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ON POTENTIAL KERNELS ASSOCIATED WITH RANDOM DYNAMICAL SYSTEMS

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Abstract. Let (θ,φ) be a continuous random dynamical system defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and taking values on a locally compact Hausdorff space E. The associated potential kernel V is given by

$$Vf(\omega,x)=\int\limits_0^\infty f(\theta_t\omega,\varphi(t,\omega)x)dt,\quad \omega\in\Omega, x\in E.$$

In this paper, we prove the equivalence of the following statements:

1. The potential kernel of (θ,φ) is proper, i.e. Vf is x-continuous for each bounded, x-continuous function f with uniformly random compact support.
2. (θ,φ) has a global Lyapunov function, i.e. a function $L:\Omega\times E\to(0,\infty)$ which is x-continuous and $L(\theta_t\omega,\varphi(t,\omega)x)\downarrow 0$ as $t\uparrow\infty$.

In particular, we provide a constructive method for global Lyapunov functions for gradient-like random dynamical systems. This result generalizes an analogous theorem known for deterministic dynamical systems.

Keywords: dynamical system, random dynamical system, random differential equation, stochastic differential equation, potential kernel, domination principle, Lyapunov function.

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1. INTRODUCTION

The notion of *Lyapunov function* is a fundamental tool in the *second Lyapunov's method* which consists of studying qualitative, asymptotic or long-term behavior of orbits of dynamical systems. Moreover, a construction of such functions seems to be an important (and difficult) problem for applications.

We propose in this paper a contribution to this aim in the framework of random dynamical systems, inspired from probabilistic potential theory.

A continuous random dynamical system (RDS) with state space E, is defined as a pair (θ, φ) where $\theta : \mathbb{R} \times \Omega \to \Omega$ is a metric dynamical system (DS) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\varphi : \mathbb{R} \times \Omega \times E \to E$ is a continuous cocycle over θ. Standard examples of RDS are solutions of random or stochastic differential equations (cf. [1] and Appendices B and C for more details).

For a given RDS (θ, φ), we associate the potential kernel V defined (as for the deterministic case) by

$$Vf(\omega, x) = \int_0^\infty f(\theta_t\omega, \varphi(t, \omega)x)dt, \quad \omega \in \Omega, x \in E,$$

for any positive measurable function $f : \Omega \times E \to [0, +\infty]$. According to the general theory of semigroups of kernels, V is in fact the potential kernel of the measurable semigroups (K_t), where

$$K_t f(\omega, x) = f(\theta_t\omega, \varphi(t, \omega)x).$$

In this paper, we prove essentially the ensuing result.

Theorem 1.1. *The following assertions are equivalent:*

1. *The potential kernel V of the RDS (θ, φ) is proper, i.e. $(\omega, x) \mapsto Vf(\omega, x)$ is x-continuous for each function $f : \Omega \times E \to [0, \infty)$ which is bounded, x-continuous and $\{f > 0\} \subset \Omega \times K$ for some compact $K \subset E$.*
2. *(θ, φ) has a global Lyapunov function L, i.e. $L : \Omega \times E \to (0, \infty)$ is measurable, x-continuous and $L(\theta_t\omega, \varphi(t, \omega)x) \downarrow 0$ as $t \uparrow \infty$ for all $\omega \in \Omega$ and $x \in E$.*

This equivalence establishes a connection between *Lyapunov functions* and a fundamental concept in the framework of probabilistic potential theory, namely the *potential kernel*. In particular, we provide a constructive method for Lyapunov functions and thus, we give an application of the so-called *domination principle*, another important notion in potential theory.

This theorem generalizes a result established by the first author for deterministic dynamical systems (cf. Appendix A). Moreover, as for the deterministic case, this characterization may be applied for systems theory or hybrid systems (cf. [4, 5] for more details).

For the proof, we use those of the deterministic result as some guideline for ours, although many additional difficulties arise in the random case.

The paper is organized as follows: In the next section, we collect some useful preliminaries, then we introduce the notion of the potential kernel associated with a RDS. In Section 3, we present the proof of the main result cited above. Section 4 contains some applications in the framework of random differential equations and stochastic differential equations. The last section contains some appendices about the classical result which is generalized here (Appendix A), the notion of random dynamical systems (Appendix B) and the standard examples of RDS (Appendix C).

2. PRELIMINARIES

In this paper $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. The notation a.e. (almost everywhere) is understood with respect to $\mathbb{P}$. $\mathbb{R}$ is the real line endowed with its σ-algebra $\mathcal{R}$. Moreover, E is a locally compact Hausdorff space endowed with its borel σ-algebra $\mathcal{E}$, d is a metric on E which generates $\mathcal{E}$ and $\mathcal{P}(E)$ denotes the set of all subsets of E. The product space $\Omega \times E$ is always endowed with the usual product σ-algebra $\mathcal{F} \otimes \mathcal{E}$.

If $A \subset \Omega \times E$, let $A(\omega) := \{x \in E : (\omega, x) \in A\}$ the ω-section of A. Notice that A is identified to the map $A : \Omega \to \mathcal{P}(E)$, $\omega \to A(\omega)$. A random set is a measurable set $A \in \mathcal{F} \otimes \mathcal{E}$ such that $\omega \to d(x, A(\omega))$ is measurable for any $x \in E$. A random set A is said to be closed (resp. compact) if $A(\omega)$ is a closed (resp. compact) subset of E for every $\omega \in \Omega$.

If A is a random set of $\Omega \times E$, the closure $\overline{A}$ of A is defined by the map $\overline{A} : \Omega \to \mathcal{P}(E)$, $\omega \to \overline{A(\omega)}$.

