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MAILLET TYPE THEOREM FOR SINGULAR FIRST ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF TOTALLY CHARACTERISTIC TYPE. PART II

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Abstract. In this paper, we study the following nonlinear first order partial differential equation:

$$f(t,x,u,\partial_t u,\partial_x u)=0 \quad \text{with} \quad u(0,x)\equiv 0.$$

The purpose of this paper is to determine the estimate of Gevrey order under the condition that the equation is singular of a totally characteristic type. The Gevrey order is indicated by the rate of divergence of a formal power series. This paper is a continuation of the previous papers [*Convergence of formal solutions of singular first order nonlinear partial differential equations of totally characteristic type*, Funkcial. Ekvac. 45 (2002), 187–208] and [*Maillet type theorem for singular first order nonlinear partial differential equations of totally characteristic type*, Surikaiseki Kenkyujo Kokyuroku, Kyoto University 1431 (2005), 94–106]. Especially the last-mentioned paper is regarded as part I of this paper.

Keywords: singular partial differential equations, totally characteristic type, nilpotent vector field, formal solution, Gevrey order, Maillet type theorem.

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1. INTRODUCTION AND MAIN RESULTS

Let $\mathbb{C}$ be the set of complex numbers or a variable, $t=(t_1,\dots,t_d)\in\mathbb{C}^d$ and $x=(x_1,\dots,x_n)\in\mathbb{C}^n$. We consider the following first order nonlinear partial differential equation:

$$\begin{cases} f(t,x,u,\partial_t u,\partial_x u)=0, \\ u(0,x)\equiv 0, \end{cases} \tag{1.1}$$

where $u(t,x)$ denotes the unknown function, $\partial_t u = (\partial_{t_1} u, \dots, \partial_{t_d} u)$ and $\partial_x u$ is defined similarly. Here, we assume that the function $f(t,x,u,\tau,\xi)$ ($\tau = (\tau_1, \dots, \tau_d) \in \mathbb{C}^d$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$) is holomorphic in a neighborhood of the origin of $\mathbb{C}^d \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^d \times \mathbb{C}^n$, and is an entire function in τ variables for any fixed t, x, u and ξ.

The purpose of this paper is to characterize the rate of divergence of formal solutions by using the "Gevrey order", and such a characterization theorem is called "Maillet type theorem". In order to study the Maillet type theorem for the above equation, we assume the following three assumptions.

Assumption 1.1 (Singular equation)**.** The function $f(t,x,u,\tau,\xi)$ is singular in t variables in the sense that

$$f(0,x,0,\tau,0) \equiv 0 \quad (\text{for all } x \in \mathbb{C}^n \text{ near } x=0, \text{ and all } \tau \in \mathbb{C}^d). \tag{1.2}$$

Assumption 1.2 (Existence of formal solutions)**.** The equation (1.1) has a formal solution of the form

$$u(t,x) = \sum_{j=1}^{d} \varphi_j(x) t_j + \sum_{|\alpha| \geq 2, |\beta| \geq 0} u_{\alpha,\beta} t^\alpha x^\beta \quad \text{for some} \quad \{\varphi_j(x)\}_{j=1}^d \in \mathbb{C}\{x\}^d, \tag{1.3}$$

where $\mathbb{C}\{x\}$ denotes the set of holomorphic functions at $x=0$, for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ ($\mathbb{N} = \{0,1,2,\dots\}$) we define $|\alpha| = \alpha_1 + \dots + \alpha_d$ and $t^\alpha = t_1^{\alpha_1} \dots t_d^{\alpha_d}$, and $|\beta|$ and x^β are defined similarly for $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$.

Assumption 1.3 (Totally characteristic type)**.** The equation (1.1) is of totally characteristic type with respect to $\{\varphi_j(x)\}$ in (1.3), which means that the following conditions hold:

$$\begin{cases} f_{\xi_k}(0,x,0,\{\varphi_j(x)\},0) \not\equiv 0 \\ f_{\xi_k}(0,0,0,\{\varphi_j(0)\},0) = 0 \end{cases} \quad \text{for} \quad k = 1,2,\dots,n. \tag{1.4}$$

Remark 1.4. The functions $\{\varphi_j(x)\}$ in (1.3) are obtained as a solutions of the following d-system of equations:

$$\begin{aligned} &\frac{\partial}{\partial t_i} f(t,x,u(t,x),\partial_t u(t,x),\partial_x u(t,x))\Big|_{t=0} \\ &= f_{t_i}(0,x,0,\{\varphi_j(x)\},0) + f_u(0,x,0,\{\varphi_j(x)\},0)\varphi_i(x) \\ &\quad + \sum_{k=1}^{n} f_{\xi_k}(0,x,0,\{\varphi_j(x)\},0) \frac{\partial \varphi_i}{\partial x_k}(x) = 0, \quad i = 1,2,\dots,d. \end{aligned} \tag{1.5}$$

In the case $d=1$, a sufficient condition that the formal solution of (1.5) to be convergent is obtained by Miyake and Shirai ([6]). In the case $d \geq 2$, a sufficient condition obtained by Shirai ([13]).

Now we put $\boldsymbol{\varphi}(x) = (0, x, 0, \{\varphi_j(x)\}, 0)$ for simplicity of notation. We define functions $a_{i,j}(x)$ $(i, j = 1, \ldots, d)$ and $b_k(x)$ $(k = 1, \ldots, n)$ by

$$a_{i,j}(x) := f_{t_i,\tau_j}(\boldsymbol{\varphi}(x)) + f_{u,\tau_j}(\boldsymbol{\varphi}(x))\varphi_i(x) + \sum_{k=1}^{n} f_{\tau_j,\xi_k}(\boldsymbol{\varphi}(x))\frac{\partial \varphi_i}{\partial x_k}(x), \tag{1.6}$$

$$b_k(x) := f_{\xi_k}(\boldsymbol{\varphi}(x)). \tag{1.7}$$

Remark 1.5. By the assumption of totally characteristic type, $b_k(x)$ satisfies $b_k(x) \not\equiv 0$, $b_k(0) = 0$ for all $k = 1, 2, \ldots, n$.

Let M_1 and M_2 be the Jordan canonical forms of $(a_{i,j}(0))$ and $J(b_1, \ldots, b_n)(0)$ respectively, where $J(b_1, \ldots, b_n)(x)$ denotes the Jacobi matrix of $(b_1(x), \ldots, b_n(x))$, we denote them by

$$(a_{i,j}(0)) \sim M_1, \quad J(b_1, \ldots, b_n)(0) \sim M_2.$$

Then the following two cases were already studied by the author's previous papers.

(a) M_1 and M_2 are regular matrices with Poincaré condition (see Theorem 2.1 in §2 or [13]).
(b) M_1 is a regular matrix with Poincaré condition and M_2 is a nilpotent matrix (see Theorem 2.2 in §2 or [15]).

In this paper we shall study the following two cases, and the main results are stated as Theorem 1.6 and Theorem 1.7.

(a) M_1 is a nilpotent matrix and M_2 is a regular matrix with Poincaré condition.
(b) M_1 and M_2 are nilpotent matrices.

In order to state our main theorems, we prepare some notations.
In case (c), we put M_1 and M_2 by

$$M_1 = \begin{pmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_I \end{pmatrix}, \quad \text{where} \quad N_j = \begin{pmatrix} 0 & & & \\ \delta & 0 & & \\ & \ddots & \ddots & \\ & & \delta & 0 \end{pmatrix} \quad \text{of size } d_j (\geq 1),$$

$$M_2 = \begin{pmatrix} \mu_1 & & & \\ \nu_1 & \mu_2 & & \\ & \ddots & \ddots & \\ & & \nu_{n-1} & \mu_n \end{pmatrix}, \quad \delta = 1, \quad \nu_j = 0 \text{ or } 1.$$

We note that $d_1 + d_2 + \ldots + d_I = d$. If the size $d_j = 1$, then $N_j = (0)$.

Theorem 1.6. *We consider case* (c). *Let Assumptions 1.1, 1.2, 1.3 and* $f_u(\boldsymbol{\varphi}(0)) \neq 0$ *be satisfied, and* M_1 *and* M_2 *be as above. Moreover, we assume the nonresonance-Poincaré condition for* M_2*, that is,*

$$\left| \sum_{k=1}^{n} \mu_k \beta_k + f_u(\boldsymbol{\varphi}(0)) \right| \geq C(|\beta| + 1) \tag{1.8}$$

by a positive constant C independent $\beta \in \mathbb{N}^n$ for all β. Then the formal solution $u(t,x)$ belongs to the Gevrey class of order at most $(2d_0, d_0+1)$ by $d_0 := \max\{d_1, \dots, d_I\}$ (which is greater or equal to 1). This means that for the formal solution

$$u(t,x) = \sum_{|\alpha|\geq 1, |\beta|\geq 0} u_{\alpha,\beta} t^\alpha x^\beta,$$

the power series

$$\sum_{|\alpha|\geq 1, |\beta|\geq 0} \frac{u_{\alpha,\beta}}{|\alpha|!^{2d_0-1}|\beta|!^{d_0}} t^\alpha x^\beta$$

is convergent in a neighborhood of the origin.

In case (d), we have M_1 and M_2

$$M_1 = \begin{pmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_I \end{pmatrix}, \quad \text{where} \quad N_j = \begin{pmatrix} 0 & & & \\ \delta & 0 & & \\ & \ddots & \ddots & \\ & & \delta & 0 \end{pmatrix} \quad \text{of size } d_j (\geq 1),$$

$$M_2 = \begin{pmatrix} \hat{N}_1 & & & \\ & \hat{N}_2 & & \\ & & \ddots & \\ & & & \hat{N}_J \end{pmatrix}, \quad \text{where} \quad \hat{N}_k = \begin{pmatrix} 0 & & & \\ \delta & 0 & & \\ & \ddots & \ddots & \\ & & \delta & 0 \end{pmatrix} \quad \text{of size } n_k (\geq 1).$$

We note that $d_1 + d_2 + \ldots + d_I = d$ and $n_1 + n_2 + \ldots + n_J = n$.

