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ON THE SUMMABILITY OF DIVERGENT POWER SERIES SOLUTIONS FOR CERTAIN FIRST-ORDER LINEAR PDES

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Abstract. This article is concerned with the study of the Borel summability of divergent power series solutions for certain singular first-order linear partial differential equations of nilpotent type. Our main purpose is to obtain conditions which coefficients of equations should satisfy in order to ensure the Borel summability of divergent solutions. We will see that there is a close affinity between the Borel summability of divergent solutions and global analytic continuation properties for coefficients of equations.

Keywords: partial differential equation, divergent power series, summability, asymptotic expansion, analytic continuation, integro-differential equation, integral equation.

Mathematics Subject Classification: 35C20, 35C10, 35C15.

1. INTRODUCTION AND MAIN RESULT

In this paper, we study the following first-order linear partial differential equation with two complex variables:

$$\begin{gathered} Pu(x,y) = f(x,y), \\ P = \{1 + x^2 + \beta(x,y)\}yD_x + \{x + b(x,y)\}y^2D_y + 1, \end{gathered} \tag{1.1}$$

where x, $y \in \mathbb{C}$, $D_x = \partial/\partial x$, $D_y = \partial/\partial y$. The coefficients β, b and f are holomorphic at $(x,y) = (0,0) \in \mathbb{C}^2$. Moreover, β and b satisfy

$$\beta(x,0) \equiv b(x,0) \equiv 0. \tag{1.2}$$

The equation (1.1) is called of "nilpotent type" (cf. Remark 1.4). We expound the incentive to consider (1.1) in Subsection 1.2.

First of all, let us consider the existence of formal power series solutions $\hat{u}(x,y) = \sum_{m,n=0}^{\infty} u_{mn}x^m y^n$ around $(x,y) = (0,0)$. Then, we can prove the unique existence of $\hat{u}(x, y)$. Moreover, we see that it takes the form of $\hat{u}(x, y) = \sum_{n=0}^{\infty} u_n(x) y^n$, where $u_n(x)$ are holomorphic in a common neighborhood of $x = 0$. However, this formal power series solution $\hat{u}(x, y)$ with respect to the y-variable diverges in general and the rate of divergence is characterized in terms of the Gevrey index (cf. Definition 1.1, (1)–(3) and Theorem 1.5). So, we are interested in the Borel summability of such a divergent solution (cf. Definition 1.1, (4)–(6)). Our main purpose in this paper is to obtain the conditions under which the divergent solution is Borel summable.

The content of this paper is as follows. In Subsection 1.1, we give the definitions of divergent power series of the Gevrey type and Borel summability (Definition 1.1). Some necessary and sufficient condition for the Borel summability is also stated (Definition 1.2 and Theorem 1.3). In Subsection 1.2, we state the theorem which assures the unique existence of divergent power series solutions (Theorem 1.5). Moreover, we explain the incentive to consider (1.1). Theorems 1.3 and 1.5 play basic roles throughout this paper. In Subsection 1.3, we state the main theorem (Theorem 1.6); that is, we give conditions which the coefficients should satisfy in order to ensure the Borel summability of the divergent solution. Some global analytic continuation properties for coefficients will be required. In Subsection 1.4 we introduce literature studying related topics. The proof of Theorem 1.6 is given in Sections 2–4. In Section 2, the proof of Theorem 1.6 is reduced to that of the global solvability of some integral equation (cf. (2.11) and Proposition 2.1). In Sections 3 and 4, we prove the global solvability of that integral equation by applying an iteration method, and complete the proof of Theorem 1.6.

1.1. DEFINITIONS AND FUNDAMENTAL RESULT

Definition 1.1.

(1) $\mathcal{O}[R]$ denotes the ring of holomorphic functions on the closed ball $B(R) = \{x \in \mathbb{C};\ |x| \leq R\}$, where R is a positive number.

(2) The ring of formal power series in y ($\in \mathbb{C}$) over the ring $\mathcal{O}[R]$ is denoted by $\mathcal{O}[R][[y]]$:
$$\mathcal{O}[R][[y]] = \left\{ \hat{u}(x, y) = \sum_{n=0}^{\infty} u_n(x) y^n;\ u_n(x) \in \mathcal{O}[R] \right\}. \tag{1.3}$$

(3) We say that $\hat{u}(x, y) = \sum_{n=0}^{\infty} u_n(x) y^n$ ($\in \mathcal{O}[R][[y]]$) *belongs to* $\mathcal{O}[R][[y]]_2$ if there exist some positive constants C and K such that
$$\max_{|x| \leq R} |u_n(x)| \leq C K^n n! \tag{1.4}$$
for all $n = 0, 1, 2, \ldots$. The suffix 2 of $\mathcal{O}[R][[y]]_2$ expresses the Gevrey index of power series. Elements of $\mathcal{O}[R][[y]]_2$ are divergent power series in general.

(4) For $\theta \in \mathbb{R}$, $\kappa > 0$ and $0 < \rho \leq +\infty$, the sector $S(\theta, \kappa, \rho)$ in the universal covering space of $\mathbb{C} \setminus \{0\}$ is defined as
$$S(\theta, \kappa, \rho) = \left\{ y;\ |\arg(y) - \theta| < \frac{\kappa}{2},\ 0 < |y| < \rho \right\}. \tag{1.5}$$

We refer to θ, κ and ρ as the *bisecting direction*, the *opening angle* and the *radius* of $S(\theta,\kappa,\rho)$, respectively.

(5) Let $\hat{u}(x,y)=\sum_{n=0}^{\infty}u_n(x)y^n \in \mathcal{O}[R][[y]]_2$ and let $u(x,y)$ be a holomorphic function on $X=B(R)\times S(\theta,\kappa,\rho)$. Then we say that *$u(x,y)$ has $\hat{u}(x,y)$ as an asymptotic expansion of the Gevrey order 2 in X* if the following asymptotic estimates hold: there exist some positive constants $\mathcal{C}$ and $\mathcal{K}$ such that

$$\max_{|x|\le R}\left|u(x,y)-\sum_{n=0}^{N-1}u_n(x)y^n\right|\le \mathcal{C}\mathcal{K}^N N!|y|^N \tag{1.6}$$

for all $y\in S(\theta,\kappa,\rho)$ and $N=1,\,2,\,\ldots$. Then we write this as

$$u(x,y)\cong_2 \hat{u}(x,y) \quad \text{in} \quad X.$$

(6) Let $\hat{u}(x,y)=\sum_{n=0}^{\infty}u_n(x)y^n \in \mathcal{O}[R][[y]]_2$. We say that *$\hat{u}(x,y)$ is Borel summable in a direction θ* if there exists a holomorphic function $u(x,y)$ on $X=B(r)\times S(\theta,\kappa,\rho)$ for some r $(0<r\le R)$, $\kappa>\pi$ and ρ $(0<\rho\le+\infty)$ which satisfies $u(x,y)\cong_2\hat{u}(x,y)$ in X. A given divergent power series $\hat{u}(x,y)\in\mathcal{O}[R][[y]]_2$ is not necessarily Borel summable in general. However, if $\hat{u}(x,y)$ is Borel summable in a direction θ, then we see that the above holomorphic function $u(x,y)$ is unique (cf. Balser [1, 3]). So we call this unique $u(x,y)$ the Borel sum of $\hat{u}(x,y)$ in a direction θ.

As mentioned above, a given divergent power series $\hat{u}(x,y)\in\mathcal{O}[R][[y]]_2$ is not necessarily Borel summable. When we would like to check the Borel summability of a given $\hat{u}(x,y)$, the following Theorem 1.3 is used frequently. Theorem 1.3 gives the necessary and sufficient condition for the Borel summability, and gives the explicit formula of the Borel sum.

Definition 1.2. For $\hat{u}(x,y)=\sum_{n=0}^{\infty}u_n(x)y^n \in \mathcal{O}[R][[y]]_2$, we define the convergent power series $\hat{\mathcal{B}}[\hat{u}](x,\eta)$ in a neighborhood of $(x,\eta)=(0,0)$ as

$$\hat{\mathcal{B}}[\hat{u}](x,\eta)=\sum_{n=0}^{\infty}u_n(x)\frac{\eta^n}{n!}. \tag{1.7}$$

We call $\hat{\mathcal{B}}[\hat{u}](x,\eta)$ the *formal Borel transform of $\hat{u}(x,y)$*.

Theorem 1.3 ([1, 3]). *For a given formal power series $\hat{u}(x,y)=\sum_{n=0}^{\infty}u_n(x)y^n \in \mathcal{O}[R][[y]]_2$, let us put $v(x,\eta)=\hat{\mathcal{B}}[\hat{u}](x,\eta)$. Then the following two statements are equivalent:*

(i) *$\hat{u}(x,y)$ is Borel summable in a direction θ.*

(ii) *$v(x,\eta)$ can be continued analytically to $B(r_0)\times S(\theta,\kappa_0,+\infty)$ for some $r_0>0$ and $\kappa_0>0$, and has the following exponential growth estimate for some positive constants C and δ:*

$$\max_{|x|\le r_0}|v(x,\eta)|\le Ce^{\delta|\eta|}, \quad \eta\in S(\theta,\kappa_0,+\infty). \tag{1.8}$$

When condition (i) *or* (ii) (*therefore both*) *is satisfied, the Borel sum* $u(x,y)$ *in the direction* θ *is given by*

$$u(x,y) = \frac{1}{y} \int_0^{\infty e^{i\theta}} e^{-\eta/y} v(x,\eta)\, d\eta. \tag{1.9}$$

Theorem 1.3 plays a significant role in the proof of the main theorem (Theorem 1.6) of this paper. Throughout this paper, we call condition (ii) in Theorem 1.3 condition (BS).

1.2. INCENTIVE

In this subsection, we state the problem, precisely. First, we consider the problem in a more general framework. Let us consider the following equation:

$$\{A(x,y)D_x + B(x,y)D_y + 1\}u(x,y) = F(x,y), \tag{1.10}$$

where all coefficients are holomorphic at the origin. Moreover, we assume the following three fundamental conditions:

$$A(x,0) \equiv 0, \tag{1.11}$$

$$\frac{\partial A}{\partial y}(0,0) \neq 0, \tag{1.12}$$

$$B(x,0) \equiv \frac{\partial B}{\partial y}(x,0) \equiv 0. \tag{1.13}$$

Remark 1.4. Conditions (1.11) and (1.13) imply $A(0,0) = B(0,0) = 0$, which means that (1.10) is *singular at the origin*. Moreover, it follows from (1.11)–(1.13) that the Jacobi matrix $\partial(A,B)/\partial(x,y)|_{(x,y)=(0,0)}$ is a nilpotent matrix

$$\begin{pmatrix} 0 & (\partial A/\partial y)(0,0) \\ 0 & 0 \end{pmatrix}. \tag{1.14}$$

In this sense, our equation is called of nilpotent type.

Now, on the existence of formal power series solutions, we already know the following fact, which will be fundamental in the argument below.

Theorem 1.5 ([6])**.** *Let us assume* (1.11)–(1.13)*. Then* (1.10) *has a unique formal power series solution* $\hat{u}(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$ *for some* $R > 0$.

A short proof of Theorem 1.5 is also given in [7]. On the basis of Theorem 1.5, we can study the coming problem; the Borel summability of the formal solution: When is the formal solution $\hat{u}(x,y)$ Borel summable in a given bisecting direction θ, that is, when does the formal Borel transform $v(x,\eta) = \hat{\mathcal{B}}[\hat{u}](x,\eta)$ of $\hat{u}(x,y)$ satisfy condition (BS)?

