \\
&= \left\langle \sum_y \xi_y v_y, \sum_x \eta_x v_x \right\rangle_{\mathscr{H}_E} \\
&\underset{(\text{by } (5.2))}{=} \left\langle \sum_y \xi_y v_y, K\eta \right\rangle_{\mathscr{H}_E} = (\mathrm{RHS})_{(5.6)}.
\end{aligned}$$

□

6. THE FRIEDRICHS EXTENSION

Below we fix a conductance function c which turns the system (V, E, c) into a connected network (see Section 2), and we will study the c-graph Laplacian Δ in $\mathscr{H}_E$, the energy Hilbert space.

Notice that Δ will then be densely defined in $\mathscr{H}_E$, see Definition 4.1 and Lemma 4.2. Below we study the Friedrichs extension of Δ when it is defined on its natural dense domain $\mathscr{D}_E$ in $\mathscr{H}_E$.

Let $(V, E, c, o, \{v_x\}, \Delta)$ be as above, i.e.,

- $G \begin{cases} V = \text{set of vertices, assumed countable infinite } \aleph_0, \\ E = \text{edges, } V \text{assumed } E\text{−connected,} \end{cases}$
- $c : E \longrightarrow \mathbb{R}_+ \cup \{0\}$ a fixed conductance function,
- $\Delta\,(:= \Delta_c)$ the graph Laplacian,
- $o \in V$ a fixed base-point such that $\Delta v_x = \delta_x - \delta_o$,
- $V' := V \backslash \{o\}$,
- $\mathscr{H}_E := \operatorname{span}\{v_x \mid x \in V'\}$.

Recall we proved in Section 2 that Δ is a semibounded Hermitian operator with dense domain $\mathscr{D}_E$ in $\mathscr{H}_E$.

In this section, we shall be concerned with its Friedrichs extension, now denoted Δ_{Fri}; for details on the Friedrichs extension, see e.g., [1, 10]; and, in the special case of $(\Delta, \mathscr{H}_E)$, see [19, 21].

In all cases, we have that Δ, Δ_{Fri}, and Δ^* as operators in $\mathscr{H}_E$ act on subspaces of functions on V via the following formula:

$$(\Delta u)(x) = \sum_{y \sim x} c_{xy} \left(u(x) - u(y)\right), \tag{6.1}$$

where u is a function on the vertex set V.

Lemma 6.1. *As an operator in $\mathscr{H}_E$, the graph Laplacian Δ (with domain $\mathscr{D}_E$) may, or may not, be essentially selfadjoint. Its deficiency indices are (m,m), where*

$$m = \dim \left\{ u \in \mathscr{H}_E \,\middle|\, \Delta u = -u \right\}. \tag{6.2}$$

Proof. Recall that if S is a densely defined operator in a Hilbert space $\mathscr{H}$ such that

$$\langle u, Su \rangle \geq 0 \text{ for all } u \in dom\,(S), \tag{6.3}$$

then S will automatically have indices (m,m) where $m = \dim\left(\mathscr{N}\left(S^* + I\right)\right)$, and where S^* denotes the adjoint operator, i.e.,

$$\begin{aligned} dom\,(S^*) = \Big\{ u \in \mathscr{H} \,\Big|\, & \text{there exists}\, C < \infty \\ & \text{such that } \left|\langle u, S\varphi \rangle\right| \leq C \left\|\varphi\right\| \text{ for all}\, \varphi \in dom\,(S) \Big\}. \end{aligned} \tag{6.4}$$

We may apply this to $\mathscr{H} = \mathscr{H}_E$, and $S := \Delta$ on the domain $\mathscr{D}_E$. One checks that, if $u \in dom\,(S^*)$, i.e., $u \in dom((\Delta|_{\mathscr{D}_E})^*)$, then

$$(S^* u)(x) = \sum_{y \in E(x)} c_{xy} \left(u(x) - u(y)\right),$$

(i.e., the pointwise action of Δ on functions) and the conclusion in (6.2) follows from the assertion about $\mathscr{N}\left(S^* + I\right)$. □

Corollary 6.2. *Let $p_{xy} := \frac{c_{xy}}{c(x)}$ be the transition-probabilities in Remark 2.4 (see also Figure 2.1), and let*

$$(Pu)(x) := \sum_{y \sim x} p_{xy} u(y) \tag{6.5}$$

be the corresponding transition operator, accounting for the p-random walk on (V, E). Let $(\Delta, \mathscr{H}_E)$ be the $\mathscr{H}_E$-symmetric operator with domain $\mathscr{D}_E$, see Definition 4.1. Then $(\Delta, \mathscr{H}_E)$ has deficiency indices (m,m), $m > 0$, if and only if there is a function u on V, $u \neq 0$, $u \in \mathscr{H}_E$ satisfying

$$\left(1 + \frac{1}{c(x)}\right) u(x) = (Pu)(x) \text{ for all } x \in V. \tag{6.6}$$

Proof. Since $(\Delta|_{\mathscr{D}_E})^*$ acts pointwise on functions on V (see Lemma 6.1), we only need to verify that the equation $-u = \Delta u$ translates into (6.6), but we have:

$$-u(x) = c(x)u(x) - \sum_{y\sim x} c_{xy}u(y) \Longleftrightarrow \left(1 + \frac{1}{c(x)}\right)u(x) = \sum_{y\sim x} \frac{c_{xy}}{c(x)}u(y)$$

which is the desired eq. (6.6).

For u from (6.6) to be in $dom((\Delta|_{\mathscr{D}_E})^*)$ we must have

$$\sum_{(xy)\in E} c_{xy} |u(x) - u(y)|^2 < \infty$$

as asserted. □

In the discussion below, we use that both operators Δ and P take real valued functions on V to real valued functions, and that P is positive, satisfying $P\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the constant function "one" on V.

Lemma 6.3. *If u is a non-zero real valued function on V satisfying* (6.2)*, or equivalently* (6.6)*, and if $p \in V$ satisfies $u(p) \neq 0$, then there is an infinite path of edges $(x_i x_{i+1}) \in E$ such that $x_0 = p$, and*

$$u(x_{k+1}) \geq \prod_{i=0}^{k} \left(1 + \frac{1}{c(x_i)}\right) u(p). \tag{6.7}$$

Proof. We may assume without loss of generality that $u(p) > 0$. Set $x_0 = p$, and $x_1 := \arg\max\{u(y) \mid y \sim x_0\}$, so $u(x_1) = \max u\big|_{E(x_0)}$. Then

$$u(x_1) \geq \sum_{y\sim x_0} p_{x_0 y}u(y) = (Pu)(x_0) = \left(1 + \frac{1}{c(x_0)}\right)u(x_0),$$

where we used (6.6) in the last step.