Below, we present a basic result about random sets, namely the measurable selection theorem (cf. [6]).

Lemma 2.1. *A set $D \subset \Omega \times E$ is a random closed set if and only if there exists a sequence $\{v_n : n \in \mathbb{N}\}$ of measurable maps $v_n : \Omega \to E$ such that*

$$v_n(\omega) \in D(\omega) \quad \text{and} \quad D(\omega) = \overline{\{v_n(\omega) : n \in \mathbb{N}\}}$$

for all $\omega \in \Omega$. In particular, if D is a random closed set, there exists a measurable selection, i.e., a measurable map $v : \Omega \to E$ such that $v(\omega) \in D(\omega)$ for all $\omega \in \Omega$.

Next, denote by $\mathcal{M}(\Omega \times E)$ the set of all functions $f : \Omega \times E \to [0, +\infty)$ which are measurable. For $f \in \mathcal{M}(\Omega \times E)$, let $S(f) := \overline{\{f > 0\}}$ the random support of f. We denote by $\mathcal{C}(\Omega \times E)$ the set of all functions $f \in \mathcal{M}(\Omega \times E)$ which are x-continuous, that is $x \to f(\omega, x)$ is continuous for each $\omega \in \Omega$. Moreover, let $\mathcal{C}_b(\Omega \times E)$ be the set of all x-continuous and x-bounded functions, i.e. for every $\omega \in \Omega$, there exists $m(\omega) \in (0, +\infty)$ such that $|f(\omega, x)| \leq m(\omega)$ for all $x \in E$. For $f \in \mathcal{C}_b(\Omega \times E)$ denote by $\|f(\omega, \cdot)\| = \sup_{x \in E} |f(\omega, x)|$.

We denote by $\mathcal{C}_k(\Omega \times E)$ the space of bounded functions $f \in \mathcal{C}(\Omega \times E)$ such that $S(f) \subset \Omega \times K$ for some compact subset K of E. In particular, $S(f)$ is random compact and $S(f)(\omega) \subset K$ for all $\omega \in \Omega$.

Throughout the paper, we use sometimes the notation $(\alpha \otimes u)(\omega, x) := \alpha(\omega) \cdot u(x)$ if $\alpha : \Omega \to \mathbb{R}$ and $u : E \to \mathbb{R}$.

Remark 2.2. For all definitions which depend on $\omega \in \Omega$, we may replace the quantifier "for all $\omega \in \Omega$" by "for $\mathbb{P}$-almost every (a.e.) $\omega \in \Omega$".

Lemma 2.3. *For each $f \in \mathcal{C}_b(\Omega \times E)$, $\omega \to \|f(\omega, \cdot)\|$ is a random variable.*

Proof. Let $f \in \mathcal{C}_b(\Omega \times E)$, $S(f)$ its random support and $S(f)(\omega)$ the ω-section of $S(f)$. Then, for $\omega \in \Omega$, we get

$$\begin{aligned}\|f(\omega,\cdot)\| &= \sup\{|f(\omega,x)| : x \in E\}\\ &= \sup\{|f(\omega,x)| : x \in S(f)(\omega)\}\\ &= \sup\{|f(\omega,x)| : x \in \overline{\{v_n(\omega) : n \in \mathbb{N}\}}\}\\ &= \sup\{|f(\omega,x)| : x \in \{v_n(\omega) : n \in \mathbb{N}\}\}\\ &= \sup\{|f(\omega,v_n(\omega))| : n \in \mathbb{N}\}.\end{aligned}$$

The third equality is got by the selection theorem (i.e. lemma 2.1) while the fourth is due to the continuity of $x \to f(\omega,x)$ for all $\omega \in \Omega$. Hence $\omega \to \|f(\omega,\cdot)\|$ is a random variable. □

3. POTENTIAL KERNELS ASSOCIATED WITH RDS

The definition and the elementary properties of random dynamical systems (RDS) are given in Appendix B.

Definition 3.1. Let (θ,φ) be a continuous RDS. The associated Koopman family $(K_t)_{t\geq 0}$ is defined by

$$K_t f(\omega,x) := f(\theta_t\omega, \varphi(t,\omega)x) \tag{3.1}$$

for $f \in \mathcal{M}(\Omega \times E), \omega \in \Omega$ and $x \in E$.

Proposition 3.2. *The family (K_t) is a measurable semigroup of Markovian kernels on $\Omega \times E$. Moreover, (K_t) is x-Fellerian, i.e.*

$$\lim_{t\to 0} \|K_t f(\omega,\cdot) - f(\omega,\cdot)\| = 0, \quad f \in \mathcal{C}_k(\Omega \times E). \tag{3.2}$$

Proof. It is obvious that each K_t is a Markovian kernel on $(\Omega \times E, \mathcal{F} \otimes \mathcal{E})$. Moreover, for each $f \in \mathcal{M}(\Omega \times E)$, the function $(t,\omega,x) \mapsto \varphi(t,\omega)x$ is measurable, then so is $(\omega,x) \to K_t f(\omega,x)$. On the other hand, by (B.3) and (B.5), we have $K_t f = f \circ \Phi_t$ and therefore $K_{s+t} = K_s \circ K_t$. By continuity of the $(t,x) \mapsto \varphi(t,\omega)x$, it is straightforward that $K_t f \in \mathcal{C}_b(\Omega \times E)$ whenever $f \in \mathcal{C}_b(\Omega \times E)$.