Hereafter, we consider the above problem. To begin with, let us rewrite (1.10) to state the main result. It follows from (1.11)–(1.13) that (1.10) is rewritten in the following form:

$$[\{\alpha(x)+\beta(x,y)\}yD_x+\{a(x)+b(x,y)\}y^2D_y+1]u(x,y)=f(x,y), \tag{1.15}$$

where each coefficient is holomorphic at the origin. Moreover, α, β and b satisfy

$$\alpha(0)\neq 0, \tag{1.16}$$

$$\beta(x,0)\equiv b(x,0)\equiv 0. \tag{1.17}$$

In [7, 8], we dealt with the case where $a(x)\equiv a$ (constant), and obtained the conditions under which the formal solution is Borel summable. In [10, 11], we studied the case where $\alpha(x)=\alpha_0+\alpha_1 x$ (cf. also [9]). So, in this paper we study the case where

$$\alpha(x)=1+x^2,\quad a(x)=x, \tag{1.18}$$

that is, we consider (1.1) satisfying (1.2). The main purpose of this paper is to give the conditions which coefficients β, b and f of (1.1) should satisfy in order to assure the Borel summability of the formal solution in a given direction θ.

1.3. MAIN THEOREM

Assumptions: First of all, let us define the region $\Omega_{r,\theta,\kappa}$ by

$$\Omega_{r,\theta,\kappa}=\{-\tan[\arcsin(\tau)];\ \tau\in B(r)\cup S(\theta,\kappa,+\infty)\}. \tag{1.19}$$

In order to ensure the well-definedness of $\Omega_{r,\theta,\kappa}$, we assume the following:

(A1) $\theta\neq 0$, $\theta\neq\pi$.

Under (A1) we obtain the following fact: by taking suitably small $r>0$ and $\kappa>0$, it holds that

$$\big|\cos[\arcsin(\tau)]\big|\geq \mathsf{M}\quad(\tau\in B(r)\cup S(\theta,\kappa,+\infty)) \tag{1.20}$$

for some positive constant M. This implies the well-definedness of $\Omega_{r,\theta,\kappa}$. Hereafter, we always take such $r>0$ and $\kappa>0$.

Next, for the inhomogeneity term $f(x,y)$ we assume the following:

(A2) $f(x,y)$ can be continued analytically to $\Omega_{r,\theta,\kappa}\times\{y\in\mathbb{C};\ |y|\leq c\}$ for some $c>0$. Moreover, it has the following growth estimate there. There exist some positive constants C and δ such that

$$\max_{|y|\leq c}|f(x,y)|\leq C\exp[\delta|\sin(\arctan x)|],\quad x\in\Omega_{r,\theta,\kappa}. \tag{1.21}$$

Finally, we impose the following conditions for the coefficients $\beta(x,y)$ and $b(x,y)$:

(A3) $\beta(x,y)$ and $b(x,y)$ can be continued analytically to $\Omega_{r,\theta,\kappa} \times \{y \in \mathbb{C};\ |y| \le c\}$. Moreover, there exist some positive constants K, $L > 0$ and $p > 1$, which are independent of m, such that

$$\left|\frac{1}{1+x^2}\frac{\partial^m \beta}{\partial y^m}(x,0)\right| \le KL^m m! |\cos(\arctan x)|^m, \tag{1.22}$$

$$\left|(1+x^2)^{3/2}\frac{\partial}{\partial x}\left\{\frac{1}{1+x^2}\frac{\partial^m \beta}{\partial y^m}(x,0)\right\}\right| \le KL^m m! |\cos(\arctan x)|^m, \tag{1.23}$$

$$\left|\frac{x}{1+x^2}\frac{\partial^m \beta}{\partial y^m}(x,0)\right| \le \frac{KL^m m! |\cos(\arctan x)|^{m+1}}{\{1+|\sin(\arctan x)|\}^p}, \tag{1.24}$$

$$\left|\frac{\partial^m b}{\partial y^m}(x,0)\right| \le \frac{KL^m m! |\cos(\arctan x)|^{m+1}}{\{1+|\sin(\arctan x)|\}^p} \tag{1.25}$$

for $x \in \Omega_{r,\theta,\kappa}$, and $m = 1, 2, \ldots$.

Let us state the main theorem in this paper.

Theorem 1.6. *Under assumptions* (A1)–(A3) *the formal solution* $\hat{u}(x,y)$ *of* (1.1) *is Borel summable in the direction* θ.

1.4. SOME REMARKS ON RELATED TOPICS

In the theory of ordinary differential equations, there are many studies concerning the summability of divergent power series solutions, and we can see many significant results in Balser's book [1,3] (cf. also Malgrange [13]).

On the other hand, in the theory of partial differential equations, there are not so many studies. The first contribution is rendered by Lutz-Miyake-Schäfke [12], where complex heat equations are dealt with. Balser [2,4], Balser and Miyake [5] and Miyake [14] generalized the result in [12]. In Ōuchi [15] also, we can find some interesting results for greatly general linear partial differential equations. We remark that our equation (1.1) is a different type of equation from theirs, and that in the above articles we can see quite different phenomena from ours.

2. PRELIMINARIES TO PROOF

By Theorem 1.3, in order to prove Theorem 1.6, it is sufficient to prove that the formal Borel transform $v(x,\eta) = \hat{\mathcal{B}}[\hat{u}](x,\eta)$ of the formal solution $\hat{u}(x,y)$ satisfies condition (BS) under assumptions (A1)–(A3). In order to do that, in Subsection 2.1 we write down the equation which $\hat{\mathcal{B}}[\hat{u}](x,\eta)$ should satisfy. Some integro-differential equation (cf. (2.4)) will be obtained. Furthermore, in Subsection 2.2, we transform that intego-differential equation into some integral equation.

2.1. FORMAL BOREL TRANSFORM OF EQUATION

Let us operate the formal Borel transform to (1.1). Then we obtain the following equality:

$$
\begin{aligned}
&(1+x^2)\int_0^\eta \hat{\mathcal{B}}[D_x\hat{u}](x,t)\,dt + \int_0^\eta \hat{\mathcal{B}}[\beta](x,\eta-t)\hat{\mathcal{B}}[D_x\hat{u}](x,t)\,dt \\
&\quad + x\int_0^\eta \hat{\mathcal{B}}[yD_y\hat{u}](x,t)\,dt + \int_0^\eta \hat{\mathcal{B}}[b](x,\eta-t)\hat{\mathcal{B}}[yD_y\hat{u}](x,t)\,dt + \hat{\mathcal{B}}[\hat{u}](x,\eta) \\
&= \hat{\mathcal{B}}[f](x,\eta),
\end{aligned}
\tag{2.1}
$$

where $\hat{\mathcal{B}}[\beta](x,\eta)$, $\hat{\mathcal{B}}[b](x,\eta)$ and $\hat{\mathcal{B}}[f](x,\eta)$ are the formal Borel transforms of $\beta(x,y)=\sum_{n=1}^\infty \beta_n(x)y^n$, $b(x,y) = \sum_{n=1}^\infty b_n(x)y^n$ and $f(x,y) = \sum_{n=0}^\infty f_n(x)y^n$, that is,

$$
\hat{\mathcal{B}}[\beta](x,\eta)=\sum_{n=1}^\infty \beta_n(x)\frac{\eta^n}{n!},\quad \hat{\mathcal{B}}[b](x,\eta)=\sum_{n=1}^\infty b_n(x)\frac{\eta^n}{n!},\quad \hat{\mathcal{B}}[f](x,\eta)=\sum_{n=0}^\infty f_n(x)\frac{\eta^n}{n!}.
$$

(2.1) is obtained by applying the following equality:

$$
\begin{aligned}
\hat{\mathcal{B}}[y^{m+n+1}](\eta) &= \frac{1}{(m+n+1)!}\eta^{m+n+1} = B(m+1,n+1)\frac{\eta^{m+n+1}}{m!n!} \quad \text{(Beta integral)} \\
&= \int_0^1 (1-s)^m s^n\,ds \cdot \frac{\eta^{m+n+1}}{m!n!} = \int_0^\eta (\eta-t)^m t^n\,dt \cdot \frac{1}{m!n!} \\
&= \int_0^\eta \hat{\mathcal{B}}[y^m](\eta-t)\hat{\mathcal{B}}[y^n](t)\,dt.
\end{aligned}
$$

Next, let us calculate $\hat{\mathcal{B}}[D_x\hat{u}](x,\eta)$ and $\hat{\mathcal{B}}[yD_y\hat{u}](x,\eta)$ in (2.1). It is clear that $\hat{\mathcal{B}}[D_x\hat{u}](x,\eta) = D_x\hat{\mathcal{B}}[\hat{u}](x,\eta)$. On $\hat{\mathcal{B}}[yD_y\hat{u}](x,\eta)$, it follows from the same argument as above that $\hat{\mathcal{B}}[yD_y\hat{u}](x,\eta) = \int_0^\eta \hat{\mathcal{B}}[D_y\hat{u}](x,t)\,dt$. Moreover, by noting the commutative diagram

$$
\begin{array}{ccc}
y^n & \xrightarrow{\text{formal Borel tr.}} & \dfrac{\eta^n}{n!} \\
{\scriptstyle D_y}\big\downarrow & & \big\downarrow{\scriptstyle D_\eta \eta D_\eta} \\
ny^{n-1} & \xrightarrow[\text{formal Borel tr.}]{} & n\dfrac{\eta^{n-1}}{(n-1)!}
\end{array}
$$

we have $\hat{\mathcal{B}}[D_y\hat{u}](x,\eta) = D_\eta\eta D_\eta\hat{\mathcal{B}}[\hat{u}](x,\eta)$. Hence, it holds that

$$
\hat{\mathcal{B}}[yD_y\hat{u}](x,\eta) = \int_0^\eta \hat{\mathcal{B}}[D_y\hat{u}](x,t)\,dt = \int_0^\eta D_t t D_t\hat{\mathcal{B}}[u](x,t)\,dt = \eta D_\eta\hat{\mathcal{B}}[\hat{u}](x,\eta). \tag{2.2}
$$

(The Euler operator is transformed to the Euler operator.) By adopting (2.2) we obtain

$$\begin{aligned}
&\int_0^\eta \hat{\mathcal{B}}[b](x,\eta-t)\hat{\mathcal{B}}[yD_y\hat{u}](x,t)\,dt \\
&= \int_0^\eta \hat{\mathcal{B}}[b](x,\eta-t)\cdot t\cdot D_t\hat{\mathcal{B}}[\hat{u}](x,t)\,dt \\
&= \left[\hat{\mathcal{B}}[b](x,\eta-t)\cdot t\cdot \hat{\mathcal{B}}[\hat{u}](x,t)\right]_{t=0}^{t=\eta} - \int_0^\eta \frac{\partial}{\partial t}\{\hat{\mathcal{B}}[b](x,\eta-t)\cdot t\}\cdot \hat{\mathcal{B}}[\hat{u}](x,t)\,dt \\
&= \int_0^\eta \hat{\mathcal{B}}[b]_\eta(x,\eta-t)\cdot t\cdot \hat{\mathcal{B}}[\hat{u}](x,t)\,dt - \int_0^\eta \hat{\mathcal{B}}[b](x,\eta-t)\hat{\mathcal{B}}[\hat{u}](x,t)\,dt.
\end{aligned}$$

Therefore, we see that $\hat{\mathcal{B}}[\hat{u}](x,\eta)$ is a solution of the following equation:

$$\begin{aligned}
&(1+x^2)\int_0^\eta v_x(x,t)\,dt + \int_0^\eta \hat{\mathcal{B}}[\beta](x,\eta-t)v_x(x,t)\,dt + x\int_0^\eta t\cdot v_\eta(x,t)\,dt \\
&\quad + \int_0^\eta \hat{\mathcal{B}}[b]_\eta(x,\eta-t)\cdot t\cdot v(x,t)\,dt - \int_0^\eta \hat{\mathcal{B}}[b](x,\eta-t)v(x,t)\,dt + v(x,\eta) \\
&= \hat{\mathcal{B}}[f](x,\eta).
\end{aligned} \tag{2.3}$$

Finally, let us operate D_η to (2.3) from the left. Then we see that $\hat{\mathcal{B}}[\hat{u}](x,\eta)$ is a solution of the following initial value problem:

$$\begin{cases}
\mathscr{L}v(x,\eta) = -\displaystyle\int_0^\eta \hat{\mathcal{B}}[\beta]_\eta(x,\eta-t)v_x(x,t)\,dt - \hat{\mathcal{B}}[b]_\eta(x,0)\cdot\eta\cdot v(x,\eta) \\
\qquad\qquad\quad - \displaystyle\int_0^\eta \hat{\mathcal{B}}[b]_{\eta\eta}(x,\eta-t)\cdot t\cdot v(x,t)\,dt + \int_0^\eta \hat{\mathcal{B}}[b]_\eta(x,\eta-t)v(x,t)\,dt \\
\qquad\qquad\quad + g(x,\eta), \\
v(x,0) = f(x,0),
\end{cases} \tag{2.4}$$

where $\mathscr{L}$ is the first-order linear partial differential operator defined by

$$\mathscr{L} = (1+x^2)D_x + (1+x\eta)D_\eta, \tag{2.5}$$

and $g(x,\eta) = \hat{\mathcal{B}}[f]_\eta(x,\eta)$. It is easy to prove that $\hat{\mathcal{B}}[\hat{u}](x,\eta)$ is the unique locally holomorphic solution of (2.4). Hence, the proof of Theorem 1.6 has been reduced to that of the following fact: *the solution* $v(x,\eta)$ *of* (2.4) *satisfies* (BS). However, we do not deal with the integro-differential equation (2.4) immediately. In the next subsection, we transform (2.4) into the integral equation.