Now for the induction: Suppose $x_1, \ldots, x_k$ have been found as specified; then set $x_{k+1} := \arg\max\{u(y) \mid y \sim x_k\}$, so

$$u(x_{k+1}) \geq (Pu)(x_k) = \left(1 + \frac{1}{c(x_k)}\right)u(x_k).$$

A final iteration then yields the desired conclusion (6.7). □

Remark 6.4. In Section 7.1 below, we illustrate a family of systems (V, E, c), where Δ in $\mathscr{H}_E$ has indices $(1, 1)$. In these examples, $V = \mathbb{Z}_+ \cup \{0\}$, and the edges E consists of nearest neighbor links, i.e., if $x \in \mathbb{Z}_+$,

$$E(x) = \{x - 1, x + 1\}, \text{ while } E(0) = \{1\}.$$

Lemma 6.5. *A function u on V is in the domain of Δ_{Fri} (the Friedrichs extension) if and only if u is in the completion of $\mathscr{D}_E$ with respect to the quadratic form*

$$\mathscr{D}_E \ni \varphi \longmapsto \langle \varphi, \Delta\varphi \rangle_{\mathscr{H}_E} \in \mathbb{R}_+ \cup \{0\} \tag{6.8}$$

and

$$\Delta u \in \mathscr{H}_E. \tag{6.9}$$

Proof. The assertion follows from an application of the characterization of Δ_{Fri} in [1, 10] combined with the following fact: If $\varphi = \sum_{x \in V} \xi_x v_x$ is a finite sum with coefficient (ξ_x) satisfying $\sum_x \xi_x = 0$, then

$$\langle \varphi, \Delta\varphi \rangle_{\mathscr{H}_E} = \sum_{x \in V'} |\xi_x|^2 . \tag{6.10}$$

Moreover, eq. (6.9) holds if and only if

$$\|\Delta u\|^2_{\mathscr{H}_E} := \frac{1}{2} \sum_{(x,y) \in E} c_{xy} \left|(\Delta u)(x) - (\Delta u)(y)\right|^2 < \infty.$$

We now prove formula (6.10). Assume $\varphi = \sum_{x \in V'} \xi_x v_x$ is as stated. Then

$$\begin{aligned}
\langle \varphi, \Delta\varphi \rangle_{\mathscr{H}_E} &= \Big\langle \sum\nolimits_x \xi_x v_x, \Delta\big(\sum\nolimits_x \xi_y v_y\big) \Big\rangle_{\mathscr{H}_E} \\
&= \Big\langle \sum\nolimits_x \xi_x v_x, \sum\nolimits_y \xi_y \left(\delta_y - \delta_o\right) \Big\rangle_{\mathscr{H}_E} \\
&= \sum\nolimits_x \sum\nolimits_y \overline{\xi_x} \xi_y \left\langle v_x, \delta_y \right\rangle_{\mathscr{H}_E} \text{ (since } \textstyle\sum_y \xi_y = 0) \\
&= \sum\nolimits_x |\xi_x|^2 .
\end{aligned}$$

□

Lemma 6.6. *Let $\mathscr{H}_E$ denote the completion of $\mathscr{D}_E$, and let $\mathscr{D}'_{l^2}$ be the dense subspace in $l^2(V')$, given by $\sum_{x \in V'} \xi_x = 0$, $\sum_x |\xi_x|^2 < \infty$. Set*

$$L(\xi_x) := \sum_x \xi_x \delta_x. \tag{6.11}$$

Then $L : l^2(V') \longrightarrow \mathscr{H}_E$ is a closable operator with dense domain $\mathscr{D}'_{l^2}$ and the corresponding adjoint operator

$$L^* : \mathscr{H}_E \longrightarrow l^2(V')$$

satisfies

$$L^*\Big(\sum\nolimits_{x \in V'} \xi_x v_x\Big) = \xi. \tag{6.12}$$

Proof. To prove the assertion, we must show that, if ξ is a finitely supported function on V' such that $\sum_{x\in V'}\xi_x = 0$, then

$$\langle L(\xi), u\rangle_{\mathscr{H}_E} = \langle \xi, \eta\rangle_{l^2}, \tag{6.13}$$

where

$$u = \sum_{y\in V'} \eta_y v_y. \tag{6.14}$$

We prove (6.13) as follows:

$$\begin{aligned} (\text{LHS})_{(6.13)} &= \left\langle \Delta(\textstyle\sum_x \xi_x v_x), \sum_y \eta_y v_y \right\rangle_{\mathscr{H}_E} \\ &= \left\langle \textstyle\sum_x \xi_x v_x, \Delta(\sum_y \eta_y v_y) \right\rangle_{\mathscr{H}_E} \\ &= \textstyle\sum_x \overline{\xi_x}\eta_x = (\text{RHS})_{(6.13)}, \end{aligned}$$

where we used formula (6.10) in the last step of the computation. □

Remark 6.7. To understand Δ as an operator in the respective subspaces of $\mathscr{H}_E$ recall that $\mathscr{H}_E$ contains two systems of vectors $\{\delta_x\}$ and $\{v_x\}$ both indexed by V'.

Neither of the two systems is orthogonal in the inner product of $\mathscr{H}_E$.

We have $\langle v_x, v_y\rangle_{\mathscr{H}_E} = v_x(y) = v_y(x)$. Recall our normalization $v_x(o) = 0$, where o is the fixed base-point. Moreover (see Corollary 4.7),

$$\langle \delta_x, \delta_y\rangle_{\mathscr{H}_E} = \begin{cases} -c_{xy} & \text{if } (x,y)\in E, \\ c(x) & \text{if } x = y, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\langle \delta_x, v_y\rangle_{\mathscr{H}_E} = \begin{cases} \delta_{xy} & \text{if } x,y\in V', \\ -1 & \text{if } x = o, y\in V'. \end{cases}$$

Corollary 6.8. *If $x\in V'$, then $\delta_x\in\mathscr{H}_E$ and*

$$\delta_x(\cdot) = c(x)v_x(\cdot) - \sum_{y\sim x} c_{xy}v_y(\cdot). \tag{6.15}$$

Proof. To show this, it is enough to check equality of

$$\left\langle \text{LHS}_{(6.15)}, v_x\right\rangle_{\mathscr{H}_E} = \left\langle \text{RHS}_{(6.15)}, v_x\right\rangle_{\mathscr{H}_E} \text{ for all } x\in V,$$

and this follows from an application of the formulas in Remark 6.7 above. □

Theorem 6.9. *Let $(V, E, c, \Delta(=\Delta_c), \mathscr{H}_E, \Delta_{Fri})$ be as above. Let L and L^* be the closed operators from Lemma 6.6. Then*

(i) LL^* *is selfadjoint, and*
(ii) $LL^* = \Delta_{Fri}$.