Let $f \in \mathcal{C}_k(\Omega \times E)$, K be compact of E such that $S(f) \subset \Omega \times K$ and $\omega \in \Omega$. For $a > 0$, by the uniform continuity of $(t,x) \mapsto f(\theta_t\omega, \varphi(t,\omega)x)$ on $[0,1] \times K$, there exists a constant $b \in (0,1)$ such that

$$\forall s \in [0,b],\ x \in K \ : \ |f(\theta_s\omega, \varphi(s,\omega)x) - f(\omega,x)| < a. \tag{3.3}$$

It can be easily seen that (3.3) implies (3.2) by using standard arguments. □

Definition 3.3. Let (θ,φ) be a continuous RDS. The associated potential kernel is defined by

$$Vf(\omega,x) := \int_0^\infty f(\theta_t\omega, \varphi(t,\omega)x)\, dt \tag{3.4}$$

for $f \in \mathcal{M}(\Omega \times E), \omega \in \Omega$ and $x \in E$. The potential kernel V is said to be *proper* if $V(\mathcal{C}_k(\Omega \times E)) \subset \mathcal{C}(\Omega \times E)$.

Remark 3.4. The following notions are familiar in the framework of potential theory defined by a measurable semigroup of sub-Markovian kernels. We refer the reader to [12, IX,3]. According to the general theory, V is in fact the potential kernel of the measurable semigroup (K_t). Moreover, the cone of the (K_t)-supermedian functions is defined by

$$S((K_t)) := \{u \in \mathcal{M}(\Omega \times E) : K_t u \leq u, t \geq 0\}. \tag{3.5}$$

It is obvious that $V(\mathcal{M}(\Omega \times E)) \subset S((K_t))$ and that $S((K_t))$ is stable by $\liminf$.

It is also known that each (K_t)-supermedian function u is V-dominant (cf. [12, T. 68]), that is, for all $f \in \mathcal{M}(\Omega \times E)$

$$(Vf \leq u \quad \text{on} \quad \{f > 0\}) \quad \Rightarrow \quad (Vf \leq u). \tag{3.6}$$

In fact, the domination principle (3.6) is satisfied for general sub-Markovian resolvents of kernels (R_ℓ). In this paper, we are only concerned with the resolvent defined by the measurable semigroup (K_t), i.e.

$$R_\ell := \int_0^\infty e^{-\ell t} K_t \, dt, \quad \ell \geq 0. \tag{3.7}$$

4. PROOF OF THE MAIN RESULT

For the proof of Theorem 1.1, we need some auxiliary results.

Let (θ, φ) be a continuous RDS and let V be the associated potential kernel. An immediate application of the domination principle, gives the following useful result.

Lemma 4.1. *If V is proper, then $V(\mathcal{C}_k(\Omega \times E)) \subset \mathcal{C}_b(\Omega \times E)$.*

Proof. Let $f \in \mathcal{C}_k(\Omega \times E)$ and K be a compact subset of E such that $S(f) \subset \Omega \times K$. For each $\omega \in \Omega$, $(t, x) \to Vf(\theta_t\omega, \varphi(t, \omega)x)$ is continuous, then

$$\sup_{x \in K} Vf(\theta_t\omega, \varphi(t, \omega)x) := c(\theta_t\omega) < \infty. \tag{4.1}$$

Since

$$c(\omega) = \sup_{x \in K} Vf(\omega, x), \tag{4.2}$$

then c is a random variable (cf. Lemma 2.3).

Moreover, by (B.1), (B.2), (3.4), (4.1) and (4.2) we get

$$c(\theta_t\omega) \leq c(\omega), \quad t \geq 0, \omega \in \Omega. \tag{4.3}$$

In particular, the function $(c \otimes 1)$ is (K_t)-supermedian. However, from (4.2) we have

$$0 \le Vf(\omega, x) \le (c \otimes 1)(\omega, x), \quad (\omega, x) \in \{f > 0\}, \tag{4.4}$$

and therefore by the domination principle (3.6), we get

$$0 \le Vf(\omega, x) \le (c \otimes 1)(\omega, x), \quad (\omega, x) \in \Omega \times E. \tag{4.5}$$

Hence $Vf \in \mathcal{C}_b(\Omega \times E)$. □

The following result is of the main importance in this paper since it contains a construction of global Lyapunov functions.

Proposition 4.2. *If V is proper, then there exists $g \in \mathcal{C}_b(\Omega \times E)$ such that $g > 0$, $Vg \in \mathcal{C}_b(\Omega \times E)$ and $Vg > 0$.*

Proof. Since $(E, \mathcal{E})$ is a locally compact Hausdorff space, there exists an exhaustion (K_n) of E by compact subsets, i.e. (i) Each K_n is a compact subset of E, (ii) $K_n \subset K_{n+1}$ for all $n \in \mathbb{N}$ and (iii) $E = \bigcup_n K_n$.

Consider now the function

$$u_n(x) := \frac{d(x, E \setminus K_{n+1})}{d(x, E \setminus K_{n+1}) + d(x, K_n)}, \quad x \in E. \tag{4.6}$$

From (4.6) we see that u_n is continuous and

1. $u_n(x) = 1$ if $x \in K_n$,
2. $u_n(x) = 0$ if $x \in E \setminus K_{n+1}$,
3. $0 < u_n(x) < 1$ if $x \in K_{n+1} \setminus K_n$.

