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ON THE QUASILINEAR CAUCHY PROBLEM FOR A HYPERBOLIC FUNCTIONAL DIFFERENTIAL EQUATION

Elżbieta Puźniakowska-Gałuch

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Abstract. The Cauchy problem for hyperbolic functional differential equations is considered. Volterra and Fredholm dependence are considered. A theorem on the local existence of generalized solutions defined on the Haar pyramid is proved. A result on differentiability of a solution with respect to initial data is proved.

Keywords: functional differential equations, Haar pyramid, differentiability of solutions, Fredholm type of equation.

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1. INTRODUCTION

For any metric spaces Y and Z we denote by $C(Y,Z)$ the class of all continuous functions from Y into Z. If $Y_0 \subset Y$ and $\alpha \in C(Y,Z)$, then $\alpha|_{Y_0}$ denotes the restriction of α to the set Y_0.

We will use vectorial inequalities understanding that the same inequalities hold between their corresponding components.

Suppose that $M = (M_1, \dots, M_n) \in AC([0,a], \mathbb{R}^n_+)$, $a > 0$, $\mathbb{R}_+ = [0, +\infty)$, and the function M is nondecreasing, $b = (b_1, \dots, b_n) \in \mathbb{R}^n_+$ and $b > M(a)$. Let E be the Haar pyramid

$$E = \big\{(t,x) \in \mathbb{R}^{1+n} : t \in [\,0, a\,], -b + M(t) \le x \le b - M(t)\big\},$$

where $x = (x_1, \dots, x_n)$. Suppose that $b_0 \in \mathbb{R}_+$ and

$$E_0 = [-b_0, 0] \times [-b, b].$$

For $(t,x) \in E$, we define the set $D[t,x]$ as follows:

$$D[t,x] = \{(\tau, y) \in \mathbb{R}^{1+n} : (t+\tau, x+y) \in E_0 \cup E\}.$$

Write
$$V[t,x] = \{(\tau, s) \in D[t,x] \,:\, \tau \le 0\}, \quad (t,x) \in E,$$
and
$$A = [-r_0, 0] \times [-2b, 2b], \quad B = [-r_0, a] \times [-2b, 2b], \quad r_0 = b_0 + a.$$
Then $V[t,x] \subset A$ and $D[t,x] \subset B$ for $(t,x) \in E$.

Given a function $z\colon E_0 \cup E \to \mathbb{R}$ and a point $(t,x) \in E$. We consider functions $z_{[t,x]}\colon D[t,x] \to \mathbb{R}$ and $z_{(t,x)}\colon V[t,x] \to \mathbb{R}$ defined by
$$z_{[t,x]}(\tau, y) = z(t+\tau, x+y), \quad (\tau, y) \in D[t,x],$$
and
$$z_{(t,x)}(\tau, y) = z(t+\tau, x+y), \quad (\tau, y) \in V[t,x].$$

Put $\Omega = E \times C(A, \mathbb{R}) \times C(B, \mathbb{R})$ and suppose that $F\colon \Omega \to \mathbb{R}$ is a given function of the variables (t,x,v,w). Suppose that
$$f\colon \Omega \to \mathbb{R}^n, \quad f = (f_1, \ldots, f_n),$$
$$\varphi_0\colon [0,a] \to \mathbb{R}, \quad \tilde{\varphi}\colon E \to \mathbb{R}^n, \quad \tilde{\varphi} = (\varphi_1, \ldots, \varphi_n),$$
and
$$\kappa\colon E_0 \to \mathbb{R}$$
are given functions. The requirements on φ_0 and $\tilde{\varphi}$ are that $0 \le \varphi_0(t) \le t$ for $t \in [0,a]$ and $(\varphi_0(t), \tilde{\varphi}(t,x)) \in E$ for $(t,x) \in E$. Write
$$\varphi(t,x) = (\varphi_0(t), \tilde{\varphi}(t,x)), \quad (t,x) \in E.$$

We will say that F and f satisfies the condition (V) if for each $(t,x) \in E$ and for $v, \bar{v} \in C(A, \mathbb{R})$, $w, \bar{w} \in C(B, \mathbb{R})$ such that
$$v\big|_{V[\varphi(t,x)]} = \bar{v}\big|_{V[\varphi(t,x)]} \quad \text{and} \quad w\big|_{D[t,x]} = \bar{w}\big|_{D[t,x]}$$
we have
$$F(t,x,v,w) = F(t,x,\bar{v},\bar{w}) \quad \text{and} \quad f(t,x,v,w) = f(t,x,\bar{v},\bar{w}).$$
Note that the condition (V) means that the value of F and f at the point $(t,x,v,w) \in \Omega$ depends on (t,x) and on the restrictions of v, w to the set $V[\varphi(t,x)]$, $D[t,x]$ only.