Theorem 1.7. *We consider case* (d). *Let Assumptions 1.1, 1.2, 1.3 and $f_u(\boldsymbol{\varphi}(0)) \neq 0$ be satisfied and M_1 and M_2 be as above. Then the formal solution $u(t,x)$ belongs to the Gevrey class of order at most $2n_0$ by $n_0 := \max\{d_1, \dots, d_I, n_1, \dots, n_J\} (\geq 1)$. This means that for the formal solution*

$$u(t,x) = \sum_{|\alpha|\geq 1, |\beta|\geq 0} u_{\alpha,\beta} t^\alpha x^\beta,$$

the power series

$$\sum_{|\alpha|\geq 1, |\beta|\geq 0} \frac{u_{\alpha,\beta}}{(|\alpha|+|\beta|)!^{2n_0-1}} t^\alpha x^\beta$$

is convergent in a neighborhood of the origin.

2. RELATED RESULTS

For the formal solution $u(t,x)$, we put $v(t,x) = u(t,x) - \sum_{j=1}^{d} \varphi_j(x) t_j = O(|t|^K)$ $(K \geq 2)$ as a new known function. By substituting $u = v + \sum_{j=1}^{d} \varphi_j(x) t_j$ into the equation (1.1), $v(t,x)$ satisfies the following singular first order nonlinear partial differential equation:

$$\begin{cases} \left(\sum_{i,j=1}^{d} a_{i,j}(x)t_i\partial_{t_j} + \sum_{k=1}^{n} b_k(x)\partial_{x_k} + c(x)\right) v(t,x) \\ \quad = \sum_{|\alpha|=K} d_\alpha(x)t^\alpha + f_{K+1}(t,x,v(t,x),\partial_t v(t,x),\partial_x v(t,x)), \\ v(t,x) = O(|t|^K), \end{cases} \tag{2.1}$$

where $c(x) = f_u(\varphi(x))$, $d_\alpha(x)$ is holomorphic in a neighborhood of the origin, and $f_{K+1}(t,x,v,\tau,\xi)$ is also holomorphic in a neighborhood of the origin with the Taylor expansion

$$f_{K+1}(t,x,v,\tau,\xi) = \sum_{V(\alpha,p,q,r)\geq K+1} f_{\alpha pqr}(x)t^\alpha v^p \tau^q \xi^r. \tag{2.2}$$

Here we used the following notation:

$$V(\alpha,p,q,r) = |\alpha| + Kp + (K-1)|q| + K|r|, \tag{2.3}$$

which denotes the order of zeros in t for each terms $t^\alpha v(t,x)^p(\partial_t v(t,x))^q(\partial_x v(t,x))^r$.

For the equation (2.1), if $b_k(x) \equiv 0$ $(k = 1,2,\dots,n)$, (2.1) is written as follows:

$$\begin{cases} \left(\sum_{i,j=1}^{d} a_{i,j}(x)t_i\partial_{t_j} + c(x)\right) v(t,x) \\ \quad = \sum_{|\alpha|=K} d_\alpha(x)t^\alpha + f_{K+1}(t,x,v(t,x),\partial_t v(t,x),\partial_x v(t,x)), \\ v(t,x) = O(|t|^K). \end{cases}$$

This equation is called *the Fuchsian equation with respect to t variables.* To this equation, a lot of Maillet type theorems have been studied by many mathematicians. For example, Gérard-Tahara, and Miyake-Shirai study the nonlinear case, which is found in the book or papers [4,6,7] and [8]. They obtained the Maillet type theorem, which include the convergent case.

The case of $b_k(x) \not\equiv 0$ and $b_k(0) = 0$, which is the case of totally characteristic type, Chen-Tahara studied the convergence of a formal solution in the case when $(t,x) \in \mathbb{C}^2$ and $b_k(x) = O(x)$ with Poincaré condition ([3]). This result was extended to the case of several space variables by Chen-Luo ([1]). Moreover, these results were generalized by the author to the case of several time-space variables ([13]) (see Theorem 2.1).

On the other hand, Chen-Luo-Tahara studied the Maillet type theorem in the case of $(t,x) \in \mathbb{C}^2$ and $b_k(x) = O(x^K)$ $(K \geq 2)$ ([2]), and they obtained that the formal solution belongs to the Gevrey class of order $K/(K-1)$. Their Maillet type theorem was generalized by the author to the case of several time-space variables ([15]) (see Theorem 2.2).

The statements of [13] and [15] are written as follows.

Theorem 2.1 ([13]). *If all eigenvalues $\{\lambda_j\}_{j=1,2,\dots,d}$ of $(a_{i,j}(0))_{i,j=1,2,\dots,d}$ and all eigenvalues $\{\mu_k\}_{k=1,\dots,n}$ of the Jacobi matrix $J(b_1,\dots,b_n)(0)$ satisfy the Poincaré condition $\mathrm{Ch}(\{\lambda_j\},\{\mu_k\}) \not\ni 0$ (convex hull of points $\{\lambda_j\}$ and $\{\mu_k\}$), then the formal solution converges in a neighborhood of the origin.*

Theorem 2.2 ([15]). *If all eigenvalues $\{\lambda_j\}_{j=1,\dots,d}$ of $(a_{i,j}(0))_{i,j=1,2,\dots,d}$ satisfy the Poincaré condition $\mathrm{Ch}(\{\lambda_j\}) \not\ni 0$, and $J(b_1,\dots,b_n)(0)$ is nilpotent, then the formal solution belongs to the Gevrey class of order at most $2d_0$ in (t,x), where d_0 denotes the maximum of size of nilpotent Jordan blocks of $J(b_1,\dots,b_n)(0)$.*

3. REFINEMENT OF THEOREM 1.6

In order to prove Theorem 1.6, we shall estimate the Gevrey order in each variables $(t_1,\dots,t_d,x_1,\dots,x_n)$ of formal solution of (2.1). To do so, we reduce (2.1) to a more exact form.

First, we set $\hat{a}_{i,j}(x) = a_{i,j}(0) - a_{i,j}(x) = O(|x|)$. Then the vector field with respect to t variables is written by

$$\sum_{i,j=1}^{d} a_{i,j}(x) t_i \partial_{t_j} = (t_1,\dots,t_d) \begin{pmatrix} a_{1,1}(0) & \cdots & a_{1,d}(0) \\ \vdots & \ddots & \vdots \\ a_{d,1}(0) & \cdots & a_{d,d}(0) \end{pmatrix} \begin{pmatrix} \partial_{t_1} \\ \vdots \\ \partial_{t_d} \end{pmatrix} - \sum_{i,j=1}^{d} \hat{a}_{i,j}(x) t_i \partial_{t_j}.$$

Here we introduce new variables $\tau = (\tau^{(1)},\dots,\tau^{(I)}) \in \mathbb{C}^d$, $(\tau^{(j)} = (\tau_{j,1},\dots,\tau_{j,d_j}) \in \mathbb{C}^{d_j}, d = d_1 + \ldots + d_I)$ by

$$(\tau^{(1)},\dots,\tau^{(I)}) = (t_1,\dots,t_d)P, \qquad P^{-1}(a_{i,j}(0))P = M_1.$$

By this linear change of variables, the above vector field is reduced to

$$(\tau^{(1)},\dots,\tau^{(I)}) \begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_I \end{pmatrix} \begin{pmatrix} \partial_{\tau^{(1)}} \\ \vdots \\ \partial_{\tau^{(I)}} \end{pmatrix} - \sum_{i,j,k,l} \alpha_{ijkl}(x) \tau_{i,j} \partial_{\tau_{k,l}},$$

where $\sum_{i,j,k,l}$ is a summation taken over

$$1 \le i \le I,\ 1 \le j \le d_i,\ 1 \le k \le I,\ 1 \le l \le d_k.$$

Next, we write the differential operator with respect to x variables by the following form:

$$\sum_{k=1}^{n} b_k(x) \partial_{x_k} = (x_1,\dots,x_n) J(b_1,\dots,b_n)(0) \begin{pmatrix} \partial_{x_1} \\ \vdots \\ \partial_{x_n} \end{pmatrix} - \sum_{k=1}^{n} \hat{b}_k(x) \partial_{x_k},$$

where $\hat{b}_k(x) = O(|x|^2)$ $(k = 1, \dots, n)$. Then we introduce new variables $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ by

$$(\xi_1, \dots, \xi_n) = (x_1, \dots, x_n)Q, \quad Q^{-1}J(b_1, \dots, b_n)(0)Q = M_2.$$

By this linear change of variables x, the above vector field is reduced to

$$\sum_{k=1}^{n} \mu_k \xi_k \partial_{\xi_k} + \sum_{k=1}^{n-1} \nu_k \xi_{k+1} \partial_{\xi_k} - \sum_{k=1}^{n} \beta_k(\xi) \partial_{\xi_k},$$

where $\beta_k(\xi) = O(|\xi|^2)$ $(k = 1, \dots, n)$.

Hereafter we rewrite (τ, ξ) by (t, x) again. Then the equation (2.1) is reduced to the following one.

$$\begin{cases} (\mathcal{N} + \mathcal{D} + \Delta)v = \displaystyle\sum_{i,j,k,l} \alpha_{ijkl}(x) t_{i,j} \partial_{t_{k,l}} v + \sum_{k=1}^{n} \beta_k(x) \partial_{x_k} v \\ \qquad\qquad\qquad +\eta(x)v + \displaystyle\sum_{|\alpha|=K} \zeta_\alpha(x) t^\alpha + g_{K+1}(t, x, v, \partial_t v, \partial_x v), \\ v(t, x) = O(|t|^K), \end{cases} \tag{3.1}$$

where the operators $\mathcal{N}$, $\mathcal{D}$ and Δ are

$$\mathcal{N} = \sum_{j=1}^{I} \sum_{k=1}^{d_j - 1} \delta t_{j,k+1} \partial_{t_{j,k}}, \tag{3.2}$$

$$\mathcal{D} = \sum_{k=1}^{n} \mu_k x_k \partial_{x_k} + c(0), \quad (c(0) = f_u(\boldsymbol{\varphi}(0))), \tag{3.3}$$

$$\Delta = \sum_{k=1}^{n-1} \nu_k x_{k+1} \partial_{x_k}, \tag{3.4}$$

respectively. Moreover, $\eta(x) = c(0) - c(x)(= f_u(\boldsymbol{\varphi}(0)) - f_u(\boldsymbol{\varphi}(x))) = O(|x|)$, and $g_{K+1}(t, x, v, \tau, \xi)$ has the similar Taylor expansion with (2.2).