In particular, for $n \in \mathbb{N}$, $(1 \otimes u_n) \in \mathcal{C}_k(\Omega \times E)$ and therefore $V(1 \otimes u_n) \in \mathcal{C}_b(\Omega \times E)$ by Lemma 4.1. Hence $c_n(\omega) := \|V(1 \otimes u_n)(\omega, \cdot)\| < \infty$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$. Define

$$g(\omega, x) := \sum_{n \in \mathbb{N}} \frac{(1 \otimes u_n)(\omega, x)}{2^n \sup(1, c_n(\omega))}, \quad \omega \in \Omega, x \in E. \tag{4.7}$$

Obviously, $g > 0$ and $g \in \mathcal{C}_b(\Omega \times E)$. Since $(1 \otimes u_n) = 1$ on $\Omega \times K_n$, we deduce from (4.7) that $Vg > 0$ and $\|Vg(\omega, \cdot)\| \le 1$ for each $\omega \in \Omega$. Using again Lemma 4.1, we conclude Vg is x-continuous as the limit of a normally convergent series of functions. Finally, $Vg \in \mathcal{C}_b(\Omega \times E)$. □

The following notion is known even for RDS, cf. [2] for a more general definition.

Definition 4.3. A global Lyapunov function associated to (θ,φ), is a mapping $L:\Omega\times E\to(0,+\infty)$ which is x-continuous, strictly decreasing along all orbits of (θ,φ), i.e.

$$L(\theta_t\omega,\varphi(t,\omega)x)<L(\omega,x),\quad t>0,x\in E,\omega\in\Omega \tag{4.8}$$

and

$$\lim_{t\to+\infty}L(\theta_t\omega,\varphi(t,\omega)x)=0,\quad x\in E,\omega\in\Omega. \tag{4.9}$$

The following useful result is adapted from our paper [8].

Proposition 4.4. *Let (θ,φ) be a continuous RDS which admits a global Lyapunov function. Then there exists a function $L:\Omega\times E\to(0,+\infty)$ which is x-continuous and satisfies*

$$L(\theta_t\omega,\varphi(t,\omega)x)=\exp(-t)\cdot L(\omega,x),\quad t\in\mathbb{R},\omega\in\Omega,x\in E. \tag{4.10}$$

In particular, L is a global Lyapunov function.

Proof. Let $\tilde{L}:\Omega\times E\to(0,\infty)$ be a global Lyapunov function of (θ,φ). For $(\omega,x)\in\Omega\times E$, put

$$\hat{L}(\omega,x):=\int_0^\infty\exp(-r)\tilde{L}(\theta_r\omega,\varphi(r,\omega)x)\,dr. \tag{4.11}$$

While $\tilde{L}$ is measurable and x-continuous then so is $\hat{L}$. On the other hand, by the translation and the cocycle equations (B.1) and (B.2), it follows from (4.11) that

$$\exp(-t)\hat{L}(\theta_t\omega,\varphi(t,\omega)x)=\hat{L}(\omega,x)-\int_0^t\exp(-r)L(\theta_r\omega,\varphi(r,\omega)x)\,dr \tag{4.12}$$

for $t\in\mathbb{R},\omega\in\Omega$ and $x\in E$. Let

$$\Upsilon(t,\omega,x):=\exp(-t)\hat{L}(\theta_t\omega,\varphi(t,\omega)x). \tag{4.13}$$

From (4.12) we obtain first that $t\mapsto\Upsilon(t,\omega,x)$ is strictly decreasing, second that $(r,x)\to\Upsilon(r,\omega,x)$ is continuous and third

$$\lim_{t\to+\infty}\Upsilon(t,\omega,x)=0,\quad\lim_{t\to-\infty}\Upsilon(t,\omega,x)=+\infty. \tag{4.14}$$

Therefore, for each $(\omega,x)\in\Omega\times E$, there exists by (4.13), a unique $\tau(\omega,x)\in\mathbb{R}$ such that

$$\Upsilon(\tau(\omega,x),\omega,x)=1. \tag{4.15}$$

Let us now prove that $x\mapsto\tau(\omega,x)$ is continuous. It can be easily verified using standard arguments. Indeed, let $\omega\in\Omega$, $x\in E$ and (x_n) be a sequence of E which tends to x.

Suppose first that the sequence $(\tau(\omega,x_n))$ is bounded. Let (y_n) and (z_n) be two subsequences of (x_n) such that $(\tau(\omega,y_n))$ and $(\tau(\omega,z_n))$ are convergent to s and t, respectively. Applying (4.15) to (ω,x), (ω,x_n) and (ω,z_n) we get

$$\Upsilon(\tau(\omega,x),\omega,x)=\Upsilon(\tau(\omega,y_n),\omega,y_n)=\Upsilon(\tau(\omega,z_n),\omega,z_n). \tag{4.16}$$

Letting $n\to\infty$ in (4.16), we conclude by the continuity of $(r,x)\mapsto\Upsilon(r,\omega,x)$ and by injectivity of $r\mapsto\Upsilon(r,\omega,x)$ that $s=t=\tau(\omega,x)$.