We consider the functional differential equation
$$\partial_t z(t,x) + f(t,x,z_{\varphi(t,x)}, z_{[t,x]}) \cdot \partial_x z(t,x) = F(t,x,z_{\varphi(t,x)}, z_{[t,x]}) \tag{1.1}$$
with the initial condition
$$z(t,x) = \kappa(t,x) \quad \text{on } E_0, \tag{1.2}$$
where $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$. We assume that F and f satisfy the condition (V).

Write

$$S_t = [-b + M(t), b - M(t)], \quad t \in [0, a],$$
$$I[x] = \{t \in [0, a] \, : \, -b + M(t) \le x \le b - M(t)\},$$
$$E_t = (E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n), \quad t \in [0, a].$$

We consider Carathéodory solutions of the initial problem. A function $\tilde{z}\colon (E_0 \cup E) \to \mathbb{R}$ is a solution to (1.1), (1.2) provided that

(i) derivatives $\partial_t \tilde{z}$, $\partial_x \tilde{z}$ exist almost everywhere on E,
(ii) $\tilde{z}$ satisfies equation (1.1) almost everywhere on E,
(iii) initial condition (1.2) holds.

The following problems are considered in the paper. We prove that under natural assumptions on given functions there exists exactly one Carathéodory solution to (1.1), (1.2) and the solution is defined on E. We will show that there is a Fréchet derivative to the solution of (1.1), (1.2) with respect to initial data.

There is a wide literature on first order partial functional differential problems, we wish to mention some references on existence results.

There are various concepts of a solution concerning initial or mixed problems for functional differential equations. Continuous functions satisfying integral systems obtained by integrating original equations along bicharacteristics were considered in [20]. Generalized solutions in the Carathéodory sense are investigated in [6, 19]. Results on the existence of solutions are obtained in these papers by using a method of bicharacteristics. Classical solutions in the functional setting are studied in [3, 10]. Cinquini Cibrario solutions to nonlinear differential functional equations were first treated in [4]. This class of solutions is placed between classical solutions and solutions in the Carathéodory sense and both inclusions are strict. Existence results for initial problems for semilinear equations can be found in [5]. Sufficient conditions for the existence of classical solutions defined on the Haar pyramid are given in [15, 16]. Classical solutions and differentiability with respect to initial data for Volterra type equations are studied in [14].

Theorems on the continuous dependence of solutions on initial or initial boundary conditions are given in Chapters 4 and 5 of [11]. We expand Kamont's theory. We investigate functional differential equations with both Volterra and Fredholm functional arguments. Moreover, the differentiability with respect to initial functions for partial functional differential equations is proved in these papers. The monograph [9] contains results on the differentiability of solutions for ordinary functional differential equations.

In [12, 13] the existence and differentiability of classical solutions with respect to initial functions for semilinear partial functional differential systems with arguments of both Volterra and Fredholm type are considered. The initial problem is transformed into a functional integral systems. The existence of global solutions is proved by a method of successive approximations. Although the techniques used in our investigations are based on the methods used in the above papers, there is a necessity to prove more complicated conditions on the characteristics. The series of Gronwall type of integral inequalities is proved in Section 2.1. The existence of solutions is proved locally.

Until now there have not been results concerning existence and differentiability of solutions with respect to initial functions for quasilinear partial functional differential equations with arguments of both Volterra and Fredholm type.

Suppose that $G\colon E \times C(E_0 \cup E, \mathbb{R}) \to \mathbb{R}$ and $g\colon E \times C(E_0 \cup E, \mathbb{R}) \to \mathbb{R}^n$ are given functions. Let us consider the functional differential equation

$$\partial_t z(t,x) + g(t,x,z) \cdot \partial_x z(t,x) = G(t,x,z), \tag{1.3}$$

where z is the functional variable. It is clear that (1.1) is a particular case of (1.3).