2.2. TRANSFORMATION TO INTEGRAL EQUATION

In order to transform (2.4) into the integral equation, we apply the following formula. The solution $V(x,\eta)$ of the initial value problem of the following first-order linear partial differential equation

$$\begin{cases} \mathscr{L}V(x,\eta) = k(x,\eta), \\ V(x,0) = l(x) \end{cases} \tag{2.6}$$

is given by

$$\begin{aligned} V(x,\eta) &= l(-\tan(\mathscr{A}(x,\eta))) \\ &\quad + \int_0^\eta k(-\tan(\mathscr{A}(x,\eta-z)), \mathscr{E}(x,\eta-z)\cdot z)\cdot \mathscr{E}(x,\eta-z)\,dz, \end{aligned} \tag{2.7}$$

where

$$\mathscr{A}(x,\eta) = \arcsin\{-\sin(\arctan x) + \cos(\arctan x)\cdot\eta\}, \tag{2.8}$$

$$\mathscr{E}(x,\eta) = \frac{\cos(\arctan x)}{\cos(\mathscr{A}(x,\eta))}. \tag{2.9}$$

We explain the derivation of the formula (2.7) in Section A. By (2.7), we see that (2.4) is equivalent to the following equation:

$$\begin{aligned} v(x,\eta) &= f(-\tan(\mathscr{A}(x,\eta)), 0) \\ &\quad + \int_0^\eta g(-\tan(\mathscr{A}(x,\eta-z)), \mathscr{E}(x,\eta-z)\cdot z)\cdot \mathscr{E}(x,\eta-z)\,dz \\ &\quad + \sum_{i=1}^4 \mathcal{I}_i v(x,\eta), \end{aligned}$$

where each operator $\mathcal{I}_i$ $(i = 1–4)$ is given by

$$\mathcal{I}_1 v(x,\eta) = -\int_0^{\eta} \int_0^{\mathscr{E}(x,\eta-z)\cdot z} \hat{\mathcal{B}}[\beta]_\eta(-\tan(\mathscr{A}(x,\eta-z)), \mathscr{E}(x,\eta-z)\cdot z - t)$$

$$\times v_x(-\tan(\mathscr{A}(x,\eta-z)), t)\, dt\, \mathscr{E}(x,\eta-z)\, dz,$$

$$\mathcal{I}_2 v(x,\eta) = -\int_0^{\eta} \hat{\mathcal{B}}[b]_\eta(-\tan(\mathscr{A}(x,\eta-z)), 0)\cdot \mathscr{E}(x,\eta-z)\cdot z$$

$$\times v(-\tan(\mathscr{A}(x,\eta-z)), \mathscr{E}(x,\eta-z)\cdot z)\cdot \mathscr{E}(x,\eta-z)\, dz,$$

$$\mathcal{I}_3 v(x,\eta) = -\int_0^{\eta} \int_0^{\mathscr{E}(x,\eta-z)\cdot z} \hat{\mathcal{B}}[b]_{\eta\eta}(-\tan(\mathscr{A}(x,\eta-z)), \mathscr{E}(x,\eta-z)\cdot z - t)\cdot t$$

$$\times v(-\tan(\mathscr{A}(x,\eta-z)), t)\, dt\, \mathscr{E}(x,\eta-z)\, dz,$$

$$\mathcal{I}_4 v(x,\eta) = \int_0^{\eta} \int_0^{\mathscr{E}(x,\eta-z)\cdot z} \hat{\mathcal{B}}[b]_\eta(-\tan(\mathscr{A}(x,\eta-z)), \mathscr{E}(x,\eta-z)\cdot z - t)$$

$$\times v(-\tan(\mathscr{A}(x,\eta-z)), t)\, dt\, \mathscr{E}(x,\eta-z)\, dz.$$

Furthermore, for $\mathcal{I}_1 v(x,\eta)$, $\mathcal{I}_3 v(x,\eta)$ and $\mathcal{I}_4 v(x,\eta)$, we practice an integration by substitution

$$t(s) = \mathscr{E}(x,\eta-z)\cdot s. \tag{2.10}$$

In addition, we transform $\mathcal{I}_1 v(x,\eta)$ as follows: Let us apply equality $\mathscr{A}_\eta(x,\eta) = \mathscr{E}(x,\eta)$ and $\mathscr{E}_\eta(x,\eta) = \mathscr{E}(x,\eta)^2 \cdot \tan(\mathscr{A}(x,\eta))$. By an integration by substitution (2.10), we have

$$\begin{aligned}\mathcal{I}_1 v(x,\eta) &= -\int_0^{\eta} \int_0^{z} \hat{\mathcal{B}}[\beta]_\eta(-\tan(\mathscr{A}(x,\eta-z)), \mathscr{E}(x,\eta-z)\cdot(z-s)) \\ &\quad \times v_x(-\tan(\mathscr{A}(x,\eta-z)), \mathscr{E}(x,\eta-z)\cdot s)\, ds\, \mathscr{E}(x,\eta-z)^2\, dz \\ &= \mathcal{I}_1{}' v(x,\eta) + \mathcal{I}_1{}'' v(x,\eta),\end{aligned}$$

where

$$
\begin{aligned}
&\mathcal{I}_1{}'v(x,\eta)\\
&= -\int_0^\eta\int_0^z \cos^2\left(\mathscr{A}(x,\eta-z)\right)\cdot\hat{\mathcal{B}}[\beta]_\eta(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot(z-s))\\
&\quad\times\mathscr{E}(x,\eta-z)\cdot\frac{\partial}{\partial z}\Big\{v\big(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot s\big)\Big\}\,ds\,dz,\\
&\mathcal{I}_1{}''v(x,\eta)\\
&= -\int_0^\eta\int_0^z \cos^2\left(\mathscr{A}(x,\eta-z)\right)\cdot\tan\left(\mathscr{A}(x,\eta-z)\right)\cdot s\\
&\quad\times\hat{\mathcal{B}}[\beta]_\eta(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot(z-s))\\
&\quad\times\mathscr{E}(x,\eta-z)^2\cdot\frac{\partial}{\partial s}\Big\{v(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot s)\Big\}\,ds\,dz.
\end{aligned}
$$

By changing the order of integration, we write $\mathcal{I}_1{}'v(x,\eta)$ as

$$
-\int_0^\eta\int_0^z \ldots\,ds\,dz = -\int_0^\eta\int_s^\eta \ldots\,dz\,ds.
$$

Thereupon, we can transform $\mathcal{I}_1{}'v(x,\eta)$ by an integration by parts. Thereafter, let us change the order of integration again. $\mathcal{I}_1{}''v(x,\eta)$ can be transformed directly by an integration by parts. Ultimately, we see that (2.4) is equivalent to the following integral equation:

$$
\begin{aligned}
v(x,\eta) &= f(-\tan\left(\mathscr{A}(x,\eta)\right),0)\\
&\quad+\int_0^\eta g(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot z)\cdot\mathscr{E}(x,\eta-z)\,dz\\
&\quad+\sum_{i=1}^{8} I_i v(x,\eta),
\end{aligned}
\tag{2.11}
$$

where

$$
\begin{aligned}
I_1 v(x,\eta) &= -\frac{1}{1+x^2}\int_0^{\eta} \hat{\mathcal{B}}[\beta]_{\eta}(x,\eta-z)v(x,z)\,dz,\\
I_2 v(x,\eta) &= \int_0^{\eta} \cos^2\left(\mathscr{A}(x,\eta-z)\right)\cdot\{1-\tan\left(\mathscr{A}(x,\eta-z)\right)\cdot\mathscr{E}(x,\eta-z)\cdot z\}\\
&\quad\times \hat{\mathcal{B}}[\beta]_{\eta}(-\tan\left(\mathscr{A}(x,\eta-z)\right),0)\\
&\quad\times v(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot z)\cdot\mathscr{E}(x,\eta-z)\,dz,\\
I_3 v(x,\eta) &= \int_0^{\eta}\int_0^{z} \cos^2\left(\mathscr{A}(x,\eta-z)\right)\cdot\{1-\tan\left(\mathscr{A}(x,\eta-z)\right)\cdot\mathscr{E}(x,\eta-z)\cdot z\}\\
&\quad\times \hat{\mathcal{B}}[\beta]_{\eta\eta}(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot(z-s))\\
&\quad\times v(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot s)\cdot\mathscr{E}(x,\eta-z)^2\,ds\,dz,\\
I_4 v(x,\eta) &= 2\int_0^{\eta}\int_0^{z} \cos^2\left(\mathscr{A}(x,\eta-z)\right)\cdot\tan\left(\mathscr{A}(x,\eta-z)\right)\\
&\quad\times \hat{\mathcal{B}}[\beta]_{\eta}(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot(z-s))\\
&\quad\times v(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot s)\cdot\mathscr{E}(x,\eta-z)^2\,ds\,dz,\\
I_5 v(x,\eta) &= \int_0^{\eta}\int_0^{z} \hat{\mathcal{B}}[\beta]_{x\eta}(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot(z-s))\\
&\quad\times v(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot s)\cdot\mathscr{E}(x,\eta-z)^2\,ds\,dz,\\
I_6 v(x,\eta) &= \mathcal{I}_2 v(x,\eta)\\
&= -\int_0^{\eta} z\cdot\hat{\mathcal{B}}[b]_{\eta}(-\tan\left(\mathscr{A}(x,\eta-z)\right),0)\\
&\quad\times v(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot z)\cdot\mathscr{E}(x,\eta-z)^2\,dz,\\
I_7 v(x,\eta) &= \mathcal{I}_3 v(x,\eta)\\
&= -\int_0^{\eta}\int_0^{z} s\cdot\hat{\mathcal{B}}[b]_{\eta\eta}(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot(z-s))\\
&\quad\times v(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot s)\cdot\mathscr{E}(x,\eta-z)^3\,ds\,dz,\\
I_8 v(x,\eta) &= \mathcal{I}_4 v(x,\eta)\\
&= \int_0^{\eta}\int_0^{z} \hat{\mathcal{B}}[b]_{\eta}(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot(z-s))\\
&\quad\times v(-\tan\left(\mathscr{A}(x,\eta-z)\right),\mathscr{E}(x,\eta-z)\cdot s)\cdot\mathscr{E}(x,\eta-z)^2\,ds\,dz.
\end{aligned}
$$

We note that $\mathcal{I}_1 v(x,\eta) = \sum_{i=1}^{5} I_i v(x,\eta)$.

By the above argument, the proof of Theorem 1.6 has been reduced to that of the following proposition.