Otherwise, there exists in this case a subsequence (z_n) of (x_n) such that $(\tau(\omega,z_n))\uparrow+\infty$ or such that $(\tau(\omega,z_n))\downarrow-\infty$. If $(\tau(\omega,z_n))\uparrow+\infty$, define

$$f_n(y):=\Upsilon(\tau(\omega,z_n),\omega,y),\quad y\in E,$$

then (f_n) is a decreasing sequence of functions which converges simply to 0 in view of (4.14). Hence (f_n) converges locally uniformly to 0. While (z_n) belongs to a compact neighborhood of x and $\lim f_n(x)=0$ we get $\lim\Upsilon(\tau(\omega,z_n),\omega,z_n)=0$. But, from (4.14) we have $\lim\Upsilon(\tau(\omega,z_n),\omega,z_n)=1$, which gives a contradiction. If $(\tau(\omega,z_n))\downarrow-\infty$, we consider

$$g_n(y):=\frac{1}{\Upsilon(\tau(\omega,z_n),\omega,y)},\quad y\in E$$

instead of f_n and, by the same arguments, we obtain again a contradiction. We conclude that $\tau\in\mathcal{C}(\Omega\times E)$. Finally, using again (B.1) and (B.2), it follows from (4.13) and (4.14) that

$$\tau(\theta_t\omega,\varphi(t,\omega)x)=\tau(\omega,x)-t,\quad t\in\mathbb{R},\omega\in\Omega,x\in E.$$

Hence, by putting $L:=\exp\tau$, we obtain the desired result. □

Now, we are able to establish the proof to the main result of the paper.

Proof of Theorem 1.1. Suppose first that V is proper. Consider the function g defined by Proposition 4.2 and put $L:=Vg$. Then $L>0$ and $L\in\mathcal{C}_b(\Omega\times E)$. Moreover, by (3.4), (B.1), (B.2) and (4.7), we get

$$L(\theta_t\omega,\varphi(t,\omega)x)=\int_t^\infty g(\theta_s\omega,\varphi(s,\omega)x)ds<\int_0^\infty g(\theta_s\omega,\varphi(s,\omega)x)ds=L(\omega,x)$$

for all $t>0$, $\omega\in\Omega$ and $x\in E$. Hence $L=Vg$ is a global Lyapunov function for (θ,φ).

Conversely, let $L>0$ be a global Lyapunov function for (θ,φ). By Proposition 4.4, we may assume that L satisfies (4.10). Now, let $f\in\mathcal{C}_k(\Omega\times E)$, K a compact subset of E such that $S(f)\subset\Omega\times K$ and m a constant such that $0\le f\le m$. Define

$$\alpha(\theta_t\omega):=\inf\{L(\theta_t\omega,\varphi(t,\omega)x)\ :\ x\in K\},\quad t\ge 0,\omega\in\Omega. \tag{4.17}$$

For $t=0$, we get

$$\alpha(\omega)=\inf\{L(\omega,x):x\in K\},\quad \omega\in\Omega. \tag{4.18}$$

Using the same arguments as in the proof of Lemma 2.3 , we obtain that α is measurable. Moreover, $\alpha > 0$ since $L > 0$, K is compact and L is x-continuous. By putting $\beta := m/\alpha$, it can be easily verified that

$$f(\omega, x) \leq \beta(\omega) \cdot L(\omega, x), \quad \omega \in \Omega, x \in E. \tag{4.19}$$

On the other hand, according to (4.17), (4.18) and (4.11) we have

$$\alpha(\theta_t \omega) \leq \alpha(\omega), \quad \omega \in \Omega, t \geq 0.$$

Thus, by Lemma B.3, we get

$$\alpha(\theta_t \omega) = \alpha(\omega), \quad \text{a.e. } \omega \in \Omega, t \geq 0. \tag{4.20}$$

Finally, from (4.19), (4.20) and (4.10) we deduce that

$$f(\theta_t \omega, \varphi(t, \omega)x) \leq \exp(-t)\beta(\omega)L(\omega, x), \quad \text{a.e. } \omega \in \Omega, t \geq 0, x \in E. \tag{4.21}$$

The finiteness and the continuity of

$$x \mapsto Vf(\omega, x) := \int_0^\infty f(\theta_t \omega, \varphi(t, \omega)x)dt$$

can be easily deduced from (4.21) using standard arguments. □

5. SOME APPLICATIONS

Definition 5.1. A continuous RDS (θ, φ) is said to be *gradient-like* if there exist a random set $S \subset \Omega \times E$ and a homemorphism $h : \Omega \times E \to \mathbb{R} \times S$ such that:

1. for each $(\omega, x) \in \Omega \times E$, there exists $\bar{t} \in \mathbb{R}, (\bar{\omega}, \bar{x}) \in S$ such that

$$(\theta_t \bar{\omega}, \varphi(\bar{t}, \bar{\omega})\bar{x}) = (\omega, x), \tag{5.1}$$

2. for all $t \in \mathbb{R}, (\bar{\omega}, \bar{x}) \in S$,

$$h(\theta_t \bar{\omega}, \varphi(t, \bar{\omega})\bar{x}) = (t, (\bar{\omega}, \bar{x})). \tag{5.2}$$

In [8], we have proved that (θ, φ) is gradient-like if and only if (θ, φ) possesses a global Lyapunov function. Using Theorem 1.1, we deduce immediately the following corollary.

Corollary 5.2. *(θ, φ) is gradient-like if and only if the associated potential kernel is proper.*

Remark 5.3. Under some additional conditions on $(\Omega, \mathcal{F})$, any RDS (θ, φ) with state space E decomposes E into a random recurrent chain part and a random gradient-like part (cf. [11, Theorem 8]). Hence:

1. our main result may be applied to the gradient-like part of a general RDS,
2. the potential kernel of (θ, φ) is proper if and only if (θ, φ) has no random recurrent chains.

Next, we test our main result on some random and stochastic differential equations. These classical concepts are introduced in Appendix C.