In order to give our refinement form, we prepare notations and definitions.

Definition 3.1 (Borel transform). Let $\mathbb{R}_{\geq 1} = \{x \in \mathbb{R} \,|\, x \geq 1\}$, and let $\boldsymbol{s} = (s_1, \dots, s_d) \in (\mathbb{R}_{\geq 1})^d$, $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in (\mathbb{R}_{\geq 1})^n$. For a formal power series $u(t, x) = \sum u_{\alpha,\beta} t^\alpha x^\beta$, we define the $\boldsymbol{s}$-Borel transform in t, the $\boldsymbol{\sigma}$-Borel transform in x and the $(\boldsymbol{s}, \boldsymbol{\sigma})$-Borel transform in (t, x) as follows, respectively:

– $\boldsymbol{s}$-Borel transform in t: $\quad \mathcal{B}_t^{\boldsymbol{s}}(u)(t, x) = \sum \dfrac{u_{\alpha,\beta}|\alpha|!}{(\boldsymbol{s} \cdot \alpha)!} t^\alpha x^\beta,$

– $\boldsymbol{\sigma}$-Borel transform in x: $\quad \mathcal{B}_x^{\boldsymbol{\sigma}}(u)(t, x) = \sum \dfrac{u_{\alpha,\beta}|\beta|!}{(\boldsymbol{\sigma} \cdot \beta)!} t^\alpha x^\beta,$

– $(\boldsymbol{s}, \boldsymbol{\sigma})$-Borel transform in (t, x): $\quad \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(u)(t, x) = (\mathcal{B}_t^{\boldsymbol{s}} \circ \mathcal{B}_x^{\boldsymbol{\sigma}})(u) = (\mathcal{B}_x^{\boldsymbol{\sigma}} \circ \mathcal{B}_t^{\boldsymbol{s}})(u),$

where $\boldsymbol{s} \cdot \alpha$ denotes $\boldsymbol{s} \cdot \alpha = s_1\alpha_1 + \ldots + s_d\alpha_d$, ($\boldsymbol{\sigma} \cdot \beta$ is also a similar definition) and $a! = \Gamma(a + 1)$.

Definition 3.2 (Gevrey class)**.** We define $u(t,x) \in \mathcal{G}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}$, which is called of $(\boldsymbol{s},\boldsymbol{\sigma})$ Gevrey class in (t,x) variables, if $\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(u)(t,x)$ is convergent in a neighborhood of the origin.

By an easy calculation, the following relation holds:

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(u)(t,x) \in \mathcal{G}_{t,x}^{(\boldsymbol{s}',\boldsymbol{\sigma}')} \implies u(t,x) \in \mathcal{G}_{t,x}^{(\boldsymbol{s}+\boldsymbol{s}'-\mathbf{1}_d,\boldsymbol{\sigma}+\boldsymbol{\sigma}'-\mathbf{1}_n)},$$

where $\mathbf{1}_d = (1,\dots,1) \in \mathbb{N}^d$.

Now, we obtain the following refinement of Theorem 1.6.

Theorem 3.3. *Let* $\boldsymbol{s}^j = (1,2,\dots,d_j) \in \mathbb{N}^{d_j}$ $(j = 1,2,\dots,I)$*, and let* $d_0 = \max\{d_1,d_2,\dots,d_I\}$*. Then under the assumptions 1.1, 1.2, 1.3 and the nonresonance-Poincaré condition* (1.8)*, the formal solution of* (7.1) *belongs to the Gevrey class of order at most* $(\boldsymbol{s}',\boldsymbol{\sigma}')$ *by*

$$(\boldsymbol{s}',\boldsymbol{\sigma}') = \begin{cases} (\boldsymbol{s}+\boldsymbol{d}_0^1, \boldsymbol{d}_0^2+\mathbf{1}_n) \in \mathbb{N}^{d+n} & \text{if } \alpha_{ijkl}(x) \not\equiv 0 \text{ for some } i,j,k,l, \\ (\boldsymbol{s}+\boldsymbol{d}', \mathbf{1}_n) \in \mathbb{N}^{d+n} & \text{if } \alpha_{ijkl}(x) \equiv 0 \text{ for all } i,j,k,l, \end{cases}$$

where $\boldsymbol{s} = (\boldsymbol{s}^1,\dots,\boldsymbol{s}^I) \in \mathbb{N}^d$, $\boldsymbol{d}_0^1 = (d_0,\dots,d_0) \in \mathbb{N}^d$, $\boldsymbol{d}_0^2 = (d_0,\dots,d_0) \in \mathbb{N}^n$ *and* $\boldsymbol{d}' = (d',\dots,d') \in \mathbb{R}^d$*, where*

$$d' = \max_{\alpha,p,q,r} \left\{ \frac{d_0}{V(\alpha,p,q,r)-K} \right\} (\le d_0). \tag{3.5}$$

Theorem 1.6 is an immediate consequence of Theorem 3.3. Indeed, all the components of $(\boldsymbol{s}^1,\dots,\boldsymbol{s}^I)$ are estimated by d_0. Therefore, all the components of $\boldsymbol{s}'$ are estimated by $2d_0$, which gives the conclusion of Theorem 1.6.

4. EXAMPLES FOR THEOREM 3.3

In this section, we give typical examples for Theorem 3.3.

Example 4.1. Let $t = (t_1,t_2) \in \mathbb{C}^2$ and $x \in \mathbb{C}$. We consider the equation

$$\begin{cases} (t_2\partial_{t_1} + x\partial_x + 1)u = x(t_1+t_2)^2 + xt_1\partial_{t_2}u + (t_2\partial_{t_2}u)(\partial_x u), \\ u(t,x) = O(|t|^2). \end{cases}$$

In the linear part of derivatives in the right hand side of the equation, there exists a derivative related to t. Hence, by Theorem 3.3, the formal solution belongs to the Gevrey class of order at most $(s_1,s_2,\sigma) = (3,4,3)$.

Example 4.2. Let $t = (t_1,t_2) \in \mathbb{C}$ and $x \in \mathbb{C}$. We consider the equation

$$\begin{cases} (t_2\partial_{t_1} + x\partial_x + 1)u = x(t_1+t_2)^2 + x^2t_1\partial_x u + (t_2\partial_{t_2}u)(\partial_x u), \\ u(t,x) = O(|t|^2). \end{cases}$$

In the linear part of derivatives in the right hand side of the equation, there does not exist a derivative related to t. Hence, by Theorem 3.3, the formal solution belongs to the Gevrey class of order at most $(s_1,s_2,\sigma) = (3,4,1)$.

5. PROOF OF THEOREM 3.3

We define the set of homogeneous polynomials of degree L in t and degree M in x by

$$\mathbb{C}[t]_L[x]_M = \left\{ \sum_{|\alpha|=L,|\beta|=M} u_{\alpha,\beta} t^\alpha x^\beta \;\middle|\; u_{\alpha\beta} \in \mathbb{C} \right\}. \tag{5.1}$$

First we give a following lemma.

Lemma 5.1.

(i) *The operator* $P := \mathcal{N} + \mathcal{D} + \Delta$ *is invertible on* $\mathbb{C}[t]_L[x]_M$ *for all* $L \geq K$ *and* $M \geq 0$.

(ii) *Let* $\boldsymbol{s} = (\boldsymbol{s}^1, \ldots, \boldsymbol{s}^I) \in \mathbb{N}^d$ $(\boldsymbol{s}^j = (1, 2, \ldots, d_j) \in \mathbb{N}^{d_j})$, *and* $T = t_1 + \ldots + t_d \in \mathbb{C}$, $X = x_1 + \ldots + x_n \in \mathbb{C}$. *For* $u(t, x) \in \mathbb{C}[t]_L[x]_M$, *if a majorant relation* $\mathcal{B}_t^{\boldsymbol{s}}(u)(t, x) \ll W_{L,M} T^L X^M$ $(W_{L,M} \geq 0)$ *holds, then the following majorant relation holds by a positive constant* C_0 *independent of* L *and* M.

$$\mathcal{B}_t^{\boldsymbol{s}}(P^{-1}u)(t, x) \ll \frac{C_0}{M+1} W_{L,M} T^L X^M = C_0 (X\partial_X + 1)^{-1} W_{L,M} T^L X^M. \tag{5.2}$$

We omit the proof of Lemma 5.1, since the similar proposition is already proved in [14, Lemma 1].

By Lemma 5.1, the operator P is invertible on $\mathbb{C}[[t, x]]_K$ by

$$\mathbb{C}[[t, x]]_K = \bigcup_{L \geq K, M \geq 0} \mathbb{C}[t]_L[x]_M.$$

Here we put $U(t, x) = Pv(t, x)$ as a new unknown function. Then $U(t, x)$ satisfies the following:

$$\begin{cases} U(t,x) = \displaystyle\sum_{i,j,k,l} \alpha_{ijkl}(x) t_{i,j} \partial_{t_{k,l}} P^{-1}U + \sum_{k=1}^{n} \beta_k(x) \partial_{x_k} P^{-1}U \\ \qquad\qquad + \eta(x) P^{-1}U + \displaystyle\sum_{|\alpha|=K} \zeta_\alpha(x) t^\alpha \\ \qquad\qquad + g_{K+1}(t, x, P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U), \\ U(t,x) = O(|t|^K). \end{cases} \tag{5.3}$$

For the equation (5.3), we apply the Borel transform of order $\boldsymbol{s} = (\boldsymbol{s}^1, \ldots, \boldsymbol{s}^I) \in \mathbb{N}^d$ in t, then the equation is reduced to the following:

$$\begin{aligned} \mathcal{B}_t^{\boldsymbol{s}}(U)(t,x) = {} & \mathcal{B}_t^{\boldsymbol{s}} \left(\sum_{i,j,k,l} \alpha_{ijkl}(x) t_{i,j} \partial_{t_{k,l}} P^{-1}U \right) + \mathcal{B}_t^{\boldsymbol{s}} \left(\sum_{k=1}^{n} \beta_k(x) \partial_{x_k} P^{-1}U \right) \\ & + \mathcal{B}_t^{\boldsymbol{s}} \Big(\eta(x) P^{-1}U \Big) + \sum_{|\alpha|=K} \frac{\zeta_\alpha(x) |\alpha|!}{(\boldsymbol{s} \cdot \alpha)!} t^\alpha \\ & + \mathcal{B}_t^{\boldsymbol{s}} \Big\{ g_{K+1}(t, x, P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U) \Big\}. \end{aligned} \tag{5.4}$$

In order to estimate the Borel transforms of products and derivatives with respect to t and x, we give the following lemma.