Proof. Conclusion (i) follows for every closed operator L with dense domain, we have that LL^* is selfadjoint. To prove (ii), we must verify that

$$LL^*u = \Delta u \text{ for all } u \in \mathscr{D}'_E. \tag{6.16}$$

Let $u = \sum_{x\in V'} \xi_x v_x$ be a finite sum such that $\sum_x \xi_x = 0$. Then

$$\begin{aligned} LL^*u &\underset{(\text{by } (6.12))}{=} L\xi \\ &\underset{(\text{by } (6.11))}{=} \sum_{x\in V'} \xi_x \delta_x \\ &\underset{(\text{since } \sum \xi_x = 0)}{=} \sum_{x\in V'} \xi_x \left(\delta_x - \delta_o\right) \\ &\underset{(\text{by } (6.11))}{=} \sum_{x\in V'} \xi_x \Delta v_x \\ &\underset{\text{since finite sum}}{=} \Delta \left(\sum_x \xi_x v_x \right) \\ &= \Delta u = (\text{RHS})_{(6.16)}. \end{aligned}$$

□

Corollary 6.10. *We have the Greens-Gauss identity:*

$$\left(\Delta_y \left(\langle v_x, v_y \rangle\right)\right)(z) = \delta_{xz} \textit{ for all } x, y, z \in V'. \tag{6.17}$$

The inner products $\langle v_x, v_y \rangle := \langle v_x, v_y \rangle_{\mathscr{H}_E}$ constitute the Gramian of the frame Theorem 3.8.

Proof.

$$\begin{aligned} (\text{LHS})_{(6.17)} &= \sum_{w\sim z} c_{zw} \left(\langle v_x, v_z \rangle_{\mathscr{H}_E} - \langle v_x, v_w \rangle_{\mathscr{H}_E} \right) \\ &= \sum_{w\sim z} c_{zw} \left(v_x(z) - v_x(w) \right) \\ &= \left(\Delta v_x\right)(z) \\ &= \left(\delta_x - \delta_o\right)(z) = \delta_{xz} = (\text{RHS})_{(6.17)}, \end{aligned}$$

where we used that $z \in V' = V \setminus \{o\}$ in the last step of the verification. □

6.1. THE OPERATOR P VERSUS Δ

Let (V, E, c) be a network with vertices V, edges E, and conductance function $c : E \to \mathbb{R}_+ \cup \{0\}$. Setting

$$\begin{aligned}\widetilde{c}(x) &:= \sum_{y\sim x} c_{xy},\ p_{xy} := \frac{c_{xy}}{\widetilde{c}(x)},\ \text{and}\\ (\Delta u)(x) &= \sum_{y\sim x} c_{xy}\left(u(x) - u(y)\right), \\ (Pu)(x) &= \sum_{y\sim x} p_{xy} u(y),\end{aligned} \tag{6.18}$$

we have the connection

$$\Delta = \widetilde{c}\,(I - P)\,,\ \text{and} \tag{6.19}$$

$$P = I - \frac{1}{\widetilde{c}}\Delta \tag{6.20}$$

from (6.6) in Corollary 6.2.

Theorem 6.11. *Let $\mathscr{H}_E$ (depending on c) be the energy Hilbert space, and set $l^2(\widetilde{c}) = l^2(V, \widetilde{c}) =$ all functions on V with inner product*

$$\langle u_1, u_2\rangle_{l^2(\widetilde{c})} = \sum_{x\in V} \widetilde{c}(x)\overline{u_1(x)}u_2(x). \tag{6.21}$$

Then

1. *Δ is Hermitian in $\mathscr{H}_E$, but not in $l^2(\widetilde{c})$,*
2. *P is Hermitian in $l^2(\widetilde{c})$ and in $\mathscr{H}_E$.*

Proof. The first half of conclusion (1) is contained in Lemma 4.2. To show that P is also Hermitian in $\mathscr{H}_E$, use (6.20) and the following lemma applied to $f = \frac{1}{\widetilde{c}}$. □

Lemma 6.12. *Let Δ, P, and $\mathscr{H}_E$ be as above, and let f be a function on V, then*

$$\left\langle (f\Delta)\, v_x, v_y\right\rangle_{\mathscr{H}_E} = \left\langle v_x, (f\Delta)\, v_y\right\rangle_{\mathscr{H}_E}. \tag{6.22}$$

Proof. We compute as follows, using (3.11):

$$\begin{aligned}\text{LHS}_{(6.22)} &= \left\langle f(\cdot)\left(\delta_x - \delta_o\right), v_y\right\rangle_{\mathscr{H}_E}\\ &= \left(f(\cdot)\left(\delta_x - \delta_o\right)\right)(y) - \left(f(\cdot)\left(\delta_x - \delta_o\right)\right)(o)\\ &= f(y)\delta_{xy} + f(o)\end{aligned}$$

for all vertices $x, y \in V' = V\setminus\{o\}$, where o is a fixed choice of base-point in the vertex set V. The desired conclusion (6.22) follows. □

Proof of Theorem 6.11, part (2). We show that

$$\langle (Pu_1), u_2\rangle_{l^2(\widetilde{c})} = \langle u_1, (Pu_2)\rangle_{l^2(\widetilde{c})}$$

for all finitely supported functions u_1, u_2 on V.

But this follows from the assumption $\widetilde{c}(x)p_{xy} = \widetilde{c}(y)p_{yx}$ (reversible). Since

$$\sum_x \widetilde{c}(x)p_{xy} = \sum_x \widetilde{c}(y)p_{yx} = \widetilde{c}(y)\sum_x p_{yx} = \widetilde{c}(y),$$

i.e., the function $\widetilde{c}$ is left-invariant for P, viz: $\widetilde{c}P = \widetilde{c}$ viewing $P = (p_{xy})$ as a Markov matrix.

The final assertion, that Δ is not Hermitian in $l^2(\widetilde{c})$ follows from Lemma 4.2, and the fact that Δ does not commute with the multiplication operator $f = \frac{1}{\widetilde{c}}$. □

Corollary 6.13. *Let* $(Pu)(x) = \sum_{y\sim x} p_{xy}u(y)$ *be the transition operator, where*

$$p_{xy} = \frac{c_{xy}}{\widetilde{c}(x)} \tag{6.23}$$

is defined for $(xy) \in E$, c *is a fixed conductance on* (V, E), *and*

$$\widetilde{c}(x) = \sum_{y\sim x} c_{xy}. \tag{6.24}$$

Then P *is selfadjoint and contractive in* $l^2(V, \widetilde{c})$.

Proof. We proved in Lemma 4.2 (3) that $\langle u, \Delta u\rangle_{l^2} \geq 0$ holds for all $u \in l^2$ where $\langle\cdot,\cdot\rangle_{l^2}$ referes to the un-weighted l^2-inner product. The connection between the two inner products is as follows: $\langle u, \widetilde{c}u\rangle_{l^2} = \|u\|^2_{l^2(\widetilde{c})}$ which yields the following:

Using (6.23) and (6.24), we get

$$Pu = u - \frac{1}{\widetilde{c}}\Delta u, \tag{6.25}$$

so $\widetilde{c}Pu = \widetilde{c}u - \Delta u$, and as a point-wise identity on V. Hence

$$\begin{aligned}\langle u, Pu\rangle_{l^2(\widetilde{c})} &= \langle u, \widetilde{c}u - \Delta u\rangle_{l^2} \\ &= \|u\|^2_{l^2(\widetilde{c})} - \langle u, \Delta u\rangle_{l^2} \\ &\leq \|u\|^2_{l^2(\widetilde{c})} \text{ (by Lemma 4.2 (3))}\end{aligned}$$

holds for all $u \in l^2(\widetilde{c})$.