Example 5.4. Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a metric DS. Consider the Lorentz system in $E := \mathbb{R}^3 \setminus \{(0,0,0)\}$ perturbed by real noises :

$$(RDE) \begin{cases} \dot{u} = \sigma(\theta_t\omega)(v-u), \\ \dot{v} = \rho(\theta_t\omega)u - v - uw, \\ \dot{w} = uv - \beta(\theta_t\omega)w \end{cases}$$

such that $\sigma(\omega), \rho(\omega), \beta(\omega) > 0$ and $\rho(\omega) < \sigma(\omega) \leq 1$ for all $\omega \in \Omega$. Let φ be an RDS generated by (RDE) and let

$$L(\omega, x, y, z) := x^2 + y^2 + z^2, \quad \omega \in \Omega, (x, y, z) \in E.$$

By the derivation of L with respect to t along orbits of φ, we obtain

$$\dot{L} = -(\sigma - \rho)u^2 - (2 - \rho - \sigma)v^2 - 2\beta w^2 < 0,$$

and therefore

$$L(\theta_t\omega, \varphi_t(\omega)(x, y, z)) < L(\omega, x, y, z), \quad \omega \in \Omega, t > 0, (x, y, z) \in E.$$

(cf. [1, 11] for more details). We conclude that L is a global Lyapunov function for (θ, φ), hence by Theorem 1.1, the associated potential kernel is proper.

Example 5.5. Consider the (RDS) defined on $E := (-1, 0) \cup (0, 1)$ by

$$dX_t = \frac{3}{2}X_t(X_t - X_t^2)^2\, dt + (X_t - X_t^3)dW_t, \quad X_0 = x \in E \tag{5.3}$$

and let φ be the associated cocycle over the Wiener DS θ. In fact, φ is explicitly given (cf. [10, p. 123]) by

$$\varphi(t, \omega)x = x \exp(t + W_t(\omega)) \cdot (1 - x^2 + x^2 \exp(2t + 2W_t(\omega)))^{1/2}. \tag{5.4}$$

Moreover, by [11, pp. 288–289], (θ, φ) is gradient-like on E. We conclude by Corollary 5.2 that the associated potential kernel is proper.

Remark 5.6. A random fixed point for the RDS (θ, φ) is a random variable $X : \Omega \to \mathbb{R}$ such that

$$\varphi(t, \omega)X(\omega) = X(\omega), \quad \omega \in \Omega, t \geq 0. \tag{5.5}$$

Consider $f \in \mathcal{C}_k(\Omega \times E)$ such that $X(\omega) \in S(f)(\omega), \omega \in \Omega$. By (3.4) and (5.5) we deduce easily that $Vf(\omega, X(\omega)) = \infty$ and therefore $Vf \notin \mathcal{C}(\Omega \times E)$. Hence, if the associated potential kernel is proper, the RDS does not have random fixed points.

The following important example proves in particular, that V may be not proper even when (θ, φ) is without random fixed points.

Example 5.7. Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a metric DS and let $\xi : \Omega \to [1, \infty)$ be a random variable. Consider the linear (RDE) defined on $E := \mathbb{R}^2 \setminus \{(0,0)\}$ by

$$\begin{cases} dX_t^1(\omega) = -\xi(\theta_t\omega) X_t^1(\omega) dt, \\ dX_t^2(\omega) = \xi(\theta_t\omega) X_t^2(\omega) dt, \\ X_0^1(\omega) = x^1, \quad X_0^2(\omega) = x^2, \quad (x^1, x^2) \in E. \end{cases}$$

Using the notation $r_t(\omega) := \int_0^t \xi(\theta_s\omega) ds$, the associated cocycle is explicitly given by

$$\varphi(t,\omega)(x^1, x^2) = (x^1 \exp(-r_t(\omega)), x^2 \exp(r_t(\omega))). \tag{5.6}$$

For each $X \in E$ and $r > 0$, let

$$D(X, r) := \{Z \in E : |Z - X| \leq r\}$$

and denote the positive orbit from X by

$$O_\omega(X) := \{\varphi(t,\omega)(X) : t \geq 0\}.$$

Let $A := (1, 0)$ and $B := (0, 1)$. Using (5.6), it can be verified that

$$O_\omega(A) = (0, 1] \times \{0\}, \quad O_\omega(B) = \{0\} \times [1, \infty). \tag{5.7}$$

For each $n \in \mathbb{N}^*$, let $A_n(\omega) := (1, \exp(-r_n(\omega)))$. Then $(A_n(\omega)) \to A$ since $(r_n) \uparrow \infty$. Moreover, using (5.6), we get

$$B_n(\omega) := \varphi(n,\omega) A_n(\omega) = (\exp(-r_n(\omega)), 1) \to B \tag{5.8}$$

Now, let $u : \mathbb{R} \times (0, \infty) \to [0, \infty)$ be continuous, $u = 1$ on $D(B, 1/2)$ and $u = 0$ on $E \setminus D(B, 3/4)$. Notice first that

$$V(1 \otimes u)(\omega, x) = \int_0^\infty u(\varphi(t,\omega)x) dt, \quad \omega \in \Omega, x \in E. \tag{5.9}$$

Using again (5.9), the definition of B_n and the cocycle property (B.2), we deduce that

$$V(1 \otimes u)(\theta_n\omega, B_n(\omega)) = \int_n^\infty u(\varphi(t,\omega)A_n) dt. \tag{5.10}$$

In particular, taking $u := 1 \otimes v$ (i.e. $u(x^1, x^2) := v(x^2)$) and using (5.6), (5.8), (5.10), $\xi \geq 1$, we get

$$\begin{aligned} V(1 \otimes u)(\theta_n\omega, B_n(\omega)) &= \int_n^\infty u\Big(\exp(-r_t(\omega), \exp(r_t(\omega) - r_n(\omega)))\Big)dt \\ &= \int_n^\infty v\Big(\exp\Big(\int_n^t \xi(\theta_s)ds\Big)\Big)dt \\ &\geq \int_n^\infty v\Big(\exp\Big(\int_n^t ds\Big)\Big)dt = \int_n^\infty v(\exp(t-n))dt. \end{aligned}$$