Lemma 5.2.

(i) *For two formal power series $u(t,x), v(t,x) \in \mathbb{C}[[t,x]]$, there exists a positive constant C_1 depending only on $\boldsymbol{s}$ such that the following majorant relation holds.*

$$\mathcal{B}_t^{\boldsymbol{s}}(uv)(t,x) \ll C_1 \mathcal{B}_t^{\boldsymbol{s}}(|u|)(t,x) \times \mathcal{B}_t^{\boldsymbol{s}}(|v|)(t,x), \tag{5.5}$$

where for $u(t,x) = \sum u_{\alpha\beta} t^\alpha x^\beta$, $|u|(t,x)$ is defined by $|u|(t,x) = \sum |u_{\alpha\beta}| t^\alpha x^\beta$.

(ii) *We put $T = t_1 + \ldots + t_d$ and $X = x_1 + \ldots + x_n$. Let $W(T,X)$ be a formal power series in T and X. If $\mathcal{B}_t^{\boldsymbol{s}}(u)(t,x) \ll W(T,X)$, then the following majorant relations hold by a positive constant C_2.*

$$\begin{aligned}\mathcal{B}_t^{\boldsymbol{s}}\left(\partial_{t_{i,j}} P^{-1} u\right)(t,x) &\ll C_2 \partial_T (T\partial_T)^{j-1} (X\partial_X + 1)^{-1} W(T,X) \\ &\ll C_2 \partial_T (T\partial_T)^{j-1} W(T,X),\end{aligned} \tag{5.6}$$

$$\mathcal{B}_t^{\boldsymbol{s}}\left(\partial_{x_k} P^{-1} u\right)(t,x) \ll C_2 \partial_X (X\partial_X + 1)^{-1} W(T,X) \ll C_2 \times S(W)(T,X), \tag{5.7}$$

where $S(W)(T,X)$ is the shift function in X defined by

$$S(W)(T,X) = \frac{W(T,X) - W(T,0)}{X}. \tag{5.8}$$

We omit the proof of Lemma 5.2, since the similar proposition is already proved in [14, Lemma 2].

By Lemma 5.2, if a majorant relation $\mathcal{B}_t^{\boldsymbol{s}}(U)(t,x) \ll W(T,X)$ holds, then there exists a positive constant $C_3 > 0$ such that the following majorant relations hold.

$$\mathcal{B}_t^{\boldsymbol{s}}\left(\alpha_{ijkl}(x) t_{ij} \partial_{t_{kl}} P^{-1} U\right)(t,x) \ll C_3 |\alpha_{ijkl}|(X) (T\partial_T)^l W(T,X), \tag{5.9}$$

$$\mathcal{B}_t^{\boldsymbol{s}}\left(\beta_k(x) \partial_{x_k} P^{-1} U\right)(t,x) \ll C_3 |\beta_k|(X) S(W)(T,X), \tag{5.10}$$

$$\mathcal{B}_t^{\boldsymbol{s}}\left(\eta(x) P^{-1} U\right)(t,x) \ll C_3 |\eta|(X) W(T,X), \tag{5.11}$$

$$\sum_{|\alpha|=K} \frac{\zeta_\alpha(x)|\alpha|!}{(\boldsymbol{s}\cdot\alpha)!} t^\alpha \ll \left(\sum_{|\alpha|=K} |\zeta_\alpha|(X)\right) T^K =: \zeta(X) T^K, \tag{5.12}$$

$$\begin{aligned}&\mathcal{B}_t^{\boldsymbol{s}}\left(g_{K+1}(t,x,P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U)\right) \\ &\qquad \ll |g_{K+1}|(T,X,C_3 W, \{C_3 \partial_T (T\partial_T)^{j-1} W\}_{i,j}, \{C_3 S(W)\}_k).\end{aligned} \tag{5.13}$$

We remark that $1 \le j \le d_0$ for j in (5.13). Moreover, since $W(T,X)$ is a majorant series of $\mathcal{B}_t^{\boldsymbol{s}}(u)(t,x)$, we have $W(T,X) \gg 0$. Therefore, we obtain the following majorant relation.

$$XS(W)(T,X) = W(T,X) - W(T,0) \ll W(T,X). \tag{5.14}$$

Since $|\beta_k|(X) = O(X^2)$, we put a holomorphic function $|\hat{\beta}_k|(X)$ by $|\hat{\beta}_k|(X) := |\beta_k|(X)/X = O(X)$. Then the following majorant relation holds.

$$|\beta_k|(X)S(W) = \frac{|\beta_k|(X)}{X} \cdot XS(W) \ll \frac{|\beta_k|(X)}{X}W = |\hat{\beta_k}|(X)W.$$

We consider the following equation.

$$\begin{cases} W = \sum_{i,j,k,l} \tilde{\alpha}_{ijkl}(X)(T\partial_T)^l W + \sum_{k=1}^{n} \tilde{\beta}_k(X)W + \tilde{\eta}(X)W + \zeta(X)T^K \\ \qquad +|g_{K+1}|(T, X, C_3 W, \{C_3\partial_T(T\partial_T)^{j-1}W\}_{i,j}, \{C_3 S(W)\}_k), \\ W = O(T^K), \end{cases} \tag{5.15}$$

where $\tilde{\alpha}_{ijkl}(X) = C_3|\alpha_{ijkl}|(X)$, $\tilde{\beta}_k(X) = C_3|\hat{\beta}_k|(X)$, $\tilde{\eta}(X) = C_3|\eta|(X)$. These are all holomorphic functions in a neighborhood of $X = 0$ and vanish at $X = 0$. By the construction of this equation, it is easily seen that

$$\mathcal{B}_t^{\boldsymbol{S}}(U)(t, x) \ll W(T, X).$$

Here we put $F(X) = 1 - \sum_{k=1}^{n} \tilde{\beta}_k(X) - \tilde{\eta}(X)$. Since $F(0) = 1 \neq 0$, $1/F(X)$ is holomorphic in a neighborhood of $X = 0$. Therefore, by multiplying $1/F(X)$ for both sides, the equation (5.15) is reduced to the following.

$$\begin{aligned} W = &\sum_{i,j,k,l} \hat{\alpha}_{ijkl}(X)(T\partial_T)^l W + \hat{\zeta}(X)T^K \\ &+ G_{K+1}(T, X, C_3 W, \{C_3\partial_T(T\partial_T)^{j-1}W\}_{i,j}, \{C_3 S(W)\}_k), \end{aligned}$$

where

$$\hat{\alpha}_{ijkl}(X) = \tilde{\alpha}_{ijkl}(X)/F(X) = O(X), \quad \hat{\zeta}(X) = \zeta(X)/F(X)$$

and

$$G_{K+1}(T, X, u, \tau, \xi) = |g_{K+1}|(T, X, u, \tau, \xi)/F(X).$$

For the equation (5.16), the following lemma holds.

Lemma 5.3. *If $\hat{\alpha}_{ijkl}(X) \not\equiv 0$ for some i, j, k, l, then the formal solution $W(T, X)$ belongs to the Gevrey class $\mathcal{G}_{T,X}^{(d_0+1,d_0+1)}$. If $\hat{\alpha}_{ijkl}(X) \equiv 0$ for all i, j, k, l, then the formal solution $W(T, X)$ belongs to the Gevrey class $\mathcal{G}_{T,X}^{(d'+1,1)}$ where d' is the constant defined by* (3.5).

By Lemma 5.3, which will be proved in the next section, $W(T, X) \in \mathcal{G}_{T,X}^{(d_0+1,d_0+1)}$ if $\hat{\alpha}_{ijkl}(X) \not\equiv 0$ for some i, j, k, l or $W(T, X) \in \mathcal{G}_{T,X}^{(d'+1,1)}$ if $\hat{\alpha}_{ijkl}(X) \equiv 0$ for all i, j, k, l. On the other hand, for $\boldsymbol{s} = (\boldsymbol{s}^1, \ldots, \boldsymbol{s}^I)$ ($\boldsymbol{s}^i = (1, 2, \ldots, d_i)$), the majorant relation $\mathcal{B}_t^{\boldsymbol{S}}(U)(t, x) \ll W(T, X)$ holds. By combining these properties, we have

$$\mathcal{B}_t^{\boldsymbol{S}}(U)(t, x) = \mathcal{B}_{t,x}^{(\boldsymbol{S}, \mathbf{1}_n)}(U)(t, x) \in \mathcal{G}_{T,X}^{(d_0+1,d_0+1)} \text{ or } \mathcal{G}_{T,X}^{(d'+1,1)}.$$

Therefore, the Gevrey order $(\boldsymbol{s}', \boldsymbol{\sigma}')$ of $U(t,x)$ is obtained by

$$\boldsymbol{s}' = \begin{cases} \boldsymbol{s} + (d_0+1,\dots,d_0+1) - \mathbf{1}_d = \boldsymbol{s} + (d_0,\dots,d_0), \\ \qquad\qquad\qquad\qquad \text{if } \hat{\alpha}_{ijkl}(X) \not\equiv 0 \text{ for some } i,j,k,l, \\ \boldsymbol{s} + (d'+1,\dots,d'+1) - \mathbf{1}_d = \boldsymbol{s} + (d',\dots,d'), \\ \qquad\qquad\qquad\qquad \text{if } \hat{\alpha}_{ijkl}(X) \equiv 0 \text{ for all } i,j,k,l, \end{cases}$$

$$\boldsymbol{\sigma}' = \begin{cases} \mathbf{1}_n + (d_0+1,\dots,d_0+1) - \mathbf{1}_n = (d_0+1,\dots,d_0+1), \\ \qquad\qquad\qquad\qquad \text{if } \hat{\alpha}_{ijkl}(X) \not\equiv 0 \text{ for some } i,j,k,l, \\ \mathbf{1}_n + \mathbf{1}_n - \mathbf{1}_n = \mathbf{1}_n, \qquad \text{if } \hat{\alpha}_{ijkl}(X) \equiv 0 \text{ for all } i,j,k,l, \end{cases}$$

which proves Theorem 3.3.