Since we also proved that P (see eq. (6.25)) is $l^2(\widetilde{c})$-Hermitian, we conclude that it is contractive and selfadjoint in the Hilbert space $l^2(\widetilde{c})$, as claimed. □

Lemma 6.14. *Let* (V, E, c, p), Δ, *and* P *be as above. Then a function* u *on* V *satisfies* $\Delta u = 0$ *if and only if* $Pu = u$.

Proof. Immediate from $\Delta u = \tilde{c}\,(u - Pu)$ (eq. (6.19)). □

Corollary 6.15. *Let* (V, E, c) *be as above, and set*

$$p_{xy} := \frac{c_{xy}}{\widetilde{c}(x)}, \quad (xy) \in E, \tag{6.26}$$

where $\widetilde{c}(x) := \sum_{y \sim x} c_{xy}$. *Fix* $o \in V$ *and consider the dipoles* $(v_x)_{x \in V'}$, $V' = V \setminus \{o\}$, *where*

$$\langle v_x, u \rangle_{\mathscr{H}_E} = u(x) - u\,(o) \text{ for all } x \in V'. \tag{6.27}$$

Setting

$$(Pu)\,(x) = \sum_{y \sim x} p_{xy} u(y), \tag{6.28}$$

we get

$$Pv_x = \sum_{y \sim x} p_{xy} v_y, \tag{6.29}$$

and the following implication holds for the dense subspace $\mathscr{D}_E$ *in* $\mathscr{H}_E$*:*

$$u \in \mathscr{D}_E \Longrightarrow Pu \in \mathscr{D}_E, \tag{6.30}$$

where

$$\mathscr{D}_E := \left\{ \sum_x \xi_x v_x \;\middle|\; \xi_x \in \mathbb{C}, \text{ finitely supported on } V', \text{ such that } \sum_x \xi_x = 0 \right\}. \tag{6.31}$$

Proof. A direct computation shows that (6.29) must hold as an identity on functions on V, up to an additive constant.

Our assertion is that working in the Hilbert space $\mathscr{H}_E$ implies that the additive constant is zero. This amounts to verification of the implication (6.30), i.e., that

$$\sum_x \xi_x = 0 \Longrightarrow \sum_y \left(\sum_x \xi_x p_{xy} \right) = 0 \tag{6.32}$$

for all finitely supported functions. But we have

$$\sum_y \left(\sum_x \xi_x p_{xy} \right) = \sum_x \xi_x \left(\sum_y p_{xy} \right) = \sum_x \xi_x = 0$$

which is the desired assertion (6.32). Hence (6.30) follows. □

7. EXAMPLES

The purpose of the first example is multi-fold.

First, by picking an infinite arithmetic progression of points on the line as vertex set V, and nearest neighbors, an assignment of conductance simply amounts to a function on the edges $(n, n+1)$, and we get non-trivial models where explicit formulas are possible and transparent. For example, we can write down the dipoles v_{xy} as functions on V, and the corresponding resistance metric; see the formulas relating to Figure 7.1. Among all the conductance functions we characterize the cases of reversible Markov models where the left/right transition probabilities are the same for all vertex points. In Section 7.2 (the binomial model) we accomplish the same characterization of the cases of reversible Markov models where the left/right transition probabilities are the same for all vertex points, but now every vertex in the binary tree has three nearest neighbors.

In Section 7.4 we give a finite graph (V, E, c) as a triangular configuration, conductance c defined on the edges of the triangle, and we find the Parseval frame (thereby illustrating Theorem 3.8).

The examples below illustrate the following: When a graph network (V, E, c) is infinite, then the dipoles v_{xy} as functions on V will not lie in the Hilbert space $l^2(V)$. Hence another justification for the energy Hilbert space $\mathscr{H}_E$.

7.1. $V = \{0\} \cup \mathbb{Z}_+$

Consider $G = (V, E, c)$, where $V = \{0\} \cup \mathbb{Z}_+$. Every sequence $a_1, a_2, \ldots$ in $\mathbb{R}_+$ defines a conductance $c_{n-1,n} := a_n$, $n \in \mathbb{Z}_+$, i.e.,

$$0 \underset{a_1}{\longleftrightarrow} 1 \underset{a_2}{\longleftrightarrow} 2 \underset{a_3}{\longleftrightarrow} 3 \qquad \cdots \qquad n \underset{a_{n+1}}{\longleftrightarrow} n+1 \;\cdots$$

The dipole vectors v_{xy} (for $x, y \in \mathbb{N}$) are given by

$$v_{xy}(z) = \begin{cases} 0 & \text{if } z \le x, \\ -\sum_{k=x+1}^{z} \frac{1}{a_k} & \text{if } x < z < y, \\ -\sum_{k=x+1}^{y} \frac{1}{a_k} & \text{if } z \ge y. \end{cases}$$

See Figure 7.1.

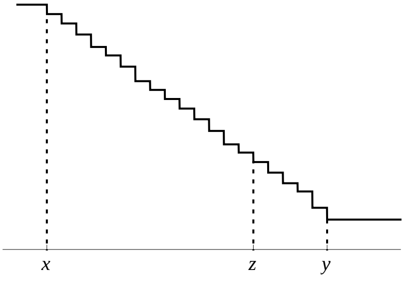


Fig. 7.1. The dipole v_{xy}

It follows from Lemma 3.6 that the resistance metric *dist* $(= d_c = d_a)$ is as follows:

$$dist\,(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \underbrace{\frac{1}{a_{x+1}} + \cdots + \frac{1}{a_y}}_{\sum_{x<k\leq y}\frac{1}{a_k}} & \text{if } x < y. \end{cases}$$

Note that $V = \mathbb{Z}_+ \cup \{0\}$ with the resistance metric described above yields a *bounded* metric space if and only if

$$\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty. \tag{7.1}$$

The corresponding graph Laplacian has the following matrix representation:

$$\begin{bmatrix} a_1 & -a_1 & & & & & & \\ -a_1 & a_1+a_2 & -a_2 & & & & 0 & \\ & -a_2 & a_2+a_3 & -a_3 & & & & \\ & & -a_3 & a_3+a_4 & \ddots & & & \\ & & & \ddots & \ddots & -a_n & & \\ & & & & -a_n & a_n+a_{n+1} & -a_{n+1} & \\ & 0 & & & & -a_{n+1} & \ddots & \ddots \\ & & & & & & \ddots & \ddots \\ & & & & & & \ddots & \ddots \end{bmatrix}, \tag{7.2}$$

that is,

$$\begin{cases} (\Delta u)_0 & = a_1\,(u_0 - u_1)\,, \\ (\Delta u)_n & = a_n\,(u_n - u_{n-1}) + a_{n+1}\,(u_n - u_{n+1})\,, \\ & = (a_n + a_{n+1})\,u_n - a_n u_{n-1} - a_{n+1}u_{n+1} \text{ for all } n \in \mathbb{Z}_+. \end{cases} \tag{7.3}$$