Hence, by putting $\ell := \exp(t - n)$, we obtain

$$V(1 \otimes u)(\theta_n\omega, B_n(\omega)) \geq \int_1^\infty \frac{v(\ell)}{\ell} d\ell, \quad n \in \mathbb{N}. \tag{5.11}$$

On the other hand, since $D(B, 3/4) \cap O_\omega(A) = \emptyset$, we can see, by (5.9), that $V(1 \otimes u)(\omega, A) = 0$. Moreover, (5.10) implies that

$$0 \leq V(1 \otimes u)(\theta_n\omega, B_n(\omega)) \leq \int_0^\infty u(\varphi(t,\omega)A_n)dt = V(1 \otimes u)(\omega, A_n(\omega)). \tag{5.12}$$

Suppose now that $X \mapsto V(1 \otimes u)(\omega, X)$ is continuous. Then we have

$$\lim_{n\to\infty} V(1 \otimes u)(\omega, A_n(\omega)) = V(1 \otimes u)(\omega, A) = 0$$

which is in contradiction with (5.11) in view of (5.12). Hence, the potential kernel V is not proper.

Let $X := (X^1, X^2) : \Omega \to E$ be a random fixed point of (θ, φ). By (5.5) and (5.6) we have

$$(X^1(\omega)\exp(-r_t(\omega)), X^2(\omega)\exp(r_t(\omega))) = (X^1(\theta_t\omega), X^2(\theta_t\omega)) \tag{5.13}$$

for $t > 0$ and $\omega \in \Omega$. By letting $t \uparrow \infty$ in (5.13), we obtain $X^1 \equiv 0$ and $X^2 \equiv \infty$ and therefore a contradiction since $(X^1, X^2) \in E$.

We conclude for this example, that the generated RDS is without random fixed points although the associated potential kernel is not proper.

APPENDICES

A. A CLASSICAL RESULT

Let ϕ be a continuous deterministic dynamical system (DS) on a locally compact space E, that is, $\phi : \mathbb{R} \times E \to E, (t,x) \mapsto \phi(t,x)$ is continuous and satisfies the translation equation, i.e.

$$\phi(0,x) = x, \quad \phi(s+t,x) = \phi(s,\phi(t,x)), \quad t \in \mathbb{R}, x \in E.$$

Standards examples are solutions of ordinary differential equations.

Following [9], the associated potential kernel U is defined for any positive measurable function v by

$$Uv(x) := \int_0^\infty v(\phi(t,x)dt, \quad x \in E.$$

It is proved in ([9, 9. Theoreme]) that the following statements are equivalent:

1. The potential kernel U of ϕ is proper, i.e. Uv is a finite continuous function whenever v is a continuous function with compact support,
2. ϕ possesses a global Lyapunov function, i.e. a continuous map $l : E \to (0,\infty)$ such that, for each $x \in E$, $l(\phi(t,x)) \downarrow 0$ as $t \uparrow \infty$.

B. RANDOM DYNAMICAL SYSTEM

For the following definitions, we refer to [1, Chap. 1].

Definition B.1. A *continuous random dynamical system* (RDS) consists of two ingredients:

(1) *A metric dynamical system* (DS) θ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. a mapping

$$\theta : \mathbb{R} \times \Omega \to \Omega, \quad (t,\omega) \mapsto \theta_t\omega := \theta(t,\omega)$$

such that:

1. θ satisfies the *translation equation*, i.e.

$$\theta_0 = I_\Omega, \quad \theta_{s+t} = \theta_s \circ \theta_t, \quad s,t \in \mathbb{R}, \tag{B.1}$$

2. $(t,\omega) \to \theta(t,\omega)$ is $(\mathcal{R} \otimes \mathcal{F}, \mathcal{F})$-measurable,
3. the probability $\mathbb{P}$ is θ-inavariant, that is $\mathbb{P}(\theta^{-1}(F)) = \mathbb{P}(F)$ for all $F \in \mathcal{F}$ and $t \in \mathbb{R}$.

(2) A *cocycle* φ *on* E *over the DS* θ, i.e. a mapping

$$\varphi : \mathbb{R} \times \Omega \times E \to E, \quad (t,\omega,x) \mapsto \varphi(t,\omega,x)$$

such that:

1. φ is $(\mathcal{R} \otimes \mathcal{F} \otimes \mathcal{E}, \mathcal{E})$-measurable,
2. $(t,x) \to \varphi(t,\omega,x)$ is continuous for each $\omega \in \Omega$,

3. the family $\varphi(t,\omega) := \varphi(t,\omega,\cdot) : E \to E$ of random mappings, satisfies the cocycle equation, i.e.

$$\varphi(0,\omega) = I_E, \quad \varphi(s+t,\omega) = \varphi(s,\theta_t\omega) \circ \varphi(t,\omega) \tag{B.2}$$

for $s,t \in \mathbb{R}, \omega \in \Omega$.