Proposition 2.1. *The solution* $v(x,\eta)$ *of* (2.11) *satisfies condition* (BS).

3. PROOF OF PROPOSITION 2.1

In order to prove Proposition 2.1, we employ the iteration method. Let us define $\{v_n(x,\eta)\}_{n=0}^{\infty}$ inductively as follows:

$$\begin{aligned} v_0(x,\eta) = {} & f(-\tan(\mathscr{A}(x,\eta)),0) \\ & + \int_0^{\eta} g(-\tan(\mathscr{A}(x,\eta-z)),\mathscr{E}(x,\eta-z)\cdot z)\cdot\mathscr{E}(x,\eta-z)\,dz, \end{aligned} \tag{3.1}$$

$$v_{n+1}(x,\eta) = v_0(x,\eta) + \sum_{i=1}^{8} I_i v_n(x,\eta) \quad (n \geq 0). \tag{3.2}$$

Next, we define $\{w_n(x,\eta)\}_{n=0}^{\infty}$ as $w_0(x,\eta) = v_0(x,\eta)$ and $w_n(x,\eta) = v_n(x,\eta) - v_{n-1}(x,\eta)$ $(n \geq 1)$. Furthermore, we define $\{W_n(x,\eta,t)\}_{n=0}^{\infty}$ as

$$W_n(x,\eta,t) = w_n(-\tan(\mathscr{A}(x,\eta-t)),\mathscr{E}(x,\eta-t)\cdot t). \tag{3.3}$$

Here, we break the proof, and provide the notation needed in the key lemma later. We can take $r_0 > 0$, $\kappa_0 > 0$ and $l > 0$ such that

$$\begin{aligned} & |x| \leq r_0,\ \zeta \in S(\theta,\kappa_0,+\infty) - le^{i\theta} \\ & \quad \Longrightarrow -\sin(\arctan x) + \cos(\arctan x)\cdot\zeta \in B(r) \cup S(\theta,\kappa,+\infty), \end{aligned} \tag{3.4}$$

where $r > 0$ and $\kappa > 0$ are the constants given in Assumptions (Subsection 1.3). Hence, it holds that

$$|x| \leq r_0,\ \ \zeta \in S(\theta,\kappa_0,+\infty) - le^{i\theta} \ \Longrightarrow\ \mathscr{A}(x,\zeta) \in \arcsin(B(r) \cup S(\theta,\kappa,+\infty)). \tag{3.5}$$

Consequently, we obtain

$$|x| \leq r_0,\ \ \zeta \in S(\theta,\kappa_0,+\infty) - le^{i\theta} \ \Longrightarrow\ \begin{cases} |\cos(\mathscr{A}(x,\zeta))| \geq \mathsf{M} \quad \text{and} \\ -\tan(\mathscr{A}(x,\zeta)) \in \Omega_{r,\theta,\kappa}, \end{cases} \tag{3.6}$$

where M is the constant given in (1.20). It follows from (1.22)–(1.25) and (3.6) that there exist some positive constants C_0 and M_0 satisfying:

$$\begin{cases} \max_{|x|\le r_0} \left|\cos^2(\mathscr{A}(x,\zeta))\cdot\beta_m(-\tan(\mathscr{A}(x,\zeta)))\cdot\mathscr{E}(x,\zeta)^m\right| \le C_0{M_0}^m, \\ \max_{|x|\le r_0} \left|\dfrac{\partial}{\partial\zeta}\Big\{\cos^2(\mathscr{A}(x,\zeta))\cdot\beta_m(-\tan(\mathscr{A}(x,\zeta)))\Big\}\cdot\mathscr{E}(x,\zeta)^m\right| \le C_0{M_0}^m, \\ \max_{|x|\le r_0} \left|\cos^2(\mathscr{A}(x,\zeta))\cdot\tan(\mathscr{A}(x,\zeta))\cdot\beta_m(-\tan(\mathscr{A}(x,\zeta)))\cdot\mathscr{E}(x,\zeta)^{m+1}\right| \\ \qquad\qquad \le \dfrac{C_0{M_0}^m}{(1+|\zeta|)^p}, \\ \max_{|x|\le r_0} \left|b_m(-\tan(\mathscr{A}(x,\zeta)))\cdot\mathscr{E}(x,\zeta)^{m+1}\right| \le \dfrac{C_0{M_0}^m}{(1+|\zeta|)^p} \quad (\le C_0{M_0}^m) \end{cases} \tag{3.7}$$

for all $m = 1, 2, \dots$ and all $\zeta \in S(\theta,\kappa_0,+\infty) - l'e^{i\theta}$, where $l' = l/2$.

Next, we give the following definition.

Definition 3.1.

(1) For $\lambda \ge 0$ and $\rho > 0$, $U_\rho[0,\lambda]$ denotes the ρ-neighborhood of $[0,\lambda]$ in $\mathbb{C}$. Precisely,

$$U_\rho[0,\lambda] = \{\tau\in\mathbb{C};\ \operatorname{dist}(\tau,[0,\lambda]) \equiv \inf\{|\tau-\sigma|;\ \sigma\in[0,\lambda]\} < \rho\}.$$

(2) For $\eta \in \mathbb{C}$, we define the function G^η as

$$G^\eta(\tau) = \tau e^{i\arg(\eta)}, \quad \tau\in\mathbb{C},$$

and define $\mathcal{G}^\eta$ and $\mathcal{G}^\eta_\rho$ as follows:

$$\mathcal{G}^\eta = \{G^\eta(R)\in\mathbb{C};\ 0\le R\le|\eta|\},$$
$$\mathcal{G}^\eta_\rho = \{G^\eta(\tau)\in\mathbb{C};\ \tau\in U_\rho[0,|\eta|]\}.$$

We remark that $\mathcal{G}^\eta$ is the segment from 0 to η and that $\mathcal{G}^\eta_\rho$ is the ρ-neighborhood of $\mathcal{G}^\eta$.

Under these preparations let us take a monotonically decreasing positive sequence $(l_n)_{n=0}^\infty$ so that

$$l' = \sum_{n=0}^\infty l_n. \tag{3.8}$$

Finally, we define $(\rho_n)_{n=0}^\infty$ as follows:

$$\rho_0 = \operatorname{dist}(S(\theta,\kappa_0,+\infty) - (l'-l_0)e^{i\theta}, \partial(S(\theta,\kappa_0,+\infty) - l'e^{i\theta})),$$

$$\rho_n = \operatorname{dist}\left(S(\theta,\kappa_0,+\infty) - \Big(l' - \sum_{j=0}^n l_j\Big)e^{i\theta}, \partial(S(\theta,\kappa_0,+\infty) - \Big(l' - \sum_{j=0}^{n-1} l_j\Big)e^{i\theta})\right) \quad (n\ge 1),$$

where $\operatorname{dist}(\mathsf{A},\mathsf{B}) = \inf\{|\mathsf{a}-\mathsf{b}|;\ \mathsf{a}\in\mathsf{A},\ \mathsf{b}\in\mathsf{B}\}$, and $\partial\mathsf{A}$ means the boundary of A. Then we obtain the following lemma.

Lemma 3.2. *$W_n(x,\eta,t)$ is continued analytically to*

$$\Big\{(x,\eta,t)\in\mathbb{C}^3;\ |x|\le r_0,\ \eta\in S(\theta,\kappa_0,+\infty)-\Big(l'-\sum_{j=0}^{n} l_j\Big)e^{i\theta},\ t\in\mathcal{G}^{\eta}_{\rho_n}\Big\}.$$

Moreover, on

$$\Big\{(x,\eta,t)\in\mathbb{C}^3;\ |x|\le r_0,\ \eta\in S(\theta,\kappa_0,+\infty)-\Big(l'-\sum_{j=0}^{n} l_j\Big)e^{i\theta},\ t\in\mathcal{G}^{\eta}\Big\}$$

we have the following estimate. For some positive constants $C_1>0$ and $\delta_1>\max\{1,M_0\}$,

$$|W_n(x,\eta,G^{\eta}(R))|\le C_1e^{\delta_1|\eta|}{M_1}^n\Big\{\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(p-1)^k}\frac{1}{(1+|\eta|-R)^{k(p-1)}}\Big\}\frac{R^n}{n!},\quad 0\le R\le|\eta|, \tag{3.9}$$

where

$$M_1=10C_0\sum_{m=1}^{\infty}\frac{{M_0}^m}{{\delta_1}^{m-1}}.$$

We prove Lemma 3.2 in Section 4. For the present, we admit it and let us accomplish the proof of Proposition 2.1.

Proof of Proposition 2.1. By Lemma 3.2, we see that $w_n(x,\eta)$ $(=W_n(x,\eta,\eta))$ is continued analytically to $B(r_0)\times\{S(\theta,\kappa_0,+\infty)-(l'-\sum_{j=0}^{n}l_j)e^{i\theta}\}$ with the estimate

$$\begin{aligned}|w_n(x,\eta)|=|W_n(x,\eta,G^{\eta}(|\eta|))|&\le C_1e^{\delta_1|\eta|}{M_1}^n\Big\{\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(p-1)^k}\Big\}\frac{|\eta|^n}{n!}\\&=C_1e^{\delta_1|\eta|}{M_1}^n\Big(1+\frac{1}{p-1}\Big)^n\frac{|\eta|^n}{n!}.\end{aligned}$$

Hence, on $B(r_0)\times S(\theta,\kappa_0,+\infty)$ we obtain

$$\sum_{n=0}^{\infty}|w_n(x,\eta)|\le C_1e^{\delta_1|\eta|}\sum_{n=0}^{\infty}{M_1}^n\Big(1+\frac{1}{p-1}\Big)^n\frac{|\eta|^n}{n!}=C_1e^{\delta_2|\eta|},$$

where $\delta_2=\delta_1+M_1(1+1/(p-1))$. This shows that $v_n(x,\eta)$ $(=\sum_{k=0}^{n}w_k(x,\eta))$ converges to the solution $V(x,\eta)$ of (2.11) uniformly on $B(r_0)\times S(\theta,\kappa_0,+\infty)$. Consequently, $V(x,\eta)$ is an analytic continuation of $v(x,\eta)$, and it holds that

$$\max_{|x|\le r_0}|V(x,\eta)|\le C_1e^{\delta_2|\eta|},\quad \eta\in S(\theta,\kappa_0,+\infty).$$

It follows from the above argument that $v(x,\eta)$ satisfies (BS). This completes the proof of Proposition 2.1. □

4. PROOF OF LEMMA 3.2

Let us prove Lemma 3.2. It is proved by induction with respect to n.

Proof of Lemma 3.2. First we consider the case $n = 0$. Let us apply equality

$$\begin{aligned}
&\mathscr{A}(-\tan(\mathscr{A}(x,\eta-t)),\mathscr{E}(x,\eta-t)\cdot t) = \mathscr{A}(x,\eta),\\
&\mathscr{A}(-\tan(\mathscr{A}(x,\eta-t)),\mathscr{E}(x,\eta-t)\cdot(t-z)) = \mathscr{A}(x,\eta-z),\\
&\mathscr{E}(-\tan(\mathscr{A}(x,\eta-t)),\mathscr{E}(x,\eta-t)\cdot(t-z)) = \frac{\mathscr{E}(x,\eta-z)}{\mathscr{E}(x,\eta-t)}.
\end{aligned}$$

Then, by (3.1) and (3.3), we see that $W_0(x,\eta,t)$ has the following form:

$$\begin{aligned}
W_0(x,\eta,t) &= f(-\tan(\mathscr{A}(x,\eta)),0)\\
&\quad+\int_0^t g(-\tan(\mathscr{A}(x,\eta-z)),\mathscr{E}(x,\eta-z)\cdot z)\cdot\mathscr{E}(x,\eta-z)\,dz\\
&\equiv J_1(x,\eta,t)+J_2(x,\eta,t).
\end{aligned}$$

Before proving the lemma for W_0, we remark the following. It follows from assumption (A2) and Cauchy's integral formula that $g(x,\eta)$ is analytic on $\Omega_{r,\theta,\kappa}\times\mathbb{C}$ with the estimate

$$|g(x,\eta)| \le C'\exp[\delta|\sin(\arctan x)|]\cdot e^{\delta'|\eta|},\quad (x,\eta)\in\Omega_{r,\theta,\kappa}\times\mathbb{C} \tag{4.1}$$

for some positive constants C' and δ'.