We will say that G and g satisfy the Volterra condition if for each $(t,x) \in E$ and for $z, \tilde{z} \in C(E_0 \cup E, \mathbb{R})$ such that $z(\tau,y) = \tilde{z}(\tau,y)$ for $(\tau,y) \in (E_0 \cup E) \cap ([-b_0,t] \times \mathbb{R}^n)$ we have $G(t,x,z) = G(t,x,\tilde{z})$ and $g(t,x,z) = g(t,x,\tilde{z})$. The Volterra condition means that the value of G, g at $(t,x,z) \in E \times C(E_0 \cup E, \mathbb{R})$ depends on (t,x) and on restrictions of z to the set $(E_0 \cup E) \cap ([-b_0,t] \times \mathbb{R}^n)$ only.

Note that equation (1.1) do not satisfy the Volterra condition.

The results presented in [1,3–6,10–16,19,20] have the following property: functional differential equations or systems considered in these papers satisfy the Volterra condition. Until now there have not been any results on functional differential equations of the form (1.3), which do not satisfy the Volterra condition.

We give examples of functional differential equations which can be obtained from (1.1) by specializing the functions F and f.

Example 1.1. Suppose that $G\colon E \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g\colon E \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ are given functions and F, f are defined by

$$F(t,x,v,w) = G(t,x,v(0,\mathbf{0}),w(0,\mathbf{0})) \quad \text{on } \Omega, \tag{1.4}$$

$$f(t,x,v,w) = g(t,x,v(0,\mathbf{0}),w(0,\mathbf{0})) \quad \text{on } \Omega, \tag{1.5}$$

where $\mathbf{0} = (0,\ldots,0) \in \mathbb{R}^n$. Then (1.1) reduces to the differential equation with deviated variables

$$\partial_t z(t,x) + g(t,x,,z(\varphi(t,x)),z(t,x)) \cdot \partial_x z(t,x) = G(t,x,z(\varphi(t,x)),z(t,x)). \tag{1.6}$$

Example 1.2. Suppose that $\varphi(t,x) = (t,x)$ for $(t,x) \in E$ and for the above G, g we put

$$F(t,x,v,w) = G\Big(t,x,\int\limits_{V[t,x]} v(\tau,s)dsd\tau, \int\limits_{D[t,x]} w(\tau,s)dsd\tau\Big) \quad \text{on } \Omega, \tag{1.7}$$

$$f(t,x,v,w) = g\Big(t,x,\int\limits_{V[t,x]} v(\tau,s)dsd\tau, \int\limits_{D[t,x]} w(\tau,s)dsd\tau\Big) \quad \text{on } \Omega. \tag{1.8}$$

Then (1.1) reduces to the differential integral equation

$$\begin{aligned} &\partial_t z(t,x) + g\Big(t,x,\int\limits_{V[t,x]} z_{(t,x)}(\tau,s)dsd\tau, \int\limits_{D[t,x]} z_{[t,x]}(\tau,s)dsd\tau\Big) \cdot \partial_x z(t,x) \\ &= G\Big(t,x,\int\limits_{V[t,x]} z_{(t,x)}(\tau,s)dsd\tau, \int\limits_{D[t,x]} z_{[t,x]}(\tau,s)dsd\tau\Big). \end{aligned} \tag{1.9}$$

It is clear that more complicated examples of differential equations with deviated variables and differential integral equations can be obtained from (1.1) by specializing the operators F, f.

Information on applications for differential integral equations and differential equations with deviated variables can be found in monographs [11, 21] and papers [2, 7, 8, 17, 18].

We remark that our work is motivated by the applications given in the above papers.

2. SEQUENCES OF SUCCESSIVE APPROXIMATIONS

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\|x\| = |x_1| + \ldots + |x_n|$. The maximum norms in the spaces $C(V[t,x],\mathbb{R})$ and $C(D[t,x],\mathbb{R})$ are denoted by $\|\cdot\|_{V[t,x]}$ and $\|\cdot\|_{D[t,x]}$, respectively. For $t \in [0,a]$ and $z \in C(E_0 \cup E, \mathbb{R})$, $v \in C(E_0 \cup E, \mathbb{R}^n)$, we define the seminorms

$$\|z\|_{(t,\mathbb{R})} = \max\{|z(\tau,x)| : (\tau,x) \in E_t\},$$
$$\|v\|_{(t,\mathbb{R}^n)} = \max\{\|v(\tau,x)\| : (\tau,x) \in E_t\}.$$

Let $\mathbb{L}(Y,Z)$ be a class of all integrable functions form Y into Z.