6. PROOF OF LEMMA 5.3

First we consider the case when $\hat{\alpha}_{ijkl}(X) \not\equiv 0$ for some i,j,k,l. Let $\hat{\alpha}_{ijkl}(X) = \sum_{M\geq 1} \alpha_{ijklM} X^M$ be the Taylor expansion of $\hat{\alpha}_{ijkl}(X)$. For the sake of simplicity of notation, we put $C_3 = 1$. By putting $W(T,X) = \sum_{L\geq K} W_L(X) T^L$ and by substituting this into (5.16), we obtain the following recurrence formula for $\{W_L(X)\}_{L\geq K}$.

$$W_K(X) = \sum_{i,j,k,l} \hat{\alpha}_{ijkl}(X) K^l W_K(X) + \hat{\zeta}(X), \tag{6.1}$$

$$\begin{aligned} W_L(X) = & \sum_{i,j,k,l} \hat{\alpha}_{ijkl}(X) L^l W_L(X) \\ & + {\sum}' G_{\alpha pqr}(X) \prod_{k=1}^{p} W_{L_k}(X) \\ & \times \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} (L_{ijl})^j W_{L_{ijl}}(X) \prod_{k=1}^{n}\prod_{l=1}^{r_k} S(W_{L_{kl}})(X), \end{aligned} \tag{6.2}$$

where the summation $\sum'$ is taken over $V(\alpha,p,q,r) \geq K+1$ and

$$|\alpha| + \sum_{k=1}^{p} L_k + \sum_{i=1}^{I}\sum_{j=1}^{d_i}\sum_{l=1}^{q_{ij}} (L_{ijl}-1) + \sum_{k=1}^{n}\sum_{l=1}^{r_k} L_{kl} = L. \tag{6.3}$$

The first recurrence formula (6.1) is an easier situation than (6.2). Therefore, in the following, we consider the case of (6.2).

We put $W_L(X) = \sum_{M\geq 0} W_{L,M} X^M$. By substituting this in the formula (6.2), we get the following recurrence formula for $\{W_{L,M}\}_{L\geq K, M\geq 0}$.

$$\begin{aligned} W_{L,M} = & \sum_{i,j,k,l}\sum_{M_1=1}^{M} L^l \alpha_{ijklM_1} W_{L,M-M_1} \\ & + {\sum}'{\sum}'' G_{\alpha pqrM'} \prod_{k=1}^{p} W_{L_k,M_k} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} (L_{ijl})^j W_{L_{ijl},M_{ijl}} \prod_{k=1}^{n}\prod_{l=1}^{r_k} W_{L_{kl},M_{kl}+1}, \end{aligned} \tag{6.4}$$

where the summation $\sum''$ is taken over

$$M' + \sum_{k=1}^{p} M_k + \sum_{i=1}^{I}\sum_{j=1}^{d_i}\sum_{l=1}^{q_{ij}} M_{ijl} + \sum_{k=1}^{n}\sum_{l=1}^{r_k} M_{kl} = M. \tag{6.5}$$

We set $Y_{L,M} = W_{L,M}/(L+M)!^{d_0}$. Then $\{Y_{L,M}\}$ satisfies the following recurrence formula.

$$\begin{aligned} Y_{L,M} = &\sum_{i,j,k,l}\sum_{M_1=1}^{M} \hat{C}_1 \alpha_{ijklM_1} Y_{L,M-M_1} \\ &+ \sum{}'\sum{}'' \hat{C}_2 G_{\alpha pqrM'} \prod_{k=1}^{p} Y_{L_k,M_k} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} Y_{L_{ijl},M_{ijl}} \prod_{k=1}^{n}\prod_{l=1}^{r_k} Y_{L_{kl},M_{kl}+1}, \end{aligned} \tag{6.6}$$

where $\hat{C}_1$, $\hat{C}_2 = \hat{C}_2(\alpha,p,q,r)$ and

$$\begin{aligned} \hat{C}_1 &= \frac{L^l (L+M-M_1)!^{d_0}}{(L+M)!^{d_0}}, \\ \hat{C}_2 &= \frac{1}{(L+M)!^{d_0}} \times \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} (L_{ijl})^j \\ &\quad \times \prod_{k=1}^{p} (L_k+M_k)!^{d_0} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} (L_{ijl}+M_{ijl})!^{d_0} \prod_{k=1}^{n}\prod_{l=1}^{r_k} (L_{kl}+M_{kl}+1)!^{d_0}. \end{aligned}$$

Since $l \le d_0$ and $M_1 \ge 1$, we get the following estimate for $\hat{C}_1$.

$$\hat{C}_1 = \frac{L^l (L+M-M_1)!^{d_0}}{(L+M)!^{d_0}} \le \frac{L^l (L+M-1)!^{d_0}}{(L+M)!^{d_0}} = \frac{L^l}{(L+M)^{d_0}} \le 1.$$

For the estimate of $\hat{C}_2$, we need the following lemma which is proved in [12, Lemma 6].

Lemma 6.1. *Let L and m_j be nonnegative integers such that $m_j \ge L$ for all $j = 1, 2, \dots, n$. Then the following inequality holds:*

$$m_1! \dots m_n! \le (L!)^{n-1} (m_1 + \dots + m_n - (n-1)L)!. \tag{6.7}$$

By using Lemma 6.1, we can estimate $\hat{C}_2 = \hat{C}_2(\alpha, p, q, r)$ as follows.

$$\begin{aligned}
\hat{C}_2 &= \frac{\prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} {L_{ijl}}^j}{(L+M)!^{d_0}} \\
&\quad \times \prod_{k=1}^{p}(L_k+M_k)!^{d_0} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ijl}+M_{ijl})!^{d_0} \prod_{k=1}^{n}\prod_{l=1}^{r_k}(L_{kl}+M_{kl}+1)!^{d_0} \\
&\le \frac{\prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} {L_{ijl}}^{j-d_0}}{(L+M)!^{d_0}} \\
&\quad \times \left\{\prod_{k=1}^{p}(L_k+M_k+1)! \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ijl}+M_{ijl}+1)! \prod_{k=1}^{n}\prod_{l=1}^{r_k}(L_{kl}+M_{kl}+1)!\right\}^{d_0} \\
&\le \frac{(K+1)!^{d_0(p+|q|+|r|)}}{(L+M)!^{d_0}} \\
&\quad \times \Bigg(\sum_{k=1}^{p}(L_k+M_k+1) + \sum_{i=1}^{I}\sum_{j=1}^{d_i}\sum_{l=1}^{q_{ij}}(L_{ijl}+M_{ijl}+1) \\
&\qquad\qquad + \sum_{k=1}^{n}\sum_{l=1}^{r_k}(L_{kl}+M_{kl}+1) - (K+1)(p+|q|+|r|-1)\Bigg)!^{d_0} \\
&= \frac{\hat{C}_3^{p+|q|+|r|}(L+M-|\alpha|-M'+p+2|q|+|r|-(K+1)(p+|q|+|r|-1))!^{d_0}}{(L+M)!^{d_0}} \\
&\quad (\text{we put } \hat{C}_3 := (K+1)!^{d_0}) \\
&= \frac{\hat{C}_3^{p+|q|+|r|}(L+M+K+1-V(\alpha,p,q,r))!^{d_0}}{(L+M)!^{d_0}} \le \hat{C}_3^{p+|q|+|r|},
\end{aligned}$$

where

$$V(\alpha,p,q,r) = |\alpha| + Kp + (K-1)|q| + K|r|.$$

By these observations, (6.6) is estimated by

$$\begin{aligned}
Y_{L,M} &\le \sum_{i,j,k,l}\sum_{M_1=1}^{M} \alpha_{ijklM_1} Y_{L,M-M_1} \\
&\quad + \sum{}' \sum{}'' G_{\alpha pqrM'} \prod_{k=1}^{p}\hat{C}_3 Y_{L_k,M_k} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}\hat{C}_3 Y_{L_{ijl},M_{ijl}} \prod_{k=1}^{n}\prod_{l=1}^{r_k}\hat{C}_3 Y_{L_{kl},M_{kl}+1}.
\end{aligned}$$

Let us consider the following recurrence formula for $\{Z_{L,M}\}_{L\geq K, M\geq 0}$:

$$Z_{K,M} = \sum_{i,j,k,l} \sum_{M_1=1}^{M} \alpha_{ijklM_1} Z_{K,M-M_1} + \hat{\zeta}_M,$$

$$\begin{aligned} Z_{L,M} = & \sum_{i,j,k,l} \sum_{M_1=1}^{M} \alpha_{ijklM_1} Z_{L,M-M_1} \\ & + \sum{}' \sum{}'' G_{\alpha pqrM'} \prod_{k=1}^{p} \hat{C}_3 Z_{L_k,M_k} \prod_{i=1}^{I} \prod_{j=1}^{d_i} \prod_{l=1}^{q_{ij}} \hat{C}_3 Z_{L_{ijl},M_{ijl}} \prod_{k=1}^{n} \prod_{l=1}^{r_k} \hat{C}_3 Z_{L_{kl},M_{kl}+1}. \end{aligned}$$

By this construction of the recurrence formula, we have $Y_{L,M} \leq Z_{L,M}$.