Let us prove that $J_1(x,\eta,t)$ and $J_2(x,\eta,t)$ are well-defined on

$$\{(x,\eta,t);\ |x|\le r_0,\ \eta\in S(\theta,\kappa_0,+\infty)-(l'-l_0)e^{i\theta},\ t\in\mathcal{G}^\eta_{\rho_0}\}.$$

Let $|x|\le r_0$, $\eta\in S(\theta,\kappa_0,+\infty)-(l'-l_0)e^{i\theta}$, $t\in\mathcal{G}^\eta_{\rho_0}$, and let us write $t\in\mathcal{G}^\eta_{\rho_0}$ as $t=G^\eta(\tau)$ $(\tau\in U_{\rho_0}[0,|\eta|])$.

Well-definedness of $J_1(x,\eta,G^\eta(\tau))$: it is clear from assumption (A2) and (3.6) (note that $S(\theta,\kappa_0,+\infty)-(l'-l_0)e^{i\theta}\subset S(\theta,\kappa_0,+\infty)-l'e^{i\theta}\subset S(\theta,\kappa_0,+\infty)-le^{i\theta}$).

Well-definedness of $J_2(x,\eta,G^\eta(\tau))$: in the integral expression of $J_2(x,\eta,G^\eta(\tau))$, by taking a path of integration as

$$z(\sigma)=\sigma e^{i\arg(\eta)}\quad(\sigma\in[0,\tau]), \tag{4.2}$$

where $[0,\tau]$ is the segment from 0 to τ, it holds that $\eta-z(\sigma)\in S(\theta,\kappa_0,+\infty)-l'e^{i\theta}$ $(\subset S(\theta,\kappa_0,+\infty)-le^{i\theta})$. Hence, from the above remark and (3.6) we obtain the well-definedness of $J_2(x,\eta,G^\eta(\tau))$.

Therefore, $W_0(x,\eta,t)$ is well-defined on

$$\{(x,\eta,t);\ |x|\le r_0,\ \eta\in S(\theta,\kappa_0,+\infty)-(l'-l_0)e^{i\theta},\ t\in\mathcal{G}^\eta_{\rho_0}\}.$$

Moreover, on

$$\{(x,\eta,t);\ |x|\le r_0,\ \eta\in S(\theta,\kappa_0,+\infty)-(l'-l_0)e^{i\theta},\ t\in\mathcal{G}^\eta\}$$

we have the following representation:

$$
\begin{aligned}
&W_0(x,\eta,G^\eta(R))\\
&= f(-\tan(\mathscr{A}(x,\eta)),0)\\
&\quad+\int_0^R g(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)})),\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})\cdot R_1e^{i\arg(\eta)})\\
&\qquad\quad\times\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})\cdot e^{i\arg(\eta)}\,dR_1\\
&\equiv \mathscr{J}_1(x,\eta,R)+\mathscr{J}_2(x,\eta,R).
\end{aligned}
$$

Let us estimate each $\mathscr{J}_1(x,\eta,R)$ and $\mathscr{J}_2(x,\eta,R)$.

On $\mathscr{J}_1(x,\eta,R)$: by (1.21), we have

$$
\begin{aligned}
|\mathscr{J}_1(x,\eta,R)| &= |f(-\tan(\mathscr{A}(x,\eta)),0)| \le C\exp[\delta|\sin(\mathscr{A}(x,\eta))|]\\
&= C\exp[\delta|-\sin(\arctan x)+\cos(\arctan x)\cdot\eta|]\\
&\le C\exp[\delta|\sin(\arctan x)|]\cdot\exp[\delta|\cos(\arctan x)|\cdot|\eta|]\le C''e^{\delta_1|\eta|},
\end{aligned}
$$

where $C''=C\exp[\delta\max_{|x|\le r_0}|\sin(\arctan x)|]$ and $\delta_1=\delta\max_{|x|\le r_0}|\cos(\arctan x)|$.

On $\mathscr{J}_2(x,\eta,R)$: it follows from (4.1) that

$$
\begin{aligned}
&|g(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}),\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})\cdot R_1e^{i\arg(\eta)})|\\
&\le C'\exp[\delta|\sin(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}))|]\\
&\quad\times\exp[\delta'|\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})\cdot R_1e^{i\arg(\eta)}|]\\
&\le C'''e^{\delta_1(|\eta|-R_1)}\cdot\exp[\delta'|\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})|R_1],
\end{aligned}
$$

where $C'''=C'\exp[\delta\max_{|x|\le r_0}|\sin(\arctan x)|]$. We remark that

$$
\max_{|x|\le r_0}|\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})|\le M\equiv\frac{1}{\mathsf{M}}\max_{|x|\le r_0}|\cos(\arctan x)| \tag{4.3}
$$

holds. Therefore, we have

$$
\begin{aligned}
&|g(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}),\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})\cdot R_1e^{i\arg(\eta)})|\\
&\quad\le C'''e^{\delta_1|\eta|}e^{-(\delta_1-\delta'M)R_1}.
\end{aligned}
$$

Here we may take $\delta>0$ so large that $\delta''\equiv\delta_1-\delta'M=\delta\max_{|x|\le r_0}|\cos(\arctan x)|-\delta'M>0$. Hence, we obtain

$$
|\mathscr{J}_2(x,\eta,R)|\le C'''Me^{\delta_1|\eta|}\int_0^R e^{-\delta''R_1}\,dR_1\le\frac{C'''M}{\delta''}e^{\delta_1|\eta|}.
$$

By the above argument, it holds that

$$
|W_0(x,\eta,G^\eta(R))|\le C_1e^{\delta_1|\eta|},
$$

where $C_1 = C'' + (C'''M)/\delta''$. We may take δ so large that $\delta_1 > \max\{1, M_0\}$. Therefore, the lemma has been proved for W_0.

Next, we assume that the claim of the lemma is proved up to n and prove it for $n+1$.

By (3.2) and (3.3), we have the following relation between W_n and W_{n+1}:

$$W_{n+1}(x,\eta,t) = \sum_{i=1}^{7} \mathscr{I}_i W_n(x,\eta,t), \tag{4.4}$$

where

$$\begin{aligned}
&\mathscr{I}_1 W_n(x,\eta,t) = (I_1 w_n)(-\tan(\mathscr{A}(x,\eta-t)), \mathscr{E}(x,\eta-t)\cdot t) \\
&= -\cos^2(\mathscr{A}(x,\eta-t)) \int_0^t \hat{\mathcal{B}}[\beta]_\eta(-\tan(\mathscr{A}(x,\eta-t)), \mathscr{E}(x,\eta-t)\cdot(t-z)) \\
&\quad \times W_n(x,\eta-t+z,z)\cdot \mathscr{E}(x,\eta-t)\,dz,
\end{aligned}$$

$$\begin{aligned}
&\mathscr{I}_2 W_n(x,\eta,t) = (I_2 w_n)(-\tan(\mathscr{A}(x,\eta-t)), \mathscr{E}(x,\eta-t)\cdot t) \\
&= \int_0^t \cos^2(\mathscr{A}(x,\eta-z))\cdot\{1-\tan(\mathscr{A}(x,\eta-z))\cdot\mathscr{E}(x,\eta-z)\cdot z\} \\
&\quad \times \hat{\mathcal{B}}[\beta]_\eta(-\tan(\mathscr{A}(x,\eta-z)), 0)\cdot W_n(x,\eta,z)\cdot\mathscr{E}(x,\eta-z)\,dz,
\end{aligned}$$

$$\begin{aligned}
&\mathscr{I}_3 W_n(x,\eta,t) = (I_3 w_n)(-\tan(\mathscr{A}(x,\eta-t)), \mathscr{E}(x,\eta-t)\cdot t) \\
&= \int_0^t \int_0^z \cos^2(\mathscr{A}(x,\eta-z))\cdot\{1-\tan(\mathscr{A}(x,\eta-z))\cdot\mathscr{E}(x,\eta-z)\cdot z\} \\
&\quad \times \hat{\mathcal{B}}[\beta]_{\eta\eta}(-\tan(\mathscr{A}(x,\eta-z)), \mathscr{E}(x,\eta-z)\cdot(z-s)) \\
&\quad \times W_n(x,\eta-z+s,s)\cdot\mathscr{E}(x,\eta-z)^2\,ds\,dz,
\end{aligned}$$

$$\begin{aligned}
&\mathscr{I}_4 W_n(x,\eta,t) = (I_4 w_n + I_5 w_n)(-\tan(\mathscr{A}(x,\eta-t)), \mathscr{E}(x,\eta-t)\cdot t) \\
&= -\int_0^t \int_0^z \frac{\partial}{\partial\zeta}\Big\{\cos^2(\mathscr{A}(x,\zeta)) \\
&\quad \times \hat{\mathcal{B}}[\beta]_\eta(-\tan(\mathscr{A}(x,\zeta)), \mathscr{E}(x,\eta-z)\cdot(z-s))\Big\}\Big|_{\zeta=\eta-z} \\
&\quad \times W_n(x,\eta-z+s,s)\cdot\mathscr{E}(x,\eta-z)\,ds\,dz,
\end{aligned}$$

$$\begin{aligned}
&\mathscr{I}_5 W_n(x,\eta,t) = (I_6 w_n)(-\tan(\mathscr{A}(x,\eta-t)), \mathscr{E}(x,\eta-t)\cdot t)\\
&= -\int_0^t z\cdot \hat{\mathcal{B}}[b]_\eta(-\tan(\mathscr{A}(x,\eta-z)),0)\cdot W_n(x,\eta,z)\cdot \mathscr{E}(x,\eta-z)^2\,dz,\\
&\mathscr{I}_6 W_n(x,\eta,t) = (I_7 w_n)(-\tan(\mathscr{A}(x,\eta-t)), \mathscr{E}(x,\eta-t)\cdot t)\\
&= -\int_0^t\int_0^z s\cdot \hat{\mathcal{B}}[b]_{\eta\eta}(-\tan(\mathscr{A}(x,\eta-z)), \mathscr{E}(x,\eta-z)\cdot(z-s))\\
&\quad\times W_n(x,\eta-z+s,s)\cdot \mathscr{E}(x,\eta-z)^3\,ds\,dz,\\
&\mathscr{I}_7 W_n(x,\eta,t) = (I_8 w_n)(-\tan(\mathscr{A}(x,\eta-t)), \mathscr{A}(x,\eta-t)\cdot t)\\
&= \int_0^t\int_0^z \hat{\mathcal{B}}[b]_\eta(-\tan(\mathscr{A}(x,\eta-z)), \mathscr{E}(x,\eta-z)\cdot(z-s))\\
&\quad\times W_n(x,\eta-z+s,s)\cdot \mathscr{E}(x,\eta-z)^2\,ds\,dz.
\end{aligned}$$

Let us prove each $\mathscr{I}_i W_n(x,\eta,t)$ $(i=1,\dots,7)$ is well-defined on

$$\{(x,\eta,t);\ |x|\le r_0,\ \eta\in S(\theta,\kappa_0,+\infty)-(l'-\textstyle\sum_{j=0}^{n+1} l_j)e^{i\theta},\ t\in\mathcal{G}^\eta_{\rho_{n+1}}\}.$$

Let $|x|\le r_0$, $\eta\in S(\theta,\kappa_0,+\infty)-(l'-\sum_{j=0}^{n+1} l_j)e^{i\theta}$, $t\in\mathcal{G}^\eta_{\rho_{n+1}}$, and let us write $t\in\mathcal{G}^\eta_{\rho_{n+1}}$ as $t=G^\eta(\tau)$ $(\tau\in U_{\rho_{n+1}}[0,|\eta|])$.