Assumption $H_0[f,\, F]$. The functions $f\colon \Omega \to \mathbb{R}^n$, $F\colon \Omega \to \mathbb{R}$ satisfies the following proprieties:

1) f, F satisfy condition (V) and $f(\,\cdot, x, v, w)\colon I[x] \to \mathbb{R}^n$, $F(\,\cdot, x, v, w)\colon I[x] \to \mathbb{R}$, are measurable for $(x, v, w) \in [-b, b] \times C(A,\mathbb{R}) \times C(B,\mathbb{R})$,
2) $f(t,\cdot)\colon S_t \times C(A,\mathbb{R}) \times C(B,\mathbb{R}) \to \mathbb{R}^n$, $F(t,\cdot)\colon S_t \times C(A,\mathbb{R}) \times C(B,\mathbb{R}) \to \mathbb{R}$ are continuous for almost all $t \in [0,a]$,
3) the following estimation is fulfilled:

$$\big(|f_1(t,x,v,w)|, \ldots, |f_n(t,x,v,w)|\big) \le M'(t), \quad (t,x) \in E, \text{ a.e.}, \tag{2.1}$$

4) there is $\gamma_0 \in \mathbb{L}([0,a],\mathbb{R}_+)$ such that

$$\|F(t,x,0,0)\| \le \gamma_0(t) \quad \text{on } E,$$

5) there are $\beta, \gamma \in \mathbb{L}([0,a],\mathbb{R}_+)$ such that

$$\|F(t,x,v,w) - F(t,x,\bar v,\bar w)\| \le \beta(t)\|v - \bar v\|_A + \gamma(t)\|w - \bar w\|_B \quad \text{on } \Omega,$$

6) $\int_0^a \gamma(\tau) e^{\int_\tau^a \beta(s)ds} d\tau < 1$.

Assumption $H[\varphi]$. The functions $\varphi_0\colon [0,a] \to \mathbb{R}$, $\tilde{\varphi}\colon E \to \mathbb{R}^n$ are continuous and

1) $0 \le \varphi_0(t) \le t$, $(\varphi_0(t), \tilde{\varphi}(t,x)) \in E$ for $(t,x) \in E$,
2) there is $\tilde{Q} \in \mathbb{R}_+$ such that

$$\|\tilde{\varphi}(t,x) - \tilde{\varphi}(t,\bar{x})\| \le \tilde{Q}\|x - \bar{x}\| \quad \text{on } E.$$

Assumption $H[\kappa]$. The functions $\kappa\colon E_0 \to \mathbb{R}$ are continuous and there is $L_0 > 0$ such that

$$|\kappa(t,x) - \kappa(t,\bar{x})| \le L_0\|x - \bar{x}\| \quad \text{on } E_0.$$

Let us denote by $\mathcal{X}$ the class of all $\kappa\colon E_0 \to \mathbb{R}$ satisfying Assumption $H[\kappa]$.

Given $\kappa \in \mathcal{X}$, we denote by $C_\kappa[d]$, $d \in \mathbb{R}_+$, the set of all $z \in C(E,\mathbb{R})$ such that $z(t,x) = \kappa(t,x)$ on E_0 and $|z(t,x) - z(t,\bar{x})| \le d\|x - \bar{x}\|$ on E.

Suppose that Assumptions $H_0[f,F]$, $H[\varphi]$ are satisfied and $\kappa \in \mathcal{X}$. Let us denote by $g[z](\cdot,t,x)$ the solution of the Cauchy problem

$$\eta'(\tau) = f(\tau, \eta(\tau), z_{\varphi(\tau,\eta(\tau))}, z_{[\tau,\eta(\tau)]}), \quad \eta(t) = x, \tag{2.2}$$

where $(t,x) \in E$. The function $g[z](\cdot,t,x)$ is the characteristic of equation (1.1).

Set

$$P[z](\tau,t,x) = \big(\tau, g[z](\tau,t,x), z_{\varphi(\tau,g[z](\tau,t,x))}, z_{[\tau,g[z](\tau,t,x)]}\big).$$

Suppose that $z \in C_\kappa[d]$. Let us denote by $\mathcal{F}[z]$ the function defined by

$$\mathcal{F}[z](t,x) = \kappa(0, g[z](0,t,x)) + \int_0^t F(P[z](\tau,t,x))d\tau \quad \text{on } E,$$

$$\mathcal{F}[z](t,x) = \kappa(t,x) \quad \text{on } E_0.$$

Consider the functional integral equation

$$z = \mathcal{F}[z]. \tag{2.3}$$

We give estimates of solutions to (1.1), (1.2).