Lemma 7.1. *Let* $G = (V, c, E)$ *be as above, where* $a_n := c_{n-1,n}$, $n \in \mathbb{Z}_+$. *Then* $u \in \mathscr{H}_E$ *is the solution to* $\Delta u = -u$ *(i.e.,* u *is a defect vector of* Δ*) if and only if* u *satisfies the following equation:*

$$\sum_{n=1}^{\infty} a_n \left\langle v_{n-1,n}, u\right\rangle_{\mathscr{H}_E} \left(\delta_{n-1}(s) - \delta_n(s) + v_{n-1,n}(s)\right) = 0 \quad \textit{for all} \quad s \in \mathbb{Z}_+, \tag{7.4}$$

where

$$\|u\|^2_{\mathscr{H}_E} = \sum_{n=1}^{\infty} a_n \left|\left\langle v_{n-1,n}, u\right\rangle_{\mathscr{H}_E}\right|^2 < \infty. \tag{7.5}$$

Proof. By Theorem 3.8, the set $\left\{\sqrt{a_n}v_{n-1,n}\right\}_{n=1}^{\infty}$ forms a Parseval frame in $\mathscr{H}_E$. In fact, the dipole vectors are

$$v_{n-1,n}(s) = \begin{cases} 0 & s \leq n-1 \\ -\frac{1}{a_n} & s \geq n \end{cases}, \quad n = 1, 2, \dots, \tag{7.6}$$

and so $\left\{\sqrt{a_n}v_{n-1,n}\right\}_{n=1}^{\infty}$ forms an ONB in $\mathscr{H}_E$, and $u \in \mathscr{H}_E$ has the representation

$$u = \sum_{n=1}^{\infty} a_n \left\langle v_{n-1,n}, u\right\rangle_{\mathscr{H}_E} v_{n-1,n}$$

(see (3.4)). Therefore, $\Delta u = -u$ if and only if

$$\sum_{n=1}^{\infty} a_n \left\langle v_{n-1,n}, u\right\rangle_{\mathscr{H}_E} \left(\delta_{n-1}(s) - \delta_n(s)\right) = -\sum_{n=1}^{\infty} a_n \left\langle v_{n-1,n}, u\right\rangle_{\mathscr{H}_E} v_{n-1,n}(s)$$

for all $s \in \mathbb{Z}_+$, which is the assertion. □

Conjecture 7.2. *Consider Δ as above as an operator in $\mathscr{H}_E$ (depending on $c_{n,n-1} = a_n$). Then Δ is essentially selfadjoint (in $\mathscr{H}_E$) if and only if $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$. If (7.1) holds, the indices are $(1,1)$.*

Remark 7.3. Below we compute the deficiency space in an example with index values $(1,1)$.

Lemma 7.4. *Let $(V, E, c = \{a_n\})$ be as above. Let $Q > 1$ and set $a_n := Q^n$, $n \in \mathbb{Z}_+$. Then Δ has deficiency indices $(1,1)$.*

Proof. Suppose $\Delta u = -u$, $u \in \mathscr{H}_E$. Then,

$$\begin{aligned} -u_1 &= Q(u_1 - u_0) + Q^2(u_1 - u_2) \Longleftrightarrow u_2 = \left(\frac{1}{Q^2} + \frac{1+Q}{Q}\right) u_1 - \frac{1}{Q} u_0, \\ -u_2 &= Q^2(u_2 - u_1) + Q^3(u_2 - u_3) \Longleftrightarrow u_3 = \left(\frac{1}{Q^3} + \frac{1+Q}{Q}\right) u_2 - \frac{1}{Q} u_1, \end{aligned}$$

and by induction,

$$u_{n+1} = \left(\frac{1}{Q^{n+1}} + \frac{1+Q}{Q}\right) u_n - \frac{1}{Q} u_{n-1}, \quad n \in \mathbb{Z}_+,$$

i.e., u is determined by the following matrix equation:

$$\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = \begin{bmatrix} \frac{1}{Q^{n+1}} + \frac{1+Q}{Q} & -\frac{1}{Q} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix}.$$

The eigenvalues of the coefficient matrix are

$$\lambda_{\pm} = \frac{1}{2}\left(\frac{1}{Q^{n+1}} + \frac{1+Q}{Q} \pm \sqrt{\left(\frac{1}{Q^{n+1}} + \frac{1+Q}{Q}\right)^2 - \frac{4}{Q}}\right)$$

$$\sim \frac{1}{2}\left(\frac{1+Q}{Q} \pm \left(\frac{Q-1}{Q}\right)\right) = \begin{cases} 1 \\ \frac{1}{Q} \end{cases} \text{as } n \to \infty.$$

Equivalently, as $n \to \infty$, we have

$$u_{n+1} \sim \left(\frac{1+Q}{Q}\right) u_n - \frac{1}{Q} u_{n-1} = \left(1 + \frac{1}{Q}\right) u_n - \frac{1}{Q} u_{n-1},$$

and so

$$u_{n+1} - u_n \sim \frac{1}{Q}\left(u_n - u_{n-1}\right).$$

Therefore, for the tail-summation, we have

$$\sum_n Q^n \left(u_{n+1} - u_n\right)^2 = \text{const} \sum_n \frac{(Q-1)^2}{Q^{n+2}} < \infty$$

which implies $\|u\|_{\mathscr{H}_E} < \infty$. □

Next, we give a random walk interpretation of Lemma 7.4. See Remark 2.4 and Figure 2.1.

Remark 7.5 (Harmonic functions in $\mathscr{H}_E$)**.** Note that in Example 7.1 (Lemma 7.4), the space of harmonic functions in $\mathscr{H}_E$ is one-dimensional. In fact if $Q > 1$ is fixed, then

$$\left\{u \in \mathscr{H}_E \;\middle|\; \Delta u = 0\right\}$$

is spanned by $u = (u_n)_{n=0}^{\infty}$, $u_n = \frac{1}{Q^n}$, $n \in \mathbb{N}$; and of course $\|1/Q^n\|^2_{\mathscr{H}_E} < \infty$.

Proof. This is immediate from Lemma 6.14. □

Remark 7.6. For the domain of the Friedrichs extension Δ_{Fri}, we have

$$dom(\Delta_{Fri}) = \left\{f \in \mathscr{H}_E \mid \left(f(x) - f(x+1)\right) Q^x \in l^2\left(\mathbb{Z}_+\right)\right\}, \tag{7.7}$$

i.e.,

$$dom(\Delta_{Fri}) = \left\{f \in \mathscr{H}_E \;\middle|\; \sum_{x=0}^{\infty} \left|f(x) - f(x+1)\right|^2 Q^{2x} < \infty\right\}.$$

Proof. By Theorem 3.8, we have the following representation, valid for all $f \in \mathscr{H}_E$:

$$\begin{aligned} f &= \sum_x \left\langle f, Q^{\frac{x}{2}} v_{(x,x+1)} \right\rangle_{\mathscr{H}_E} Q^{\frac{x}{2}} v_{(x,x+1)} \\ &= \sum_x \left(f(x) - f(x+1)\right) Q^x v_{(x,x+1)}, \end{aligned}$$

and

$$\langle f, \Delta f \rangle_{\mathscr{H}_E} = \sum_x |f(x) - f(x+1)|^2 Q^{2x}.$$