On $\mathscr{I}_1 W_n(x,\eta,G^\eta(\tau))$: let us take a path of integration as

$$z(\sigma)=\sigma e^{i\arg(\eta)}\quad(\sigma\in[0,\tau]). \tag{4.5}$$

Then we have $\eta-G^\eta(\tau)+z(\sigma)\in S(\theta,\kappa_0,+\infty)-(l'-\sum_{j=0}^{n} l_j)e^{i\theta}$ and $z(\sigma)\in\mathcal{G}^{\eta-G^\eta(\tau)+z(\sigma)}_{\rho_{n+1}}$ $(\subset G^{\eta-G^\eta(\tau)+z(\sigma)}_{\rho_n})$. Hence, $W_n(x,\eta-G^\eta(\tau)+z(\sigma),z(\sigma))$ is well-defined, which implies the well-definedness of $\mathscr{I}_1 W_n(x,\eta,G^\eta(\tau))$.

On $\mathscr{I}_2 W_n(x,\eta,G^\eta(\tau))$ and $\mathscr{I}_5 W_n(x,\eta,G^\eta(\tau))$: let us take a path of integration as (4.5). Then we have $\eta\in S(\theta,\kappa_0,+\infty)-(l'-\sum_{j=0}^{n} l_j)e^{i\theta}$ and $z(\sigma)\in\mathcal{G}^\eta_{\rho_n}$. Hence, $W_n(x,\eta,z(\sigma))$ is well-defined. Therefore, it follows that $\mathscr{I}_2 W_n(x,\eta,G^\eta(\tau))$ and $\mathscr{I}_5 W_n(x,\eta,G^\eta(\tau))$ are well-defined.

On $\mathscr{I}_i W_n(x,\eta,G^\eta(\tau))$ $(i=3,4,6,7)$: we only state paths of integration. By taking paths of integration as

$$\begin{cases} z(\sigma)=\sigma e^{i\arg(\eta)} & (\sigma\in[0,\tau]),\\ s(\lambda)=\lambda e^{i\arg(\eta)} & (\lambda\in[0,\sigma]),\end{cases} \tag{4.6}$$

we see all $\mathscr{I}_i W_n(x,\eta,G^\eta(\tau))$ $(i=3,4,6,7)$ are well-defined.

Hence, $W_{n+1}(x,\eta,t)$ is well-defined on

$$\left\{(x,\eta,t);\ |x|\le r_0,\ \eta\in S(\theta,\kappa_0,+\infty)-\left(l'-\textstyle\sum_{j=0}^{n+1} l_j\right)e^{i\theta},\ t\in\mathcal{G}^\eta_{\rho_{n+1}}\right\}.$$

Moreover, on

$$\{(x,\eta,t);\ |x|\le r_0,\ \eta\in S(\theta,\kappa_0,+\infty)-(l'-\textstyle\sum_{j=0}^{n+1}l_j)e^{i\theta},\ t\in\mathcal{G}^\eta\}$$

we have the following representations:

$$\begin{aligned}
&\mathscr{I}_1 W_n(x,\eta,G^\eta(R))\\
&=-\cos^2(\mathscr{A}(x,(|\eta|-R)e^{i\arg(\eta)}))\\
&\quad\times\int_0^R \hat{\mathcal{B}}[\beta]_\eta(-\tan(\mathscr{A}(x,(|\eta|-R)e^{i\arg(\eta)})),\\
&\quad\mathscr{E}(x,(|\eta|-R)e^{i\arg(\eta)})\cdot(R-R_1)e^{i\arg(\eta)})\\
&\quad\times\mathscr{W}_n(x,\eta,R,R_1)\cdot\mathscr{E}(x,(|\eta|-R)e^{i\arg(\eta)})\cdot e^{i\arg(\eta)}\,dR_1,\\
&\mathscr{I}_2 W_n(x,\eta,G^\eta(R))\\
&=\int_0^R\cos^2(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}))\\
&\quad\times\{1-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}))\cdot\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})\cdot R_1e^{i\arg(\eta)}\}\\
&\quad\times\hat{\mathcal{B}}[\beta]_\eta(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)})),0)\\
&\quad\times\mathscr{W}_n(x,\eta,R_1,R_1)\cdot\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})\cdot e^{i\arg(\eta)}\,dR_1,\\
&\mathscr{I}_3 W_n(x,\eta,G^\eta(R))\\
&=\int_0^R\int_0^{R_1}\cos^2(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}))\\
&\quad\times\{1-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}))\cdot\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})\cdot R_1e^{i\arg(\eta)}\}\\
&\quad\times\hat{\mathcal{B}}[\beta]_{\eta\eta}(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)})),\\
&\quad\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})\cdot(R_1-R_2)e^{i\arg(\eta)})\\
&\quad\times\mathscr{W}_n(x,\eta,R_1,R_2)\cdot\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})^2\cdot\{e^{i\arg(\eta)}\}^2\,dR_2\,dR_1,\\
&\mathscr{I}_4 W_n(x,\eta,G^\eta(R))\\
&=\int_0^R\int_0^{R_1}\frac{\partial}{\partial\zeta}\Big\{\cos^2(\mathscr{A}(x,\zeta))\cdot\hat{\mathcal{B}}[\beta]_\eta(-\tan(\mathscr{A}(x,\zeta)),\\
&\quad\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})\cdot(R_1-R_2)e^{i\arg(\eta)})\Big\}\Big|_{\zeta=(|\eta|-R_1)e^{i\arg(\eta)}}\\
&\quad\times\mathscr{W}_n(x,\eta,R_1,R_2)\cdot\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})\cdot\{e^{i\arg(\eta)}\}^2\,dR_2\,dR_1,\\
&\mathscr{I}_5 W_n(x,\eta,G^\eta(R))\\
&=-\int_0^R R_1e^{i\arg(\eta)}\cdot\hat{\mathcal{B}}[b]_\eta(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)})),0)\\
&\quad\times\mathscr{W}_n(x,\eta,R_1,R_1)\cdot\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})^2\cdot e^{i\arg(\eta)}\,dR_2\,dR_1,
\end{aligned}$$

$$
\begin{aligned}
&\mathscr{I}_6 W_n(x,\eta,G^\eta(R))\\
&= -\int_0^R\int_0^{R_1} R_2 e^{i\arg(\eta)}\cdot \hat{\mathcal{B}}[b]_{\eta\eta}(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)})),\\
&\quad \mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})\cdot(R_1-R_2)e^{i\arg(\eta)})\\
&\quad \times \mathscr{W}_n(x,\eta,R_1,R_2)\cdot \mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})^3\cdot\{e^{i\arg(\eta)}\}^2\,dR_2\,dR_1,\\
&\mathscr{I}_7 W_n(x,\eta,G^\eta(R))\\
&= \int_0^R\int_0^{R_1} \hat{\mathcal{B}}[b]_{\eta}(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)})),\\
&\quad \mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})\cdot(R_1-R_2)e^{i\arg(\eta)})\\
&\quad \times \mathscr{W}_n(x,\eta,R_1,R_2)\cdot \mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})^2\cdot\{e^{i\arg(\eta)}\}^2\,dR_1\,dR_1,
\end{aligned}
\tag{4.7}
$$

where

$$
\mathscr{W}_n(x,\eta,\mu,\nu) = W_n(x,(|\eta|-\mu+\nu)e^{i\arg(\eta)},G^{(|\eta|-\mu+\nu)e^{i\arg(\eta)}}(\nu)). \tag{4.8}
$$

Here we note the following fact: in general, let

$$
\Phi(x,y)=\sum_{m=0}^{\infty}\Phi_m(x)y^m \quad\text{and}\quad \hat{\mathcal{B}}[\Phi](x,\eta)=\sum_{m=0}^{\infty}\Phi_m(x)\frac{\eta^m}{m!}.
$$

Then it follows from integration by parts that

$$
\begin{aligned}
&\int_0^\eta \hat{\mathcal{B}}[\Phi](x,Z\cdot(\eta-t))\cdot\Psi(x,t)\,dt\\
&\qquad=\sum_{m=0}^{\infty}\frac{\Phi_m(x)Z^m}{m!}\int_0^\eta(\eta-t)^m\cdot\Psi(x,t)\,dt\\
&\qquad=\sum_{m=0}^{\infty}\Phi_m(x)Z^m\int_0^\eta\int_0^{\eta_1}\dots\int_0^{\eta_m}\Psi(x,\eta_{m+1})\,d\eta_{m+1}\,\dots\,d\eta_2\,d\eta_1.
\end{aligned}
\tag{4.9}
$$

By applying (4.9) to (4.7), we obtain the following infinite-order integral representation:

$$
W_{n+1}(x,\eta,G^\eta(R))=\sum_{i=1}^{5}\mathscr{J}_i(x,\eta,R), \tag{4.10}
$$

where

$$\mathscr{J}_1(x,\eta,R) = \mathscr{I}_1 W_n(x,\eta,G^\eta(R))$$

$$= -\sum_{m=1}^{\infty} \cos^2(\mathscr{A}(x,(|\eta|-R)e^{i\arg(\eta)}))\cdot\beta_m(-\tan(\mathscr{A}(x,(|\eta|-R)e^{i\arg(\eta)})))$$

$$\times\,\mathscr{E}(x,(|\eta|-R)e^{i\arg(\eta)})^m$$

$$\times\int_0^R\int_0^{R_1}\cdots\int_0^{R_{m-1}}\mathscr{W}_n(x,\eta,R,R_m)\,dR_m\,\ldots\,dR_2\,dR_1\cdot\{e^{i\arg(\eta)}\}^m,$$

$$\mathscr{J}_2(x,\eta,R) = \mathscr{I}_2 W_n(x,\eta,G^\eta(R)) + \mathscr{I}_3 W_n(x,\eta,G^\eta(R))$$

$$= \sum_{m=1}^{\infty}\int_0^R \cos^2(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}))$$

$$\times\{1-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}))\cdot\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})\cdot R_1 e^{i\arg(\eta)}\}$$

$$\times\,\beta_m(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)})))\cdot\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})^m$$

$$\times\int_0^{R_1}\cdots\int_0^{R_{m-1}}\mathscr{W}_n(x,\eta,R_1,R_m)\,dR_m\,\ldots\,dR_2\,dR_1\cdot\{e^{i\arg(\eta)}\}^m$$

$$= \sum_{m=1}^{\infty}\int_0^R \cos^2(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}))\cdot\beta_m(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)})))$$

$$\times\,\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})^m$$

$$\times\int_0^R\int_0^{R_1}\cdots\int_0^{R_{m-1}}\mathscr{W}_n(x,\eta,R_1,R_m)\,dR_m\,\ldots\,dR_2\,dR_1\cdot\{e^{i\arg(\eta)}\}^m$$

$$-\sum_{m=1}^{\infty}\int_0^R \cos^2(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}))\cdot\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}))$$

$$\times\,\beta_m(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)})))\cdot\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})^{m+1}\cdot R_1$$

$$\times\int_0^{R_1}\cdots\int_0^{R_{m-1}}\mathscr{W}_n(x,\eta,R_1,R_m)\,dR_m\,\ldots\,dR_2\,dR_1\cdot\{e^{i\arg(\eta)}\}^{m+1}$$

$$\equiv \mathscr{J}_2{}'(x,\eta,R) + \mathscr{J}_2{}''(x,\eta,R),$$

$$
\begin{aligned}
&\mathscr{J}_3(x,\eta,R) = \mathscr{I}_4 W_n(x,\eta,G^\eta(R)) \\
&= -\sum_{m=1}^{\infty}\int_0^R \frac{\partial}{\partial\zeta}\Big\{\cos^2(\mathscr{A}(x,\zeta))\cdot\beta_m(-\tan(\mathscr{A}(x,\zeta)))\Big\}\Big|_{\zeta=(|\eta|-R_1)e^{i\arg(\eta)}} \\
&\quad\times \mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})^m \\
&\quad\times \int_0^{R_1}\cdots\int_0^{R_m} \mathscr{W}_n(x,\eta,R_1,R_{m+1})\,dR_{m+1}\,\ldots\,dR_2\,dR_1\cdot\{e^{i\arg(\eta)}\}^{m+1}, \\
&\mathscr{J}_4(x,\eta,R) = \mathscr{I}_5 W_n(x,\eta,G^\eta(R)) + \mathscr{I}_6 W_n(x,\eta,G^\eta(R)) \\
&= -\sum_{m=1}^{\infty}\int_0^R b_m(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)})))\cdot\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})^{m+1} \\
&\quad\times \int_0^{R_1}\cdots\int_0^{R_{m-1}} R_m\cdot\mathscr{W}_n(x,\eta,R_1,R_m)\,dR_m\,\ldots\,dR_2\,dR_1\cdot\{e^{i\arg(\eta)}\}^{m+1}, \\
&\mathscr{J}_5(x,\eta,R) = \mathscr{I}_7 W_n(x,\eta,G^\eta(R)) \\
&= \sum_{m=1}^{\infty}\int_0^R b_m(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)})))\cdot\mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})^{m+1} \\
&\quad\times \int_0^{R_1}\cdots\int_0^{R_m} \mathscr{W}_n(x,\eta,R_1,R_{m+1})\,dR_{m+1}\,\ldots\,dR_2\,dR_1\cdot\{e^{i\arg(\eta)}\}^{m+1}.
\end{aligned}
$$

Before estimating each $\mathscr{J}_i(x,\eta,R)$, let us note the following inequality. We omit the proof.