We remark that these recurrence formulas are obtained by the following equation:

$$\begin{aligned} Z(T,X) = & \sum_{i,j,k,l} \hat{\alpha}_{ijkl}(X) Z + \hat{\zeta}(X) T^K \\ & + G_{K+1}\left(T, X, \hat{C}_3 Z, \left\{\frac{\hat{C}_3 Z}{T}\right\}, \left\{\hat{C}_3 S(Z)\right\}\right) \end{aligned} \tag{6.8}$$

with $Z(T,X) = O(T^K)$. By dividing both sides of the equation by $1-\sum_{i,j,k,l} \hat{\alpha}_{ijkl}(X)$, the equation is reduced to the following one:

$$Z(T,X) = \bar{\zeta}(X) T^K + H_{K+1}\left(T, X, \hat{C}_3 Z, \left\{\frac{\hat{C}_3 Z}{T}\right\}, \left\{\hat{C}_3 S(Z)\right\}\right), \tag{6.9}$$

where

$$\bar{\zeta}(X) = \hat{\zeta}(X) \Big/ \Big(1 - \sum_{i,j,k,l} \hat{\alpha}_{ijkl}(X)\Big) \in \mathbb{C}\{X\}$$

and

$$H_{K+1} = G_{K+1} \Big/ \Big(1 - \sum_{i,j,k,l} \hat{\alpha}_{ijkl}(X)\Big)$$

is holomorphic in a neighborhood of the origin.

By this construction of the equation (6.9), we have the following majorant relation between Z and Y.

$$Z(T,X) \gg Y(T,X) = \mathcal{B}_{T,X}^{(d_0+1,d_0+1)}(W)(T,X).$$

We put $\varphi(T,X) = Z(T,X)/T$ as a new unknown function. Then $\varphi(T,X)$ satisfies the following.

$$\varphi(T,X) = \bar{\zeta}(X) T^{K-1} + \frac{1}{T} H_{K+1}(T, X, \hat{C}_3 T\varphi, \{\hat{C}_3 \varphi\}, \{\hat{C}_3 T S(\varphi)\}) \tag{6.10}$$

with $\varphi(T,X) = O(T^{K-1})$.

We decompose the formal solution $\varphi(T,X)$ as follows.

$$\varphi(T,X)=\varphi_1(X)T^{K-1}+\varphi_2(X)T^K+T^K\psi(T,X),\quad \psi(0,X)\equiv 0.$$

By an easy calculation, $\varphi_1(X)$ and $\varphi_2(X)$ are given by

$$\varphi_1(X)=\bar{\zeta}(X),$$

$$\varphi_2(X)=\sum_{|\alpha|+Kp+(K-1)|q|+K|r|=K+1} H_{\alpha pqr}(X)\hat{C}_3^{p+|q|+|r|}\varphi_1(X)^{p+|q|}S(\varphi_1)(X)^{|r|}.$$

We remark that these are holomorphic in a neighborhood of $X=0$.

Moreover, $\psi(T,X)$ satisfies the following equation:

$$\begin{cases}\psi(T,X)=H(T,X,T\psi,TS(\psi)),\\ \psi(0,X)\equiv 0,\end{cases}\tag{6.11}$$

where

$$\begin{aligned}H(T,X,\eta_1,\eta_2)=\frac{1}{T^{K+1}}\Big[H_{K+1}\Big(&T,X,\hat{C}_3\varphi_1(X)T^K+\hat{C}_3\varphi_2(X)T^{K+1}+\hat{C}_3T^K\eta_1,\\ &\{\hat{C}_3\varphi_1(X)T^{K-1}+\hat{C}_3\varphi_2(X)T^K+\hat{C}_3T^K\eta_1\},\\ &\{\hat{C}_3S(\varphi_1)(X)T^K+\hat{C}_3S(\varphi_2)(X)T^{K+1}+\hat{C}_3T^K\eta_2\}\Big)\Big]\\ -\sum_{|\alpha|+Kp+(K-1)|q|+K|r|=K+1}&H_{\alpha pqr}(X)\hat{C}_3^{p+|q|+|r|}\varphi_1(X)^{p+|q|}S(\varphi_1)(X)^{|r|}.\end{aligned}$$

We remark that the order of zeros in T of $H(T,X,T\psi(T,X),TS(\psi)(T,X))$ is greater than or equal to 1.

In order to prove the convergence of $\psi(T,X)$, it is sufficient to show the following:

Lemma 6.2. *There exists a small positive constant $\varepsilon>0$ such that $\psi_\varepsilon(\rho):=\psi(\varepsilon\rho,\rho)$ is convergent in a neighborhood of $\rho=0$.*

The proof of Lemma 6.2 can be found in [13], so we omit it.

Lemma 6.2 implies that $\varphi(T,X)$ is convergent. Therefore, the following majorant relations hold: In the case $\hat{\alpha}_{ijkl}(X)\not\equiv 0$ for some i,j,k,l,

$$\mathbb{C}\{T,X\}\ni T\varphi(T,X)=Z(T,X)\gg Y(T,X)=\mathcal{B}_{T,X}^{(d_0+1,d_0+1)}(W)(T,X).$$

Next we consider the case when $\hat{\alpha}_{ijkl}(X)\equiv 0$ for all i,j,k,l. In this case, the argument follows analogously from above by removing the term $\sum_{i,j,k,l}\hat{\alpha}_{ijkl}(X)(T\partial_T)^lW$. Therefore, by putting $W(T,X)=\sum_{L\geq K}\sum_{M\geq 0}W_{L,M}T^LX^M$, we get the following recurrence formula for $\{W_{L,M}\}_{L\geq K,M\geq 0}$

$$W_{L,M}=\sum{}'\sum{}''G_{\alpha pqrM'}\prod_{k=1}^{p}W_{L_k,M_k}\prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ijl})^jW_{L_{ijl},M_{ijl}}\prod_{k=1}^{n}\prod_{l=1}^{r_k}W_{L_{kl},M_{kl}+1},\tag{6.12}$$

where the summation $\sum'$ and $\sum''$ are taken by in the same way as (6.3) and (6.5), respectively. In this case we put $Y_{L,M} = W_{L,M}/L!^{d'}$, where d' is defined by (3.5). Then $\{Y_{L,M}\}$ satisfies the following recurrence formula.

$$Y_{L,M} = \sum{}' \sum{}'' \tilde{C}_2 G_{\alpha pqrM'} \prod_{k=1}^{p} Y_{L_k,M_k} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} Y_{L_{ijl},M_{ijl}} \prod_{k=1}^{n}\prod_{l=1}^{r_k} Y_{L_{kl},M_{kl}+1}, \tag{6.13}$$

where $\tilde{C}_2 = \tilde{C}_2(\alpha,p,q,r)$ and

$$\tilde{C}_2 = \frac{\prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ijl})^j \times \prod_{k=1}^{p} L_k!^{d'} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} L_{ijl}!^{d'} \prod_{k=1}^{n}\prod_{l=1}^{r_k} L_{kl}!^{d'}}{L!^{d'}}.$$

By using Lemma 6.1, for an arbitrary $\mathcal{L} \in \mathbb{N}$, $\tilde{C}_2$ is estimated by

$$\begin{aligned}
\tilde{C}_2 &= \frac{\prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ijl})^j \times \{\prod_{k=1}^{p} L_k! \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} L_{ijl}! \prod_{k=1}^{n}\prod_{l=1}^{r_k} L_{kl}!\}^{d'}}{L!^{d'}} \\
&\le \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ijl})^{d_0 - d'\mathcal{L}} \\
&\quad \times \frac{\{\prod_{k=1}^{p}(L_k+\mathcal{L})! \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ijl}+\mathcal{L})! \prod_{k=1}^{n}\prod_{l=1}^{r_k}(L_{kl}+\mathcal{L})!\}^{d'}}{L!^{d'}} \\
&\le (K+\mathcal{L})!^{d'(p+|q|+|r|-1)} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ij})^{d_0 - d'\mathcal{L}} \frac{(L - V(\alpha,p,q,r)+K+\mathcal{L})!^{d'}}{L!^{d'}}.
\end{aligned}$$

Here we set $\Omega_1 = \left\{(\alpha,p,q,r)\,;\, \dfrac{d_0}{V(\alpha,p,q,r)-K} = d'\right\}$ (this is a finite set), $\Omega_2 = \left\{(\alpha,p,q,r)\,;\, \dfrac{d_0}{V(\alpha,p,q,r)-K} < d'\right\}$ and we set

$$\mathcal{L} = \mathcal{L}(\alpha,p,q,r) = \begin{cases} V(\alpha,p,q,r)-K & \text{if } (\alpha,p,q,r) \in \Omega_1, \\ \left[\dfrac{d_0}{d'}\right] + 1 & \text{if } (\alpha,p,q,r) \in \Omega_2. \end{cases}$$

Remark that

$$\frac{d_0}{d'} < \mathcal{L} \le \frac{d_0}{d'} + 1 < V(\alpha,p,q,r) - K + 1$$

holds for $(\alpha,p,q,r) \in \Omega_2$. By this inequality, $\mathcal{L} \le V(\alpha,p,q,r) - K$ holds for all (α,p,q,r), because $\mathcal{L}$ and $V(\alpha,p,q,r) - K + 1$ are natural numbers. By the choice of $\mathcal{L}$, we have $\tilde{C}_2 \le \tilde{C}_3^{p+|q|+|r|}$ by $\tilde{C}_3 = (K + \max \mathcal{L})!^{d'}$. Therefore, $\tilde{C}_2$ can be estimated by the same form as the case $\hat{\alpha}_{ijkl}(X) \not\equiv 0$.

By these observations, (6.13) is estimated by

$$Y_{L,M} \le \sum{}' \sum{}'' G_{\alpha pqrM'} \prod_{k=1}^{p} \tilde{C}_3 Y_{L_k,M_k} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} \tilde{C}_3 Y_{L_{ijl},M_{ijl}} \prod_{k=1}^{n}\prod_{l=1}^{r_k} \tilde{C}_3 Y_{L_{kl},M_{kl}+1}, \tag{6.14}$$

Let us consider the following recurrence formula for $\{Z_{L,M}\}_{L\geq K,M\geq 0}$: $Z_{K,M} = \hat{\zeta}_M$ and

$$Z_{L,M} = \sum{}' \sum{}'' G_{\alpha pqrM'} \prod_{k=1}^{p} \tilde{C}_3 Z_{L_k,M_k} \prod_{i=1}^{I} \prod_{j=1}^{d_i} \prod_{l=1}^{q_{ij}} \tilde{C}_3 Z_{L_{ijl},M_{ijl}} \prod_{k=1}^{n} \prod_{l=1}^{r_k} \tilde{C}_3 Z_{L_{kl},M_{kl}+1}. \tag{6.15}$$

This recurrence formula is obtained by the following equation:

$$Z(T,X) = \hat{\zeta}(X)T^K + G_{K+1}\left(T, X, \tilde{C}_3 Z, \left\{\frac{\tilde{C}_3 Z}{T}\right\}, \left\{\tilde{C}_3 S(Z)\right\}\right) \tag{6.16}$$

with $Z(T,X) = O(T^K)$. By the construction of equation (6.16), we have

$$Z(T,X) \gg Y(T,X) = \mathcal{B}_{T,X}^{(d'+1,1)}(W)(T,X),$$

and (6.16) is the same form as (6.9). Therefore, the convergence of a formal solution $Z(T,X)$ follows from the same argument in the case when $\hat{\alpha}_{ijkl}(X) \not\equiv 0$ for some i,j,k,l. This completes the proof of Lemma 5.3.