Lemma 2.1. *Suppose that Assumption $H_0[f,F]$ is satisfied and $\tilde{z} \in C_\kappa[d]$ is a solution to* (1.1), (1.2). *Then the following estimation holds true:*

$$\|\tilde{z}\|_{(t,\mathbb{R})} \le \exp\Big\{ \int_0^t \beta(\tau)d\tau \Big\} \Big(\|\kappa\| + \int_0^t \gamma_0(\tau)d\tau \Big) \Big(1 + \Lambda \int_0^t \gamma(\tau)d\tau \Big), \tag{2.4}$$

where

$$\Lambda = \exp\Big\{ \int_0^a \beta(\tau)d\tau \Big\} \Big(1 - \int_0^a \gamma(\tau) \exp\Big\{ \int_\tau^a \beta(s)ds \Big\} d\tau \Big)^{-1}. \tag{2.5}$$

Proof. It is clear that $\tilde{z}$ satisfies (2.3). Then we have

$$\|\tilde{z}\|_{(t,\mathbb{R})} \leq \|\kappa\| + \int_0^t \beta(\tau)\|\tilde{z}\|_{(\tau,\mathbb{R})}d\tau + \|\tilde{z}\|_{(a,\mathbb{R})}\int_0^t \gamma(\tau)d\tau + \int_0^t \gamma_0(\tau)d\tau. \tag{2.6}$$

Write

$$\Psi(t) = \|\kappa\| + \int_0^t \beta(\tau)\|\tilde{z}\|_{(\tau,\mathbb{R})}d\tau + \|\tilde{z}\|_{(a,\mathbb{R})}\int_0^t \gamma(\tau)d\tau + \int_0^t \gamma_0(\tau)d\tau, \quad t \in [0,c].$$

Then

$$\Psi'(t) \leq \beta(t)\Psi(t) + \|\tilde{z}\|_{(a,\mathbb{R})}\gamma(t) + \gamma_0(t)$$

for almost all $t \in [0,c]$. This gives

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big[\Psi(t)\exp\Big\{-\int_0^t \beta(\tau)d\tau\Big\}\Big] \leq \big[\|\tilde{z}\|_{(a,\mathbb{R})}\gamma(t) + \gamma_0(t)\big]\exp\Big\{-\int_0^t \beta(\tau)d\tau\Big\}$$

for almost all $t \in [0,a]$, and consequently

$$\begin{aligned}\Psi(t) \leq \|\kappa\|\exp\Big\{\int_0^t \beta(\tau)d\tau\Big\} &+ \|\tilde{z}\|_{(a,\mathbb{R})}\int_0^t \gamma(\tau)\exp\Big\{\int_0^t \beta(s)ds\Big\}d\tau \\ &+ \int_0^t \gamma_0(\tau)\exp\Big\{\int_\tau^t \beta(s)ds\Big\}d\tau\end{aligned}$$

for $t \in [0,a]$. The above inequality and (2.6) imply (2.4), (2.5). This completes the proof. $\square$

2.1. INTEGRAL INEQUALITIES

In the present paper we consider functional differential equations which do not satisfy the Volterra condition. We need a new comparison result for integral inequalities. More precisely, we consider integral inequalities generated by the equation

$$y(t) = \eta + \int_0^t \tilde{C}_0(\tau)d\tau + \int_0^t \tilde{B}(\tau)y(\tau)d\tau + y(a)\int_0^t \tilde{C}(\tau)d\tau, \tag{2.7}$$

where $\tilde{B}, \tilde{C}, \tilde{C}_0 \colon [0,a] \to \mathbb{R}_+$ and $\eta \in \mathbb{R}_+$.

Lemma 2.2. *Suppose that* $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{C}_0 \in \mathbb{L}([0,c], \mathbb{R}_+)$*,* $\tilde{\omega} \in C([0,a], \mathbb{R}_+)$ *and* $\eta \in \mathbb{R}_+$*.*

(i) *There exists exactly one solution of the integral equation* (2.7) *if*

$$\int_0^a \tilde{C}(s) e^{\int_s^a \tilde{B}(\tau)d\tau} ds < 1. \tag{2.8}$$

(ii) *If*

$$\int_0^a \tilde{C}(s) e^{\int_s^a (\tilde{A}(\tau)+\tilde{B}(\tau))d\tau} ds < 1, \tag{2.9}$$

and $\tilde{y} \colon [0,a] \to \mathbb{R}_+$ *is a solution of the integral equation*

$$y(t) = \eta + \int_0^t \tilde{C}_0(\tau)d\tau + \int_0^t [\tilde{A}(\tau) + \tilde{B}(\tau)]y(\tau)d\tau + y(a) \int_0^t \tilde{C}(\tau)d\tau, \tag{2.10}$$

and

$$\tilde{\omega}(t) \leq \eta + \int_0^t \tilde{C}_0(\tau)d\tau + \int_0^t \tilde{A}(\tau)\tilde{\omega}(\tau)d\tau + \int_0^t \tilde{B}(\tau)\tilde{y}(\tau)d\tau + \tilde{y}(a) \int_0^t \tilde{C}(\tau)d\tau, \tag{2.11}$$

then

$$\tilde{\omega}(t) \leq \tilde{y}(t) \quad \textit{for} \quad t \in [0,a]. \tag{2.12}$$