Lemma 4.1. *For $\delta>0$, it holds that*

$$
\int_0^R\int_0^{R_1}\cdots\int_0^{R_{m-1}} e^{\delta R_m}\,dR_m\,\ldots\,dR_2\,dR_1 \le \frac{1}{\delta^m}e^{\delta R} \quad (R\ge 0). \tag{4.11}
$$

Now let us estimate each $\mathscr{J}_i(x,\eta,R)$.

On $\mathscr{J}_1(x,\eta,R)$: it follows from the assumption of the induction that

$$
\begin{aligned}
&|\mathscr{W}_n(x,\eta,R,R_m)| \\
&\le C_1 e^{\delta_1|\eta|}e^{-\delta_1 R}e^{\delta_1 R_m}{M_1}^n \\
&\quad\times\left\{\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(p-1)^k}\frac{1}{\{1+(|\eta|-R+R_m)-R_m\}^{k(p-1)}}\right\}\frac{{R_m}^n}{n!} \\
&\le C_1 e^{\delta_1|\eta|}e^{-\delta_1 R}e^{\delta_1 R_m}{M_1}^n\left\{\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(p-1)^k}\frac{1}{(1+|\eta|-R)^{k(p-1)}}\right\}\frac{{R_1}^n}{n!}.
\end{aligned} \tag{4.12}
$$

Hence, by the estimate

$$\Big|\cos^2\left(\mathscr{A}(x,(|\eta|-R)e^{i\arg(\eta)})\right)\cdot\beta_m(-\tan(\mathscr{A}(x,(|\eta|-R)e^{i\arg(\eta)}))) \\ \times\mathscr{E}(x,(|\eta|-R)e^{i\arg(\eta)})^m\Big|\le C_0{M_0}^m \tag{4.13}$$

which follows from (3.7), we have

$$|\mathscr{J}_1(x,\eta,R)|\le C_1e^{\delta_1|\eta|}e^{-\delta_1R}{M_1}^nC_0\left\{\sum_{k=0}^n\binom{n}{k}\frac{1}{(p-1)^k}\frac{1}{(1+|\eta|-R)^{k(p-1)}}\right\} \\ \times\sum_{m=1}^\infty{M_0}^m\int_0^R\frac{{R_1}^n}{n!}\int_0^{R_1}\dots\int_0^{R_{m-1}}e^{\delta_1R_m}\,dR_m\,\dots\,dR_2\,dR_1.$$

Here let us adopt the estimate

$$\int_0^R\frac{{R_1}^n}{n!}\int_0^{R_1}\dots\int_0^{R_{m-1}}e^{\delta_1R_m}\,dR_m\,\dots\,dR_2\,dR_1 \\ \le\int_0^R\frac{{R_1}^n}{n!}\frac{1}{{\delta_1}^{m-1}}e^{\delta_1R_1}\,dR_1\le\frac{1}{{\delta_1}^{m-1}}e^{\delta_1R}\int_0^R\frac{{R_1}^n}{n!}\,dR_1=\frac{1}{{\delta_1}^{m-1}}e^{\delta_1R}\frac{R^{n+1}}{(n+1)!}.$$

Then we obtain

$$|\mathscr{J}_1(x,\eta,R)| \\ \le C_1e^{\delta_1|\eta|}{M_1}^n\left(C_0\sum_{m=1}^\infty\frac{{M_0}^m}{{\delta_1}^{m-1}}\right)\left\{\sum_{k=0}^n\binom{n}{k}\frac{1}{(p-1)^k}\frac{1}{(1+|\eta|-R)^{k(p-1)}}\right\}\frac{R^{n+1}}{(n+1)!}. \tag{4.14}$$

On $\mathscr{J}_2{}'(x,\eta,R)$: let us consider R_1 instead of R in (4.12). Then we have

$$|\mathscr{W}_n(x,\eta,R_1,R_m)| \\ \le C_1e^{\delta_1|\eta|}e^{-\delta_1R_1}e^{\delta_1R_m}{M_1}^n\left\{\sum_{k=0}^n\binom{n}{k}\frac{1}{(p-1)^k}\frac{1}{(1+|\eta|-R_1)^{k(p-1)}}\right\}\frac{{R_1}^n}{n!} \tag{4.15} \\ \le C_1e^{\delta_1|\eta|}e^{-\delta_1R_1}e^{\delta_1R_m}{M_1}^n\left\{\sum_{k=0}^n\binom{n}{k}\frac{1}{(p-1)^k}\frac{1}{(1+|\eta|-R)^{k(p-1)}}\right\}\frac{{R_1}^n}{n!}.$$

Hence, by a similar estimate to (4.13) it holds that

$$|\mathscr{J}_2{}'(x,\eta,R)|\le C_1e^{\delta_1|\eta|}{M_1}^nC_0\left\{\sum_{k=0}^n\binom{n}{k}\frac{1}{(p-1)^k}\frac{1}{(1+|\eta|-R)^{k(p-1)}}\right\} \\ \times\sum_{m=1}^\infty{M_0}^m\int_0^R\frac{{R_1}^n}{n!}e^{-\delta_1R_1}\int_0^{R_1}\dots\int_0^{R_{m-1}}e^{\delta_1R_m}\,dR_m\,\dots\,dR_2\,dR_1.$$

Therefore, by using the estimate

$$\begin{aligned}&\int_0^R \frac{{R_1}^n}{n!} e^{-\delta_1 R_1} \int_0^{R_1} \dots \int_0^{R_{m-1}} e^{\delta_1 R_m}\, dR_m \, \dots \, dR_2\, dR_1 \\ &\le \int_0^R \frac{{R_1}^n}{n!} e^{-\delta_1 R_1} \frac{1}{{\delta_1}^{m-1}} e^{\delta_1 R_1}\, dR_1 = \frac{1}{{\delta_1}^{m-1}} \int_0^R \frac{{R_1}^n}{n!}\, dR_1 = \frac{1}{{\delta_1}^{m-1}} \frac{R^{n+1}}{(n+1)!},\end{aligned}$$

we see that $\mathscr{J}_2{}'(x,\eta,R)$ has the same estimate as (4.14) for $\mathscr{J}_1(x,\eta,R)$.

On $\mathscr{J}_3(x,\eta,R)$: it follows from the assumption of the induction that

$$\begin{aligned}&|\mathscr{W}_n(x,\eta,R_1,R_{m+1})| \\ &\le C_1 e^{\delta_1|\eta|} e^{-\delta_1 R_1} e^{\delta_1 R_{m+1}} {M_1}^n \\ &\quad \times \left\{ \sum_{k=0}^n \binom{n}{k} \frac{1}{(p-1)^k} \frac{1}{\{1+(|\eta|-R_1+R_{m+1})-R_{m+1}\}^{k(p-1)}} \right\} \frac{{R_{m+1}}^n}{n!} \\ &\le C_1 e^{\delta_1|\eta|} e^{-\delta_1 R_1} e^{\delta_1 R_{m+1}} {M_1}^n \left\{ \sum_{k=0}^n \binom{n}{k} \frac{1}{(p-1)^k} \frac{1}{(1+|\eta|-R)^{k(p-1)}} \right\} \frac{{R_1}^n}{n!}.\end{aligned}$$

Hence, by the estimate

$$\begin{aligned}&\left| \frac{\partial}{\partial\zeta} \left\{ \cos^2\left(\mathscr{A}(x,\zeta)\right) \cdot \beta_m(-\tan\left(\mathscr{A}(x,\zeta)\right)) \right\}\Big|_{\zeta=(|\eta|-R_1)e^{i\arg(\eta)}} \right. \\ &\qquad \left. \times\, \mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})^m \right| \le C_0 {M_0}^m,\end{aligned}$$

we obtain

$$\begin{aligned}|\mathscr{J}_3(x,\eta,R)| &\le C_1 e^{\delta_1|\eta|} {M_1}^n C_0 \left\{ \sum_{k=0}^n \binom{n}{k} \frac{1}{(p-1)^k} \frac{1}{(1+|\eta|-R)^{k(p-1)}} \right\} \\ &\quad \times \sum_{m=1}^\infty {M_0}^m \int_0^R \frac{{R_1}^n}{n!} e^{-\delta_1 R_1} \int_0^{R_1} \dots \int_0^{R_m} e^{\delta_1 R_{m+1}}\, dR_{m+1} \, \dots \, dR_2\, dR_1.\end{aligned}$$

Moreover, by using the estimate

$$\begin{aligned}&\int_0^R \frac{{R_1}^n}{n!} e^{-\delta_1 R_1} \int_0^{R_1} \dots \int_0^{R_m} e^{\delta_1 R_{m+1}}\, dR_{m+1} \, \dots \, dR_2\, dR_1 \\ &\le \frac{1}{{\delta_1}^m} \frac{R^{n+1}}{(n+1)!} \le \frac{1}{{\delta_1}^{m-1}} \frac{R^{n+1}}{(n+1)!},\end{aligned}$$

we see that $\mathscr{J}_3(x,\eta,R)$ has the same estimate as (4.14) for $\mathscr{J}_1(x,\eta,R)$.

On $\mathscr{J}_5(x,\eta,R)$: by the estimate

$$\left|b_m(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}))) \cdot \mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})^{m+1}\right| \leq C_0 {M_0}^m,$$

similarly to the calculation for $\mathscr{J}_3(x,\eta,R)$ we see that $\mathscr{J}_5(x,\eta,R)$ has the same estimate as (4.14) for $\mathscr{J}_1(x,\eta,R)$.