7. REFINEMENT OF THEOREM 1.7

In this section, we shall prove Theorem 1.7. To do so, we reduce (2.1) to the more exact form.

By the same linear change of t variables as in section 3, the vector field is reduced to the following.

$$\sum_{i,j=1}^{d} a_{i,j}(x) t_i \partial_{t_j} \mapsto (\tau^{(1)}, \ldots, \tau^{(I)}) \begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_I \end{pmatrix} \begin{pmatrix} \partial_{\tau^{(1)}} \\ \vdots \\ \partial_{\tau^{(I)}} \end{pmatrix} - \sum_{i,j,k,l} \alpha_{ijkl}(x) \tau_{i,j} \partial_{\tau_{k,l}},$$

where $\alpha_{ijkl}(x) = O(|x|)$ denote holomorphic functions and N_j $(j = 1, \ldots, I)$ denotes the nilpotent Jordan block of size d_j.

Next, we write the differential operator with respect to x variables by the following form:

$$\sum_{k=1}^{n} b_k(x) \partial_{x_k} = (x_1, \ldots, x_n) J(b_1, \ldots, b_n)(0) \begin{pmatrix} \partial_{x_1} \\ \vdots \\ \partial_{x_n} \end{pmatrix} - \sum_{k=1}^{n} \hat{b}_k(x) \partial_{x_k},$$

where $\hat{b}_k(x) = O(|x|^2)$ $(k = 1, \ldots, n)$. Then we introduce new variables $\xi = (\xi^{(1)}, \ldots, \xi^{(J)}) \in \mathbb{C}^n$ $(\xi^{(k)} = (\xi_{k,1}, \ldots, \xi_{k,n_k}) \in \mathbb{C}^{n_k})$ by

$$(\xi^{(1)}, \ldots, \xi^{(J)}) = (x_1, \ldots, x_n) Q, \qquad Q^{-1} J(b_1, \ldots, b_n)(0) Q = \begin{pmatrix} \hat{N}_1 & & \\ & \ddots & \\ & & \hat{N}_J \end{pmatrix}$$

where $\hat{N}_j$ $(j = 1, \dots, J)$ denotes the nilpotent Jordan block of size n_j. By this linear change of variables x, the above vector field with respect to x is reduced to

$$(\xi^{(1)}, \dots, \xi^{(J)}) \begin{pmatrix} \hat{N}_1 & & \\ & \ddots & \\ & & \hat{N}_J \end{pmatrix} \begin{pmatrix} \partial_{\xi^{(1)}} \\ \vdots \\ \partial_{\xi^{(J)}} \end{pmatrix} - \sum_{i=1}^{J} \sum_{j=1}^{n_i} \beta_{ij}(\xi) \partial_{\xi_{i,j}},$$

where $\beta_{ij}(\xi) = O(|\xi|^2)$ $(i = 1, \dots, J; j = 1, \dots, n_i)$ denotes a holomorphic function.

Hereafter we rewrite (τ, ξ) by (t, x) again. Then the equation (2.1) is rewritten as follows.

$$\begin{cases} \Big(\mathcal{N}_1 + \mathcal{N}_2 + c(0)\Big) v = \displaystyle\sum_{i,j,k,l} \alpha_{ijkl}(x) t_{i,j} \partial_{t_{k,l}} v \\ \qquad\qquad + \displaystyle\sum_{i=1}^{J} \sum_{j=1}^{n_i} \beta_{ij}(x) \partial_{x_{i,j}} v + \eta(x) v \\ \qquad\qquad + \displaystyle\sum_{|\alpha|=K} \zeta_\alpha(x) t^\alpha + g_{K+1}(t, x, v, \partial_t v, \partial_x v), \\ v(t,x) = O(|t|^K), \end{cases} \tag{7.1}$$

where $c(x) = f_u(\boldsymbol{\varphi}(x))$, $\eta(x) = c(0) - c(x) = O(|x|)$ and

$$\mathcal{N}_1 = \sum_{j=1}^{I} \sum_{k=1}^{d_j - 1} \delta t_{j,k+1} \partial_{t_{j,k}}, \quad \mathcal{N}_2 = \sum_{j=1}^{J} \sum_{k=1}^{n_j - 1} \delta x_{j,k+1} \partial_{x_{j,k}}. \tag{7.2}$$

Moreover,

$$\begin{aligned} g_{K+1}(t, x, v, \tau, \xi) &= \sum_{V(\alpha,p,q,r) \geq K+1} g_{\alpha pqr}(x) t^\alpha v^p \tau^q \xi^r, \\ V(\alpha, p, q, r) &= |\alpha| + Kp + (K-1)|q| + K|r|. \end{aligned} \tag{7.3}$$

We put $q = (q_{ij})$ $(1 \leq i \leq I, 1 \leq j \leq d_i)$ and $r = (r_{ij})$ $(1 \leq i \leq J, 1 \leq j \leq n_i)$ which are associated with $t = (t_{ij}) \in \mathbb{C}^d$ and $x = (x_{ij}) \in \mathbb{C}^n$. By using these notations and definitions, we obtain the following result:

Theorem 7.1. *Let* $\boldsymbol{s}^j = (1, 2, \dots, d_j) \in \mathbb{N}^{d_j}$ $(j = 1, 2, \dots, I)$, $\boldsymbol{\sigma}^j = (1, 2, \dots, n_j) \in \mathbb{N}^{n_j}$ $(j = 1, 2, \dots, J)$ *and let* $n_0 = \max\{d_1, d_2, \dots, d_I, n_1, n_2, \dots, n_J\}$. *We put* $N(\alpha, p, q, r) = \max\{j\, ;\, q_{i,j} \neq 0$ *or* $r_{i,j} \neq 0\}$ *for each nonzero term* $g_{\alpha,p,q,r}(x) t^\alpha u^p \tau^q \xi^r$. *Here we define a positive constant* n' *by*

$$n' = \max_{\alpha,p,q,r} \left\{ \frac{N(\alpha,p,q,r)}{V(\alpha,p,q,r) - K} \right\}. \tag{7.4}$$

Then under Assumptions 1.1, 1.2, 1.3 and $c(0) (= f_u(\boldsymbol{\varphi}(0))) \neq 0$, *the formal solution belongs to the Gevrey class of order at most* $(\boldsymbol{s}', \boldsymbol{\sigma}')$ *by*

$$(\boldsymbol{s}', \boldsymbol{\sigma}') = \begin{cases} (\boldsymbol{s} + \boldsymbol{n}_0^1, \boldsymbol{\sigma} + \boldsymbol{n}_0^2) & \text{if } \alpha_{i,j,k,l}(x) \not\equiv 0 \text{ or } \beta_{i,j}(x) \not\equiv 0 \text{ for some } i,j,k,l, \\ (\boldsymbol{s} + \boldsymbol{n}', \boldsymbol{\sigma}) & \text{if } \alpha_{i,j,k,l}(x) \equiv 0 \text{ and } \beta_{i,j}(x) \equiv 0 \text{ for all } i,j,k,l, \end{cases}$$

where $\boldsymbol{s} = (\boldsymbol{s}^1, \dots, \boldsymbol{s}^I) \in \mathbb{N}^d$, $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^J) \in \mathbb{N}^n$, $\boldsymbol{n}_0^1 = (n_0, \dots, n_0) \in \mathbb{N}^d$, $\boldsymbol{n}_0^2 = (n_0, \dots, n_0) \in \mathbb{N}^n$ *and* $\boldsymbol{n}' = (n', \dots, n') \in \mathbb{R}^d$.

Theorem 1.7 is an immediate consequence of Theorem 7.1. Indeed, all the components of $(\boldsymbol{s}^1, \dots, \boldsymbol{s}^I)$ and $(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^J)$ are estimated by n_0 and $n' \leqq n_0$. Therefore, all the components of $\boldsymbol{s}'$ and $\boldsymbol{\sigma}'$ are estimated by $2n_0$, which gives the conclusion of Theorem 1.7.

8. EXAMPLES FOR THEOREM 7.1

Example 8.1. Let $t = (t_1, t_2) \in \mathbb{C}^2$ and $x = (x_1, x_2, x_3) \in \mathbb{C}^3$, and consider

$$\begin{cases} (t_2\partial_{t_1} + x_2\partial_{x_1} + 1)u(t,x) = (t_1 + t_2)^2 + x_1 t_1 \partial_{t_2} u + t_1 t_2 u \times \partial_{x_3} u, \\ u = O(|t|^2). \end{cases}$$

Since $\alpha_{i,j,k,l}(x) = x_1 \not\equiv 0$, the Gevrey order is estimated by

$$(\boldsymbol{s}', \boldsymbol{\sigma}') = (1,2,1,2,1) + (2,2,2,2,2) = (3,4,3,4,3).$$

Example 8.2. Let $t = (t_1, t_2) \in \mathbb{C}^2$ and $x = (x_1, x_2, x_3) \in \mathbb{C}^3$, and consider

$$\begin{cases} (t_2\partial_{t_1} + x_2\partial_{x_1} + 1)u(t,x) = (t_1 + t_2)^2 + t_1 t_2 u \times \partial_{x_3} u, \\ u = O(|t|^2). \end{cases}$$

Since $\alpha_{i,j,k,l}(x) \equiv 0$ and $\beta_{i,j}(x) \equiv 0$, the Gevrey order is estimated by

$$(\boldsymbol{s}', \boldsymbol{\sigma}') = (1,2,1,2,1) + \left(\frac{1}{4}, \frac{1}{4}, 0, 0, 0\right) = \left(\frac{5}{4}, \frac{9}{4}, 1, 2, 1\right).$$

9. PROOF OF THEOREM 7.1

In order to prove Theorem 7.1, we prepare lemmas.