The desired conclusion (7.7) now follows from Theorem 6.9 above and the characterization of Δ_{Fri} (see e.g. [1, 10]). $\square$

Definition 7.7. Let $G = (V, E, c)$ be a connected graph. The set of transition probabilities (p_{xy}) is said to be reversible if there exists $c : V \to \mathbb{R}_+$ such that

$$c(x)p_{xy} = c(y)p_{yx}, \tag{7.8}$$

and then

$$c_{xy} := c(x)p_{xy} \tag{7.9}$$

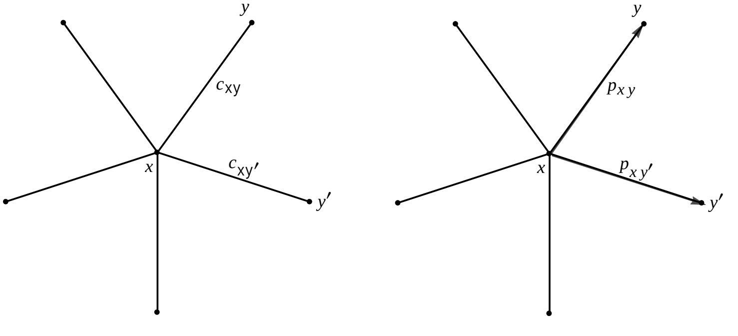
is a system of conductance. Conversely, for a system of conductance (c_{xy}) we set

$$c(x) := \sum_{y \sim x} c_{xy} \tag{7.10}$$

and

$$p_{xy} := \frac{c_{xy}}{c(x)}, \tag{7.11}$$

and so (p_{xy}) is a set of transition probabilities. See Figure 7.2 below.



c_{xy}, $y \sim x$ transition probabilities

Fig. 7.2. Neighbors of x

Recall the graph Laplacian in (7.3) can be written as

$$(\Delta u)_n = c(n)(u_n - p_-(n)u_{n-1} - p_+(n)u_{n+1}) \quad \text{for all} \quad n \in \mathbb{Z}_+, \tag{7.12}$$

where

$$c(n) := a_n + a_{n+1} \tag{7.13}$$

and

$$p_-(n) := \frac{a_n}{c(n)}, \quad p_+(n) := \frac{a_{n+1}}{c(n)} \tag{7.14}$$

are the left/right transition probabilities, as shown in Figure 7.3.

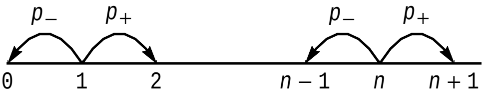


Fig. 7.3. The transition probabilities p_+, p_-, in the case of constant transition probabilities, i.e., $p_+(n) = p_+$, and $p_-(n) = p_-$ for all $n \in \mathbb{Z}_+$

In the case $a_n = Q^n$, $Q > 1$, as in Lemma 7.4, we have

$$c(n) := Q^n + Q^{n+1}, \tag{7.15}$$

and

$$p_+ := p_+(n) = \frac{Q^{n+1}}{Q^n + Q^{n+1}} = \frac{Q}{1+Q}, \tag{7.16}$$

$$p_- := p_-(n) = \frac{Q^n}{Q^n + Q^{n+1}} = \frac{1}{1+Q}. \tag{7.17}$$

For all $n \in \mathbb{Z}_+ \cup \{0\}$, set

$$(Pu)_n := p_- u_{n-1} + p_+ u_{n+1}. \tag{7.18}$$

Note $(Pu)_0 = u_1$. By (7.12), we have

$$\Delta = c(1 - P). \tag{7.19}$$

In particular, $p_+ > \frac{1}{2}$, i.e., a random walker has probability $> \frac{1}{2}$ of moving to the right. It follows that

$$\underbrace{\text{travel time}\,(n,\infty)}_{=\text{ dist to }\infty} < \infty,$$

and so Δ is not essentially selfadjoint, i.e., indices $(1,1)$.

Lemma 7.8. *Let $(V, E, \Delta(=\Delta_c))$ be as above, where the conductance c is given by $c_{n-1,n} = Q^n$, $n \in \mathbb{Z}_+$, $Q > 1$ (see Lemma 7.4). For all $\lambda > 0$, there exists $f_\lambda \in \mathscr{H}_E$ satisfying $\Delta f_\lambda = \lambda f_\lambda$.*

Proof. By (7.19), we have $\Delta f_\lambda = \lambda f_\lambda \Longleftrightarrow P f_\lambda = \left(1 - \frac{\lambda}{c}\right) f_\lambda$, i.e.,

$$\frac{1}{1+Q} f_\lambda(n-1) + \frac{Q}{1+Q} f_\lambda(n+1) = \left(1 - \frac{\lambda}{Q^{n-1}(1+Q)}\right) f_\lambda(n)$$

and so

$$f_\lambda(n+1) = \left(\frac{1+Q}{Q} - \frac{\lambda}{Q^n}\right) f_\lambda(n) - \frac{1}{Q} f_\lambda(n-1). \tag{7.20}$$

This corresponds to the following matrix equation:

$$\begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix} = \begin{bmatrix} \frac{1+Q}{Q} - \frac{\lambda}{Q^n} & -\frac{1}{Q} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}$$
$$\sim \begin{bmatrix} \frac{1+Q}{Q} & -\frac{1}{Q} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}, \text{ as } n \to \infty.$$

The eigenvalues of the coefficient matrix are given by

$$\lambda_{\pm} \sim \frac{1}{2}\left(\frac{1+Q}{Q} \pm \left(\frac{Q-1}{Q}\right)\right) = \begin{cases} 1 \\ \frac{1}{Q} \end{cases} \quad \text{as } n \to \infty.$$

That is, as $n \to \infty$,

$$f_\lambda(n+1) \sim \left(\frac{1+Q}{Q}\right) f_\lambda(n) - \frac{1}{Q} f_\lambda(n-1),$$

i.e.,

$$f_\lambda(n+1) \sim \frac{1}{Q} f_\lambda(n); \tag{7.21}$$

and so the tail summation of $\|f_\lambda\|^2_{\mathscr{H}_E}$ is finite. (See the proof of Lemma 7.4.) We conclude that $f_\lambda \in \mathscr{H}_E$. □

Corollary 7.9. *Let (V, E, Δ) be as in the lemma. The Friedrichs extension Δ_{Fri} has continuous spectrum $[0, \infty)$.*

Proof. Fix $\lambda \geq 0$. We prove that if $\Delta f_\lambda = \lambda f_\lambda$, $f \in \mathscr{H}_E$, then $f_\lambda \notin dom(\Delta_{Fri})$.

Note for $\lambda = 0$, f_0 is harmonic, and so $f_0 = k\left(\frac{1}{Q^n}\right)_{n=0}^{\infty}$ for some constant $k \neq 0$. See Remark 7.5. It follows from (7.7) that $f_0 \notin dom(\Delta_{Fri})$.

The argument for $\lambda > 0$ is similar. Since as $n \to \infty$, $f_\lambda(n) \sim \frac{1}{Q^n}$ (eq. (7.21)), so by (7.7) again, $f_\lambda \notin dom(\Delta_{Fri})$.