Proof. (i) Set

$$\bar{y}(t) = \eta e^{\int_0^t \tilde{B}(\tau)d\tau} + \int_0^t \tilde{C}_0(s) e^{\int_s^t \tilde{B}(\tau)d\tau} ds + C_\star \int_0^t \tilde{C}(s) e^{\int_s^t \tilde{B}(\tau)d\tau} ds, \quad t \in (0,a],$$

where

$$C_\star = \left[\eta e^{\int_0^a \tilde{B}(\tau)d\tau} + \int_0^a \tilde{C}_0(s) e^{\int_s^a \tilde{B}(\tau)d\tau} ds \right] \left[1 - \int_0^a \tilde{C}(s) e^{\int_s^a \tilde{B}(\tau)d\tau} ds \right]^{-1}.$$

It follows that $\bar{y}$ is the unique solution of the Cauchy problem corresponding to (2.7). This completes the proof of the first part of the lemma.

(ii) It follows from (2.9) that there exists exactly one solution $\tilde{y} \colon [0,a] \to \mathbb{R}_+$ of (2.10). Write $y_\epsilon(t) = \tilde{\omega}(t) - \tilde{\omega}_\epsilon(t)$, $t \in [0,a]$, where $\epsilon > 0$ and $\tilde{\omega}_\epsilon$ is the solution of the integral equation $\omega(t) = \epsilon + \int_0^t \tilde{A}(\tau)\omega(\tau)d\tau$. We will show that $y_\epsilon(t) < \tilde{y}(t)$ for $t \in [0,a]$. It is clear that $y_\epsilon(0) < \tilde{y}(0)$. Suppose that there is $\tilde{t} \in (0,a]$ such that $y_\epsilon(t) < \tilde{y}(t)$ for $t \in [0,\tilde{t})$ and

$$y_\epsilon(\tilde{t}) = \tilde{y}(\tilde{t}). \tag{2.13}$$

Then

$$y_\epsilon(\tilde{t}) - \tilde{y}(\tilde{t}) = \bar{y}(\tilde{t}) - \omega_\epsilon(\tilde{t}) - \tilde{y}(\tilde{t}) \leq \int_0^{\tilde{t}} A(\tau)[\bar{y}(\tau) - \tilde{y}(\tau)]d\tau - \omega_\epsilon(\tilde{t}) \leq -\epsilon$$

which contradicts (2.13). Therefore, $y_\epsilon(t) < \tilde{y}(t)$ for $t \in [0,a]$. From this inequality we obtain in the limit, letting ϵ tend to zero, estimate (2.12). □

Now we prove a lemma on an integral inequality of the Fredholm type.

Lemma 2.3. *Suppose that* $\bar{A}, \bar{B}, \bar{C} \in \mathbb{L}([0,a], \mathbb{R}_+)$ *and* $\bar{A}$ *is nondecreasing and condition* (2.8) *holds and the function* $\tilde{\omega} \in \mathbb{L}([0,a], \mathbb{R}_+)$ *is the solution of the integral inequality*

$$y(t) \leq \bar{A}(t) + \int_0^t \bar{B}(\tau) y(\tau) d\tau + y(a) \int_0^t \bar{C}(\tau) d\tau. \tag{2.14}$$

Then

$$\tilde{\omega}(t) \leq \bar{A}(t) e^{\int_0^t \bar{B}(\tau)d\tau} + \bar{A}(a)\Lambda_\star \int_0^t \bar{C}(s) e^{\int_s^t \bar{B}(\tau)d\tau} ds, \tag{2.15}$$

where

$$\Lambda_\star = e^{\int_0^a \bar{B}(\tau)d\tau} \Big[1 - \int_0^a \bar{C}(\tau) e^{\int_\tau^a \bar{B}(s)ds} d\tau\Big]^{-1}. \tag{2.16}$$