Therefore, it holds that

$$\begin{aligned}
&|\mathscr{J}_1(x,\eta,R)| + |{\mathscr{J}_2}'(x,\eta,R)| + |\mathscr{J}_3(x,\eta,R)| + |\mathscr{J}_5(x,\eta,R)| \\
&\leq C_1 e^{\delta_1|\eta|} {M_1}^n \left(4C_0 \sum_{m=1}^{\infty} \frac{{M_0}^m}{{\delta_1}^{m-1}}\right) \left\{\sum_{k=0}^{n} \binom{n}{k} \frac{1}{(p-1)^k} \frac{1}{(1+|\eta|-R)^{k(p-1)}}\right\} \frac{R^{n+1}}{(n+1)!}.
\end{aligned} \tag{4.16}$$

On ${\mathscr{J}_2}''(x,\eta,R)$: by adopting (4.15) let us estimate $\mathscr{W}_n(x,\eta,R_1,R_{m+1})$ as

$$\begin{aligned}
&|\mathscr{W}_n(x,\eta,R_1,R_m)| \\
&\leq C_1 e^{\delta_1|\eta|} e^{-\delta_1 R_1} e^{\delta_1 R_m} {M_1}^n \left\{\sum_{k=0}^{n} \binom{n}{k} \frac{1}{(p-1)^k} \frac{1}{(1+|\eta|-R_1)^{k(p-1)}}\right\} \frac{{R_1}^n}{n!} \\
&\leq C_1 e^{\delta_1|\eta|} e^{-\delta_1 R_1} e^{\delta_1 R_m} {M_1}^n \left\{\sum_{k=0}^{n} \binom{n}{k} \frac{1}{(p-1)^k} \frac{1}{(1+|\eta|-R_1)^{k(p-1)}}\right\} \frac{R^n}{n!}.
\end{aligned}$$

Then by the estimate

$$\begin{aligned}
&\Big|\cos^2(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)})) \cdot \tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)})) \\
&\quad \times \beta_m(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}))) \cdot \mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})^{m+1} \cdot R_1\Big| \\
&\leq \frac{C_0 {M_0}^m}{(1+|\eta|-R_1)^p} R_1 \leq \frac{C_0 {M_0}^m}{(1+|\eta|-R_1)^p} R
\end{aligned}$$

we have

$$\begin{aligned}
&|{\mathscr{J}_2}''(x,\eta,R)| \\
&\leq C_1 e^{\delta_1|\eta|} {M_1}^n C_0 \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(p-1)^k} \sum_{m=1}^{\infty} {M_0}^m \\
&\quad \times \int_0^R \frac{e^{-\delta_1 R_1}}{(1+|\eta|-R_1)^{(k+1)(p-1)+1}} \int_0^{R_1} \cdots \int_0^{R_{m-1}} e^{\delta_1 R_m}\, dR_m \ldots dR_2\, dR_1 \frac{R^{n+1}}{n!} \\
&\leq C_1 e^{\delta_1|\eta|} {M_1}^n \left(C_0 \sum_{m=1}^{\infty} \frac{{M_0}^m}{{\delta_1}^{m-1}}\right) \\
&\quad \times \sum_{k=0}^{n} \binom{n}{k} \frac{n+1}{(p-1)^k} \int_0^R \frac{1}{(1+|\eta|-R_1)^{(k+1)(p-1)+1}}\, dR_1 \frac{R^{n+1}}{(n+1)!}.
\end{aligned}$$

Furthermore, by noting

$$\int_0^R \frac{1}{(1+|\eta|-R_1)^{(k+1)(p-1)+1}}\,dR_1 = \left[\frac{1}{k+1}\frac{1}{p-1}\frac{1}{(1+|\eta|-R_1)^{(k+1)(p-1)}}\right]_{R_1=0}^{R_1=R}$$
$$\leq \frac{1}{k+1}\frac{1}{p-1}\frac{1}{(1+|\eta|-R)^{(k+1)(p-1)}},$$

we obtain

$$\begin{aligned}|\mathscr{J}_2{}''(x,\eta,R)| &\leq C_1 e^{\delta_1|\eta|} {M_1}^n \left(C_0 \sum_{m=1}^{\infty} \frac{{M_0}^m}{{\delta_1}^{m-1}}\right)\\ &\quad\times \left\{\sum_{k=0}^{n} \binom{n}{k} \frac{n+1}{k+1} \frac{1}{(p-1)^{k+1}} \frac{1}{(1+|\eta|-R)^{(k+1)(p-1)}}\right\} \frac{R^{n+1}}{(n+1)!}\\ &= C_1 e^{\delta_1|\eta|} {M_1}^n \left(C_0 \sum_{m=1}^{\infty} \frac{{M_0}^m}{{\delta_1}^{m-1}}\right)\\ &\quad\times \left\{\sum_{k=1}^{n+1} \binom{n}{k-1} \frac{n+1}{k} \frac{1}{(p-1)^{k}} \frac{1}{(1+|\eta|-R)^{k(p-1)}}\right\} \frac{R^{n+1}}{(n+1)!}.\end{aligned} \tag{4.17}$$

Similarly, it follows from the estimate

$$\begin{aligned}&\left|b_m(-\tan(\mathscr{A}(x,(|\eta|-R_1)e^{i\arg(\eta)}))) \cdot \mathscr{E}(x,(|\eta|-R_1)e^{i\arg(\eta)})^{m+1} \cdot R_m\right|\\ &\leq \frac{C_0 {M_0}^m}{(1+|\eta|-R_1)^p} R\end{aligned}$$

that $\mathscr{J}_4(x,\eta,R)$ has the same estimate as (4.17) for $\mathscr{J}_2{}''(x,\eta,R)$. Therefore, it holds that

$$\begin{aligned}&|\mathscr{J}_2{}''(x,\eta,R)| + |\mathscr{J}_4(x,\eta,R)|\\ &\leq C_1 e^{\delta_1|\eta|} {M_1}^n \left(2C_0 \sum_{m=1}^{\infty} \frac{{M_0}^m}{{\delta_1}^{m-1}}\right)\\ &\quad\times \left\{\sum_{k=1}^{n+1} \binom{n}{k-1} \frac{n+1}{k} \frac{1}{(p-1)^{k}} \frac{1}{(1+|\eta|-R)^{k(p-1)}}\right\} \frac{R^{n+1}}{(n+1)!}.\end{aligned} \tag{4.18}$$

Finally, let us combine (4.16) and (4.18). Then we obtain

$$\begin{aligned}
&|W_{n+1}(x,\eta,G^\eta(R))| \\
&\le \sum_{i=1}^{5} |\mathscr{J}_i(x,\eta,R)| \\
&\le C_1 e^{\delta_1|\eta|} {M_1}^n \left(5C_0 \sum_{m=1}^{\infty} \frac{{M_0}^m}{{\delta_1}^{m-1}} \right) \\
&\quad \times \Bigg[1 + \sum_{k=1}^{n} \left\{ \binom{n}{k} + \binom{n}{k-1} \frac{n+1}{k} \right\} \frac{1}{(p-1)^k} \frac{1}{(1+|\eta|-R)^{k(p-1)}} \\
&\qquad + \frac{1}{(p-1)^{n+1}} \frac{1}{(1+|\eta|-R)^{(n+1)(p-1)}} \Bigg] \frac{R^{n+1}}{(n+1)!}.
\end{aligned}$$

Moreover, by noting

$$\binom{n}{k} + \binom{n}{k-1}\frac{n+1}{k} = \binom{n}{k} + \binom{n+1}{k} \le 2\binom{n+1}{k},$$

we have

$$\begin{aligned}
&|W_{n+1}(x,\eta,G^\eta(R))| \\
&\le C_1 e^{\delta_1|\eta|} {M_1}^{n+1} \left\{ \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1}{(p-1)^k} \frac{1}{(1+|\eta|-R)^{k(p-1)}} \right\} \frac{R^{n+1}}{(n+1)!},
\end{aligned}$$

which implies the lemma for $n+1$. The proof has been completed. □

APPENDIX

A. DERIVATION OF FORMULA (2.7)

In Subsection 2.2 we gave the formula of the solution $V(x,\eta)$ of the initial value problem (2.6) in the form of (2.7). Here we explain the derivation of it. Let $(x,\eta) = (X(\mu,\nu), Y(\mu,\nu))$ be the characteristic curve of the operator $\mathscr{L}$, that is, the solution of the following initial value problem:

$$\begin{cases} \dfrac{dx}{d\mu} = 1 + x^2, & \dfrac{d\eta}{d\mu} = 1 + x\eta, \\ x(0) = \nu, & \eta(0) = 0. \end{cases} \tag{A.1}$$

Then, on this curve $V(x,\eta)$ has the following form:

$$V(X(\mu,\nu), Y(\mu,\nu)) = l(\nu) + \int_0^\mu k(X(\tau,\nu), Y(\tau,\nu))\, d\tau.$$

Hence, if $(x,\eta) \mapsto (\mu,\nu) = (M(x,\eta), N(x,\eta))$ is the inverse mapping of $(\mu,\nu) \mapsto (x,\eta) = (X(\mu,\nu), Y(\mu,\nu))$ we have

$$V(x,\eta) = l(N(x,\eta)) + \int_0^{M(x,\eta)} k(X(\tau, N(x,\eta)), Y(\tau, N(x,\eta)))\, d\tau. \tag{A.2}$$

By solving (A.1), it holds that

$$\begin{aligned} X(\mu,\nu) &= \tan(\mu + \arctan\nu), \\ Y(\mu,\nu) &= \tan(\mu + \arctan\nu) - \frac{\nu \cdot \cos(\arctan\nu)}{\cos(\mu + \arctan\nu)}, \\ M(x,\eta) &= \arctan x + \mathscr{A}(x,\eta), \\ N(x,\eta) &= -\tan(\mathscr{A}(x,\eta)). \end{aligned}$$

Therefore, it follows from (A.2) that

$$\begin{aligned} V(x,\eta) &= l(-\tan(\mathscr{A}(x,\eta))) \\ &+ \int_0^{\arctan x + \mathscr{A}(x,\eta)} k(X(\tau, -\tan(\mathscr{A}(x,\eta))), Y(\tau, -\tan(\mathscr{A}(x,\eta))))\, d\tau. \end{aligned} \tag{A.3}$$

Moreover, in (A.3) we practice an integration by substitution

$$\tau(z) = \mathscr{A}(x,\eta) - \mathscr{A}(x,\eta - z), \quad z : 0 \to \eta.$$

Then we obtain (2.7).

REFERENCES

[1] W. Balser, *From Divergent Power Series to Analytic Functions. Theory and Application of Multisummable Power Series*, Lecture Notes in Mathematics, 1582, Springer-Verlag, 1994.

[2] W. Balser, *Divergent solutions of the heat equation: on an article of Lutz, Miyake and Schäfke*, Pacific J. Math. **188** (1999), 53–63.

[3] W. Balser, *Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations*, Universitext, Springer-Verlag, 2000.

[4] W. Balser, *Multisummability of formal power series solutions of partial differential equations with constant coefficients*, J. Differential Equations **201** (2004), 63–74.

[5] W. Balser, M. Miyake, *Summability of formal solutions of certain partial differential equations*, Acta Sci. Math. (Szeged) **65** (1999), 543–552.

[6] M. Hibino, *Divergence property of formal solutions for singular first order linear partial differential equations*, Publ. Res. Inst. Math. Sci. **35** (1999), 893–919.

[7] M. Hibino, *Borel summability of divergent solutions for singular first-order partial differential equations with variable coefficients. I*, J. Differential Equations **227** (2006), 499–533.

[8] M. Hibino, *Borel summability of divergent solutions for singular first-order partial differential equations with variable coefficients. II*, J. Differential Equations **227** (2006), 534–563.

[9] M. Hibino, *Summability of formal solutions for singular first-order linear PDEs with holomorphic coefficients*, Differential Equations and Exact WKB Analysis, 47–62, RIMS Kôkyûroku Bessatsu **B10**, Res. Inst. Math. Sci., 2008.

[10] M. Hibino, *Summability of formal solutions for singular first-order partial differential equations with holomorphic coefficients. I*, in preparation.

[11] M. Hibino, *Summability of formal solutions for singular first-order partial differential equations with holomorphic coefficients. II*, in preparation.

[12] D.A. Lutz, M. Miyake, R. Schäfke, *On the Borel summability of divergent solutions of the heat equation*, Nagoya Math. J. **154** (1999), 1–29.

[13] B. Malgrange, *Sommation des séries divergentes*, Exposition. Math. **13** (1995), 163–222.

[14] M. Miyake, *Borel summability of divergent solutions of the Cauchy problem to non-Kowalevskian equations*, Partial Differential Equations and Their Applications (Wuhan, 1999), 225–239, World Sci. Publ., 1999.

[15] S. Ōuchi, *Multisummability of formal solutions of some linear partial differential equations*, J. Differential Equations **185** (2002), 513–549.

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