Lemma 9.1.

(i) *The operator* $P := \mathcal{N}_1 + \mathcal{N}_2 + c(0)$ $(c(0) \neq 0)$ *is invertible on* $\mathbb{C}[t]_L[x]_M$ *for all* $L \geq K$ *and* $M \geq 0$.

(ii) *Let* $\boldsymbol{s} = (\boldsymbol{s}^1, \dots, \boldsymbol{s}^I) \in \mathbb{N}^d$ *and* $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^J) \in \mathbb{N}^d$ *be as before, and* $T = t_1 + \dots + t_d \in \mathbb{C}$, $X = x_1 + \dots + x_n \in \mathbb{C}$. *For* $u(t,x) \in \mathbb{C}[t]_L[x]_M$, *if a majorant relation* $\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(u)(t,x) \ll W_{L,M} T^L X^M$ $(W_{L,M} \geq 0)$ *holds, then the following majorant relation holds by a positive constant* C_0 *independent of* L *and* M.

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(P^{-1}u)(t,x) \quad \ll \quad C_0 W_{L,M} T^L X^M \tag{9.1}$$

Lemma 9.2. *Let* $\boldsymbol{s} = (\boldsymbol{s}^1, \ldots, \boldsymbol{s}^I) \in \mathbb{N}^d$ *and* $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^J) \in \mathbb{N}^n$*. We put* $T = t_1 + \ldots + t_d$ *and* $X = x_1 + \ldots + x_n$*. For a formal power series* $W(T, X)$ *in* T *and* X*, if* $\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(u)(t, x) \ll W(T, X)$*, then the following majorant relations hold by a positive constant* C_2*:*

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\partial_{t_{i,j}} P^{-1}u\Big)(t, x) \ll C_2 \partial_T (T\partial_T)^{j-1} W(T, X), \tag{9.2}$$

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\partial_{x_{i,j}} P^{-1}u\Big)(t, x) \ll C_2 \partial_X (X\partial_X)^{j-1} W(T, X). \tag{9.3}$$

The proofs of Lemmas 9.1 and 9.2 are similar to those of Lemmas 5.1 and 5.2, so we omit them.

We put $U(t, x) = Pv(t, x)$ as a new unknown function. Then $U(t, x)$ satisfies the following:

$$\begin{cases} U(t, x) = \displaystyle\sum_{i,j,k,l} \alpha_{ijkl}(x) t_{i,j} \partial_{t_{k,l}} P^{-1}U + \sum_{i=1}^{J}\sum_{j=1}^{n_i} \beta_{ij}(x)\partial_{x_{ij}} P^{-1}U + \eta(x)P^{-1}U \\ \qquad\qquad + \displaystyle\sum_{|\alpha|=K} \zeta_\alpha(x) t^\alpha + g_{K+1}(t, x, P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U), \\ U(t, x) = O(|t|^K). \end{cases} \tag{9.4}$$

For equation (9.4), we apply the $(\boldsymbol{s}, \boldsymbol{\sigma})$-Borel transform, then (9.4) is reduced to the following:

$$\begin{aligned} \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(U)(t, x) = {} & \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\sum_{i,j,k,l} \alpha_{ijkl}(x) t_{i,j} \partial_{t_{k,l}} P^{-1}U\Big) \\ & + \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\sum_{k=i}^{J}\sum_{j=1}^{n_i} \beta_{ij}(x)\partial_{x_{ij}} P^{-1}U\Big) \\ & + \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\eta(x)P^{-1}U\Big) + \sum_{|\alpha|=K} \frac{\zeta_\alpha(x)|\alpha|!|\beta|!}{(\boldsymbol{s}\cdot\alpha)!(\boldsymbol{\sigma}\cdot\beta)!} t^\alpha \\ & + \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big\{g_{K+1}(t, x, P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U)\Big\}. \end{aligned} \tag{9.5}$$

By Lemma 5.2, (i) and Lemma 9.2, if a majorant relation $\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(U)(t, x) \ll W(T, X)$ is satisfied, then there exists a positive constant C_3 such that the following majorant relations hold:

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\alpha_{ijkl}(x) t_{ij} \partial_{t_{kl}} P^{-1}U\Big)(t, x) \ll C_3 |\alpha_{ijkl}|(X)(T\partial_T)^l W(T, X), \tag{9.6}$$

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\beta_{ij}(x)\partial_{x_{i,j}} P^{-1}U\Big)(t, x) \ll C_3 |\beta_{ij}|(X)\partial_X (X\partial_X)^{j-1} W(T, X), \tag{9.7}$$

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\eta(x)P^{-1}U\Big)(t, x) \ll C_3 |\eta|(X) W(T, X), \tag{9.8}$$

$$\begin{aligned} & \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(g_{K+1}(t, x, P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U)\Big) \\ & \ll |g_{K+1}|(T, X, C_3 W, \{C_3 \partial_T (T\partial_T)^{j-1} W\}_{i,j}, \{C_3 \partial_X (X\partial_X)^{j-1} W\}_{i,j}). \end{aligned} \tag{9.9}$$

On the other hand, for the Borel transform of $\sum \zeta_\alpha(x)t^\alpha$, we have

$$\sum_{|\alpha|=K} \frac{\zeta_\alpha(x)|\alpha|!|\beta|!}{(\boldsymbol{s}\cdot\alpha)!(\boldsymbol{\sigma}\cdot\beta)!}t^\alpha \ll \Bigl(\sum_{|\alpha|=K}|\zeta_\alpha|(X)\Bigr)T^K =: \zeta(X)T^K. \tag{9.10}$$

We remark that $1 \leq j \leq n_0 = \max\{d_1, \ldots, d_I, n_1, \ldots, n_J\}$ for j in (9.9).

Since $|\beta_{ij}|(X) = O(X^2)$, we put a holomorphic function $|\hat{\beta}_{ij}|(X) = |\beta_{ij}|(X)/X = O(X)$. Then the following relation holds.

$$|\beta_{ij}|(X)\partial_X(X\partial_X)^{j-1}W =: |\hat{\beta_{ij}}|(X)(X\partial_X)^j W.$$

We consider the following equation.

$$\begin{aligned} W = &\sum_{i,j,k,l}\tilde{\alpha}_{ijkl}(X)(T\partial_T)^l W + \sum_{i=1}^{J}\sum_{j=1}^{n_i}\tilde{\beta}_{ij}(X)(X\partial_X)^j W + \tilde{\eta}(X)W + \zeta(X)T^K \\ &+ |g_{K+1}|(T, X, C_3W, \{C_3\partial_T(T\partial_T)^{j-1}W\}_{i,j}, \{C_3\partial_X(X\partial_X)^{j-1}W\}_{i,j}), \end{aligned} \tag{9.11}$$

with $W = O(T^K)$, where $\tilde{\alpha}_{ijkl}(X) = C_3|\alpha_{ijkl}|(X)$, $\tilde{\beta}_{ij}(X) = C_3|\hat{\beta_{ij}}|(X)$ and $\tilde{\eta}(X) = C_3|\eta|(X)$ all vanish at $X = 0$. By the construction of this equation, it is easily seen that

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(U)(t,x) \ll W(T,X).$$

Now we put $F(X) = 1 - \tilde{\eta}(X)$. Since $F(0) = 1$, by multiplying $1/F(X)$ for both sides, the equation (9.11) is reduced to the following.

$$\begin{aligned} W = &\sum_{i,j,k,l}\hat{\alpha}_{ijkl}(X)(T\partial_T)^l W + \sum_{i=1}^{J}\sum_{j=1}^{n_i}\bar{\beta}_{ij}(X)(X\partial_X)^j W + \hat{\zeta}(X)T^K \\ &+ G_{K+1}(T, X, C_3W, \{C_3\partial_T(T\partial_T)^{j-1}W\}_{i,j}, \{C_3\partial_X(X\partial_X)^{j-1}(W)\}_{i,j}), \end{aligned} \tag{9.12}$$

where $\hat{\alpha}_{ijkl}(X) = \tilde{\alpha}_{ijkl}(X)/F(X) = O(X)$ and the others are similarly defined. Especially $\bar{\beta}_{ij}(X) = O(X)$.

For the equation (9.12), the following lemma holds.

Lemma 9.3.

(i) *If $\hat{\alpha}_{ijkl}(X) \not\equiv 0$ or $\bar{\beta}_{ij}(X) \not\equiv 0$ for some i, j, k, l, then the formal solution $W(T,X)$ of* (9.12) *belongs to the Gevrey class $\mathcal{G}_{T,X}^{(n_0+1,n_0+1)}$, where $n_0 = \max\{d_1, \ldots, d_I, n_1, \ldots, n_J\}$.*

(ii) *If $\hat{\alpha}_{ijkl}(X) \equiv 0$ and $\bar{\beta}_{ij}(X) \equiv 0$ for all i, j, k, l, then the formal solution $W(T,X)$ of* (9.12) *belongs to the Gevrey class $\mathcal{G}_{T,X}^{(n'+1,1)}$, where n' is the constant defined by* (7.4).

Since Lemma 9.3, (i) can be proved by similarly to that of Lemma 5.3, and Lemma 9.3, (ii) is a special case of Theorem 1.6 in the previous paper [12], we omit the proof of Lemma 9.3.

By Lemma 9.3, $W(T,X) \in \mathcal{G}_{T,X}^{(n_0+1,n_0+1)}$ or $\mathcal{G}_{T,X}^{(n'+1,1)}$. On the other hand, the majorant relation $\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(U)(t,x) \ll W(T,X)$. Therefore, we have

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(U)(t,x) \in \mathcal{G}_{T,X}^{(n_0+1,n_0+1)} \text{ or } \mathcal{G}_{T,X}^{(n'+1,1)},$$

which proves Theorem 3.3.

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