However, if $\lambda_0 < \lambda_1$ in $[0, \infty)$ then

$$\int_{\lambda_0}^{\lambda_1} f_\lambda(\cdot)\, d\lambda \in dom(\Delta_{Fri}) \tag{7.22}$$

and so every f_λ, $\lambda \in [0, \infty)$, is a generalized eigenfunction, i.e., the spectrum of Δ_{Fri} is purely continuous with Lebesgue measure, and multiplicity one.

The verification of (7.22) follows from eq. (7.20), i.e.,

$$f_\lambda(n+1) = \left(\frac{1+Q}{Q} - \frac{\lambda}{Q^n}\right) f_\lambda(n) - \frac{1}{Q} f_\lambda(n-1). \tag{7.23}$$

Set

$$F_{[\lambda_0, \lambda_1]} := \int_{\lambda_0}^{\lambda_1} f_\lambda(\cdot)\, d\lambda. \tag{7.24}$$

Then by (7.23) and (7.24),

$$F_{[\lambda_0,\lambda_1]}(n+1) = \frac{1+Q}{Q} F_{[\lambda_0,\lambda_1]}(n) - \frac{1}{Q^n} \int_{\lambda_0}^{\lambda_1} \lambda f_\lambda(n)\, d\lambda - \frac{1}{Q} F_{[\lambda_0,\lambda_1]}(n-1)$$

and $\int_{\lambda_0}^{\lambda_1} \lambda f_\lambda d\lambda$ is computed using integration by parts. □

Remark 7.10 (Krein extension)**.** Set

$$\begin{aligned} dom(\Delta^{Harm}) &:= dom(\Delta) + \text{Harmonic functions}, \\ \Delta^{Harm} &:= \Delta^* \big|_{dom(\Delta^{Harm})}. \end{aligned}$$

Then $\Delta^{Harm} \supset \Delta$ is a well-defined selfadjoint extension of Δ. It is semibounded, since

$$\left\langle \varphi + h, \Delta^{Harm}(\varphi + h) \right\rangle_{\mathscr{H}_E} = \langle \varphi + h, \Delta\varphi \rangle_{\mathscr{H}_E} = \langle \varphi, \Delta\varphi \rangle_{\mathscr{H}_E} \geq 0$$

for all $\varphi \in dom(\Delta)$, and h harmonic. In fact Δ^{Harm} is the Krein extension.

7.2. A REVERSIBLE WALK ON THE BINARY TREE (THE BINOMIAL MODEL)

Consider the binary tree as a set V of vertices. To get a graph $G = (V, E)$, take for edges E the nearest neighbor lines as follows:

$$V := \{o = \phi, (x_1 x_2 \cdots x_n), x_i \in \{0,1\}, 1 \leq i \leq n\}, \tag{7.25}$$

and

$$\begin{aligned} &E(\phi) = \{0,1\}, \\ &E((x_1 x_2 \cdots x_n)) = \Big\{ \underbrace{(x_1 \cdots x_{n-1})}_{=x^*}, \underbrace{(x0), (x1)}_{\text{extended words}} \Big\}, \text{ see Figure 7.4.} \end{aligned}$$

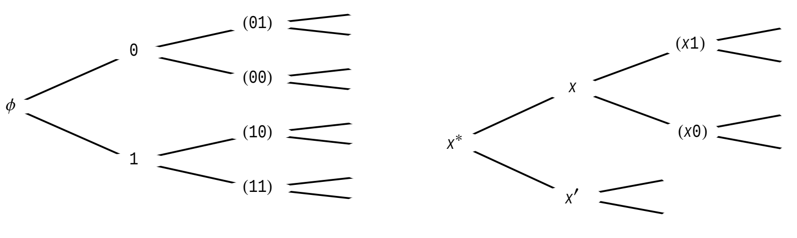


Fig. 7.4. The binary tree model, three nearest neighbors

Now fix transition probabilities at $o = \phi$, and at the vertices $V' = V \backslash \{\phi\}$ as follows:

$$\begin{cases} \text{Prob}\,(x \to (x0)) & = p_0, \\ \text{Prob}\,(x \to (x1)) & = p_1, \\ \text{Prob}\,(x \to (x^*)) & = p_- \end{cases} \tag{7.26}$$

with $0 < p_0 < 1$ (see Figure 7.4) such that $p_0 + p_1 + p_- = 1$, see Figure 7.5.

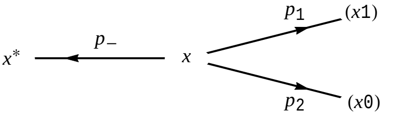


Fig. 7.5. Transition probabilities at a vertex $x \in V'$. The reversible case with three nearest neighbors

For $x \in V'$, set

$$F_0(x) = \#\{i \mid x_i = 0\}, \quad F_1(x) = \#\{i \mid x_i = 1\}, \tag{7.27}$$

and

$$|x| := x_1 + x_2 + \cdots + x_n \text{ so that } F_0(x) + F_1(x) = |x| \text{ for all } x \in V'. \tag{7.28}$$

Further, define the function $c : V \to \mathbb{R}_+$ as follows:

$$c(x) = \frac{p_0^{F_0(x)} p_1^{F_1(x)}}{p_-^{|x|}} \text{ for all } x \in V'. \tag{7.29}$$

Lemma 7.11. *With the transition probabilities defined in* (7.26)*, it follows that the corresponding walk on* V *is reversible via the function* $c : V \to \mathbb{R}_+$ *defined in* (7.29)*, i.e., we have the following identity for any edge* (xy) *in* $G = (V, E)$*:*

$$c(x) Prob\,(x \to y) = c(y) Prob\,(y \to x). \tag{7.30}$$

Proof. This follows from a direct inspection of the cases (see also Figures 7.4 and 7.5). □

For $(xy) \in E$, using (7.29)–(7.30), set $c_{xy} := c(x)\text{Prob}\,(x \to y)$, and

$$(\Delta u)(x) := \sum_{y \sim x} c_{xy} (u(x) - u(y)). \tag{7.31}$$

Corollary 7.12. *If* $\min(p_i) > p_-$*, then* Δ *in* (7.31) *has deficiency indices* $(1, 1)$*, i.e., the non-zero solution* u *to* $-u = \Delta u$ *is in* $\mathscr{H}_E$*; i.e.,* $0 < \|u\|_{\mathscr{H}_E} < \infty$.

Proof. The analysis here is analogous to the one above in Example 7.1, and so we omit the details here. □

Remark 7.13. For literature on the binomial model and its applications, see for example [7, 26, 34].