Proof. Write

$$\Psi(t) = \int_0^t \bar{B}(\tau)\tilde{\omega}(\tau)d\tau + \tilde{\omega}(a) \int_0^t \bar{C}(\tau)d\tau.$$

Then

$$\Psi'(t) \leq \bar{B}(t)[\Psi(t) + \bar{A}(t)] + \bar{C}(t)\tilde{\omega}(a)$$

for almost all $t \in (0,a]$ and

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big[\Psi(t) e^{-\int_0^t \bar{B}(\tau)d\tau}\Big] \leq e^{-\int_0^t \bar{B}(\tau)d\tau} [\bar{A}(t)\bar{B}(t) + \bar{C}(t)\tilde{\omega}(a)].$$

This gives

$$\Psi(t) \leq \bar{A}(t)\Big[e^{\int_0^t \bar{B}(\tau)d\tau} - 1\Big] + \tilde{\omega}(a) \int_0^t \bar{C}(\tau) e^{\int_\tau^t \bar{B}(s)ds} d\tau$$

and

$$\tilde{\omega}(t) \leq \bar{A}(t) e^{\int_0^t \bar{B}(\tau)d\tau} + \tilde{\omega}(a) \int_0^t \bar{C}(\tau) e^{\int_\tau^t \bar{B}(s)ds} d\tau. \tag{2.17}$$

It follows from (2.8) that $\tilde{\omega}(a) \le \Lambda_\star \bar{A}(a)$. The last inequality and (2.17) imply (2.15). This completes the proof. $\square$

2.2. CHARACTERISTICS

Denote

$$\eta(t) = \exp\Big\{ \int_0^t \beta(\tau)d\tau\Big\}\Big(\|\kappa\| + \int_0^a \gamma_0(s)ds\Big)\Big(1 + \Lambda \int_0^t \gamma(s)ds\Big).$$

Take $r = \eta(a)$. Set

$$\Omega[r] = \{(t, x, v, w) \in \Omega \,:\, \|v\|_A \le r,\ \|w\|_B \le r\}.$$

Assumption $H[f,\, F]$. Suppose that there is $\beta_0 \in \mathbb{L}([0, a], \mathbb{R}_+)$ such that

$$\|F(t, x, v, w) - F(t, \bar{x}, v, w)\| \le \beta_0(t)\|x - \bar{x}\| \quad \text{on } \Omega[r],$$

and there are $\alpha_0, \rho_0, \lambda_0 \in \mathbb{L}([0, a], \mathbb{R}_+)$ such that on $\Omega[r]$ the following estimations are fulfilled:

$$\|f(t, x, v, w) - f(t, \bar{x}, \bar{v}, \bar{w})\| \le \alpha_0(t)\|x - \bar{x}\| + \rho_0(t)\|v - \bar{v}\|_A + \lambda_0(t)\|w - \bar{w}\|_B,$$

$$\int_0^a d \cdot \lambda_0(s) \mathrm{e}^{\int_s^a (\alpha_0(\tau) + d\cdot\tilde{Q}\rho_0(\tau))d\tau} ds < 1, \tag{2.18}$$

and moreover

$$\int_0^a (\gamma(\tau) + ak(\tau)) \mathrm{e}^{\int_s^a (\beta(\tau) + k(\tau))d\tau} ds < 1, \tag{2.19}$$

where

$$k(\tau) = (\beta(\tau)\tilde{Q}d\theta(\tau) + d\gamma(\tau)\theta(a)) \int_0^a \rho_0(w)dw$$

($\theta(\tau)$ is given in Lemma 2.4).

Lemma 2.4. *If Assumptions $H_0[f, F]$, $H[f, F]$ are satisfied, then we have $(\tau, g[z](\tau, t, x)) \in E$, $\tau \in [0, t]$, and*

$$\|g[z](\,\cdot\,, t, x) - g[z](\,\cdot\,, t, \bar{x})\|_{(\tau,\mathbb{R}^n)} \le \theta(\tau)\|x - \bar{x}\|, \quad (t, x), (t, \bar{x}) \in E, \tag{2.20}$$

and

$$\begin{aligned} &\|g[z](\,\cdot\,, t, x) - g[\tilde{z}](\,\cdot\,, t, \bar{x})\|_{(\tau,\mathbb{R}^n)} \\ &\le \theta(\tau)\Big(\int_\tau^a \rho_0(s)\|z - \tilde{z}\|_{(s,\mathbb{R})}ds + a\|z - \tilde{z}\|_{(a,\mathbb{R})}\Big), \end{aligned} \tag{2.21}$$

where

$$\theta(\tau) = \mathrm{e}^{\int_0^\tau (\alpha_0(\zeta)+d\tilde{Q}\rho_0(\zeta))d\zeta} + \Lambda_\star\, d \int_0^\tau \lambda_0(s)\mathrm{e}^{\int_s^\tau (\alpha_0(\zeta)+d\tilde{Q}\rho_0(\zeta))d\zeta} ds$$

and $\Lambda_\star$ *is given by* (2.16).