In this case, we say that φ is a continuous RDS driven by the DS θ, with state space E and with time $\mathbb{R}$. It is denoted by the pair (θ,φ). Moreover, we may associate to it the skew product

$$\Phi : \mathbb{R} \times \Omega \times E \to \Omega \times E, \quad (t,\omega,x) \mapsto \Phi_t(\omega,x)$$

defined by

$$\Phi_t(\omega,x) := (\theta_t\omega, \varphi(t,\omega,x)), \quad t \in \mathbb{R}, \omega \in \Omega, x \in E. \tag{B.3}$$

Remark B.2.

(i) If Ω is reduced to one point, then the cocycle equation is reduced to the translation equation. Indeed, if $\Omega := \{\varpi\}$ then $\theta_t\varpi = \varpi$ for all $t \in \mathbb{R}$. Therefore, by putting $\varphi_t := \varphi(t,\varpi)$, the relation (B.2) becomes

$$\varphi_0 = I_E, \quad \varphi_{s+t} = \varphi_s \circ \varphi_t$$

which is the translation equation on E. Hence, RDS generalize in a natural way the deterministic DS.

(ii) It follows from (B.2) that $\varphi(t,\omega)$ is an homeomorphism on E and

$$\varphi(t,\omega)^{-1} = \varphi(-t,\theta_t\omega), \quad t \in \mathbb{R}, \omega \in \Omega. \tag{B.4}$$

(iii) Using (B.1) and (B.2), it can be easily seen that Φ satisfies the translation equation on $\Omega \times E$, i.e.

$$\Phi_{s+t} = \Phi_s \circ \Phi_t, \quad s,t \in \mathbb{R}. \tag{B.5}$$

Thus, the skew product allows a natural generalization from the deterministic case to the random case.

The following result is useful in this paper.

Lemma B.3. *Let $(\Omega,\mathcal{F},\mathbb{P},\theta)$ be a metric DS, α be a positive random variable and $t > 0$. If $\alpha(\theta_t\omega) \le \alpha(\omega)$ a.e., then $\alpha(\theta_t\omega) = \alpha(\omega)$ a.e.*

Proof. Using the θ-invariance of $\mathbb{P}$ and an approximation of α by step functions, we obtain

$$\int_\Omega (\alpha(\omega) - \alpha(\theta_t\omega))\, d\mathbb{P}(\omega) = 0. \tag{B.6}$$

By the hypothesis on α, we can write $\mathbb{P}(\{\alpha\circ\theta_t < \alpha\} \cup \{\alpha\circ\theta_t = \alpha\}) = 1$ and therefore (B.6) reads

$$\int_{\{\alpha\circ\theta_t<\alpha\}} (\alpha(\omega) - \alpha(\theta_t\omega))\, d\mathbb{P}(\omega) = 0$$

In other words, $\mathbb{P}(\alpha\circ\theta_t < \alpha) = 0$. □

C. STANDARD EXAMPLES OF RDS

For the following examples of RDS, we refer to [1, Chap. 2].

Example C.1. *Random differential equations:* Let θ be a DS on $(\Omega, \mathcal{F}, \mathbb{P})$, let E be an open subset of $\mathbb{R}^d$ and $h : \Omega \times E \to E$ be measurable such that, for each $\omega \in \Omega$, $(t, x) \to h(\theta_t\omega, x)$ is continuous and $x \to h(\theta_t\omega, x)$ is locally-Lipschitz. Then the random differential equation

$$dX_t(\omega) = h(\theta_t\omega, X_t(\omega))dt, \quad X_0(\omega) = x \in E$$

admits a unique solution φ which is a continuous RDS over θ. The solution φ is given implicitly by

$$\varphi(t, \omega)x = x + \int_0^t h(\theta_s\omega, \varphi(s, \omega)x)\, ds$$

In particular, $t \to \varphi(t, \omega)x$ is absolutely continuous.

Example C.2. *Stochastic differential equations:* For this example, let us recall first the Wiener DS on $\mathbb{R}^d$. Let $\Omega = \{\omega : \mathbb{R} \to \mathbb{R}^d : \omega \text{ is continuous and } \omega(0) = 0\}$ equipped with the compact open topology, let $\mathcal{F} := \mathcal{B}(\Omega)$ the associated Borel σ-algebra and for $t \in \mathbb{R}$, let $W_t : \Omega \to \mathbb{R}^d$, $\omega \to \omega(t)$. There exists, by a classical result (the Kolmogorov extension theorem), a unique probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that the processus $(W_t)_{t\in\mathbb{R}}$ is with stationary and independent increments and $(W_t - W_s)$ has the normal distribution with mean 0 and variance $|t - s| I_{\mathbb{R}^d}$. For $t \in \mathbb{R}$, define the Wiener shift $\theta_t : \Omega \to \Omega$ by

$$\theta_t\omega(s) := \omega(s + t) - \omega(t).$$

It can be easily verified that $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is a metric DS, called the Wiener or Brownian DS.

The Wiener DS is the appropriate sample space in order to interpret an stochastic differential equation (SDE) as an RDS. More precisely, consider the standard SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0(\omega) = x \in \mathbb{R}^d, \quad t \in \mathbb{R}, \tag{C.1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ are such that

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \le D|x - y|, \quad x, y \in \mathbb{R}^d,$$

for some constant $D > 0$. Then (C.1) admits a unique solution

$\varphi : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ which is a continuous RDS over the Wiener DS θ.

As for the RDE, φ is given implicitly by

$$\varphi(t, \omega)x = x + \int_0^t b(\varphi(s, \omega)x)\, ds + \int_0^t \sigma(\varphi(s, \omega)x)\, dW_s(\omega).$$

The quantity "$\int_0^t \sigma(\varphi(s, \omega)x)\, dW_s(\omega)$" stands for the Itô stochastic integral (cf. [13]).

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