7.3. A 2D LATTICE

Let $G = (V, E, c)$ the graph in Figure 7.6. Fix $Q, \overline{Q} > 1$, and set the conductance as

$$a_n := c_{n-1,n} = Q^n,$$
$$\overline{a}_n := \overline{c}_{n-1,n} = \overline{Q}^n$$

for all $n \in \mathbb{Z}_+$. We get the set of transition probabilities as follows:

$$\begin{cases} p_+ = \frac{Q}{1+Q}, \ \overline{p}_+ = \frac{\overline{Q}}{1+\overline{Q}}, \\ p_- = \frac{1}{1+Q}, \ \overline{p}_- = \frac{1}{1+\overline{Q}}, \\ p_d = \text{vertical transitions.} \end{cases}$$

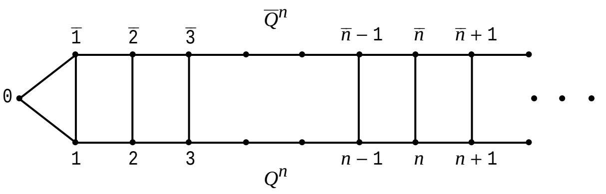


Fig. 7.6. Conductance: $c_{n-1,n} = Q^n$ and $\overline{c}_{n-1,n} = \overline{Q}^n$

7.4. A PARSEVAL FRAME THAT IS NOT AN ONB IN $\mathscr{H}_E$

Let c_{01}, c_{02}, c_{12} be positive constants, and assign conductances on the three edges (see Figure 7.7) in the triangle network.

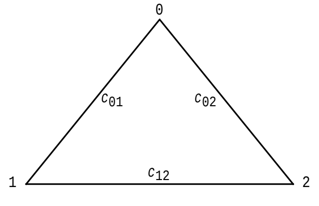


Fig. 7.7. The set $\{v_{xy} : (xy) \in E\}$ is not orthogonal

We show that $w_{ij} = \sqrt{e_{ij}} v_{ij}$, $i < j$, in the cyclic order is a Parseval frame but not an ONB in $\mathscr{H}_E$.

Note the corresponding Laplacian $\Delta \, (= \Delta_c)$ has the following matrix representation:

$$M := \begin{bmatrix} c\,(0) & -c_{01} & -c_{02} \\ -c_{01} & c\,(1) & -c_{12} \\ -c_{02} & -c_{12} & c\,(2) \end{bmatrix}. \tag{7.32}$$

The dipoles $\left\{v_{xy} : (xy) \in E^{(ori)}\right\}$ as 3-D vectors are the solutions to the equation

$$\Delta v_{xy} = \delta_x - \delta_y.$$

Hence,

$$\begin{aligned} Mv_{01} &= \begin{bmatrix}1 & -1 & 0\end{bmatrix}^{tr}, \\ Mv_{02} &= \begin{bmatrix}1 & 0 & -1\end{bmatrix}^{tr}, \\ Mv_{12} &= \begin{bmatrix}0 & 1 & -1\end{bmatrix}^{tr}. \end{aligned}$$

We check directly eq. (3.17) holds, and so $\left\{v_{xy} : (xy) \in E^{(ori)}\right\}$ is not orthonormal. For example, we have

$$v_{01} = \left[\frac{c_{12}}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}}, -\frac{c_{02}}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}}, 0\right]^{tr}$$

and

$$c_{01}\left(v_{01}(0) - v_{01}(1)\right) = \frac{c_{01}\left(c_{12} + c_{02}\right)}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}} < 1;$$

see (3.17). Hence the voltage drop across (01) is strictly smaller then the (01) resistance, i.e.,

$$v_{01}(0) - v_{01}(1) < \frac{1}{c_{01}} = \mathrm{Res}_{(01)}.$$

In this example, the Parseval frame from Lemma 3.2 is

$$\begin{aligned} w_{01} &= \sqrt{c_{01}}v_{01} = \left[\frac{\sqrt{c_{01}}\,c_{12}}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}}, -\frac{\sqrt{c_{01}}\,c_{02}}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}}, 0\right]^{tr}, \\ w_{12} &= \sqrt{c_{12}}v_{12} = \left[0, \frac{\sqrt{c_{12}}\,c_{02}}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}}, -\frac{\sqrt{c_{12}}\,c_{01}}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}}\right]^{tr}, \\ w_{20} &= \sqrt{c_{20}}v_{20} = \left[\frac{-\sqrt{c_{20}}\,c_{12}}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}}, 0, \frac{\sqrt{c_{20}}\,c_{01}}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}}\right]^{tr}. \end{aligned}$$

Remark 7.14. The dipole v_{xy} is unique in $\mathscr{H}_E$ as an equivalence class, not a function on V. Note $\ker M$ = harmonic functions = constant (see (7.32)), and so $v_{xy} + \text{const} = v_{xy}$ in $\mathscr{H}_E$. Thus, the above frame vectors have non-unique representations as functions on V. Also see Remark 3.5.

Introduce the vector system of conductance as follows:

$$\begin{aligned} \widetilde{c}_1 &= \left(c_{01}, c_{01}, c_{02}\right), \\ \widetilde{c}_2 &= \left(c_{02}, c_{12}, c_{12}\right), \\ \widetilde{c}_0 &= \left(c_{01}, c_{02}, c_{12}\right). \end{aligned}$$

We arrive at the following formula for the spectrum of the system (V, E, c, Δ), where (V, E) is the triangle in Figure 7.7.

$$\lambda_1 = 0,$$
$$\lambda_2 = \mathrm{tr}\widetilde{c}_0 - \sqrt{\|\widetilde{c}_0\|^2 - \langle \widetilde{c}_1, \widetilde{c}_2 \rangle},$$
$$\lambda_3 = \mathrm{tr}\widetilde{c}_0 + \sqrt{\|\widetilde{c}_0\|^2 - \langle \widetilde{c}_1, \widetilde{c}_2 \rangle},$$

and so the spectral gap

$$\lambda_3 - \lambda_2 = 2\sqrt{\|\widetilde{c}_0\|^2 - \langle \widetilde{c}_1, \widetilde{c}_2 \rangle}$$

is a function of the coherence for $\widetilde{c}_1$ and $\widetilde{c}_2$.

8. OPEN PROBLEMS

Let $G = (V, E, c)$ be a graph, with vertices V, edges E, and conductance c; V is countable infinite. The graph-Laplacian Δ is essentially selfadjoint as an $l^2(V)$-operator, but not as a $\mathscr{H}_E$-operator. It is known that Δ, as a $\mathscr{H}_E$-operator, has deficiency indices (m, m), $m > 0$, when the conductance function c is of exponential growth.

1. Compare the deficiency indices of Δ (in $\mathscr{H}_E$) for various cases: $V = \mathbb{Z}^d$, $d = 1, 2, 3, \ldots$; nearest neighbor. If $d = 1$ must the indices then be (m, m) with $m = 0$ or 1? Are there examples with $m = 2$ in any of the classes of examples?
2. What is the spectral representation of the Friedrichs extension Δ_{Fri} as a $\mathscr{H}_E$-operator? Find the $\mathscr{H}_E$ projection valued measure.
3. Find the spectrum of the transition operator P, considered as an operator in $l^2(V, \widetilde{c})$. When is there point-spectrum? If so, what are the two top eigenvalues? What is the connection between this spectrum (the spectrum of P), and the spectrum of Δ_{Fri} in $\mathscr{H}_E$, and of Δ in $l^2(V)$?
4. It is not known whether or not the transition operator P is bounded as an operator of $\mathscr{H}_E$ into itself.

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