Proof. The existence and uniqueness of solutions of (2.2) follows from classical existence theorems. We prove that the graph of the characteristic $g[z](\,\cdot, t, x)$ is in the E for $\tau \in [0, t]$, $(t, x) \in E$. Suppose that $[0, t]$ is an interval on which the characteristic $g[z](\,\cdot, t, x)$ is defined. Then we have

$$-M'(\tau) \leq \frac{d}{d\tau} g[z](\tau, t, x) \leq M'(\tau) \quad \text{for} \quad \tau \in [0, t],$$

and consequently

$$-b + M(\tau) \leq g[z](\tau, t, x) \leq b - M(\tau) \quad \text{for} \quad \tau \in [0, t].$$

This gives $(\tau, g[z](\tau, t, x)) \in E$ for $\tau \in [0, t]$.

For $(t, x), (t, \bar{x}) \in E$, it follows that the functions $g[z](\,\cdot\,, t, x) - g[z](\,\cdot\,, t, \bar{x})$ satisfy the integral inequalities

$$\begin{aligned} &\|g[z](\cdot, t, x) - g[z](\,\cdot, t, \bar{x})\|_{(\tau,\mathbb{R}^n)} \\ &\leq \|x - \bar{x}\| \\ &\quad + \Big| \int_\tau^t \big[(\alpha_0(s) + \rho_0(s)d\tilde{Q})\|g[z](\,\cdot, t, x) - g[z](\,\cdot, t, \bar{x})\|_{(s,\mathbb{R}^n)} \\ &\quad + d\,\lambda_0(s)\|g[z](\,\cdot, t, x) - g[z](\,\cdot, t, \bar{x})\|_{(a,\mathbb{R}^n)}\big] ds \Big|. \end{aligned}$$

From Lemma 2.3 we obtain (2.20). Now we will show (2.21). We have that

$$\begin{aligned} &\|g[z](\,\cdot, t, x) - g[\tilde{z}](\,\cdot, t, x)\|_{(\tau,\mathbb{R}^n)} \\ &\leq \Big| \int_\tau^t (\alpha_0(s) + \rho_0(s)d\tilde{Q})\|g[z](\,\cdot, t, x) - g[\tilde{z}](\,\cdot, t, x)\|_{(s,\mathbb{R}^n)} ds \\ &\quad + d\|g[z](\,\cdot, t, x) - g[\tilde{z}](\,\cdot, t, x)\|_{(a,\mathbb{R}^n)} \int_\tau^t \lambda_0(s) ds \\ &\quad + \int_\tau^t \rho_0(s)\|z - \tilde{z}\|_{(s,\mathbb{R})} ds + \|z - \tilde{z}\|_{(a,\mathbb{R})} a \Big|. \end{aligned}$$

From Lemma 2.3 we obtain (2.21). This completes the proof. □

3. EXISTENCE OF SOLUTIONS

The proof of the existence of weak solutions to (2.3) is based on the following method of successive approximations. Suppose that $\kappa \in \mathcal{X}$ and Assumptions $H_0[f, F]$, $H[f, F]$, $H[\varphi]$ are satisfied. We consider the sequence $\{z^{(m)}\}$, where

$$z^{(m)} \colon E_0 \cup E \to \mathbb{R},$$

defined in the following way. We put

$$z^{(0)}(t,x) = \kappa(t,x) \quad \text{on } E_0, \quad z^{(0)}(t,x) = \kappa(0,x) \quad \text{on } E. \tag{3.1}$$

If $z^{(m)} \colon E_0 \cup E \to \mathbb{R}$ is given, then $z^{(m+1)}$ is defined by

$$z^{(m+1)} = \mathcal{F}[z^{(m)}]. \tag{3.2}$$

Suppose that Assumptions $H_0[f, F]$, $H[f, F]$ are satisfied. Write

$$\tilde{L}(t) = L_0\theta(a) \exp\Big[\int_0^t Z(\tau)d\tau\Big] + \int_0^t \exp\Big[\int_\tau^t Z(s)ds\Big]\beta_0(\tau)\theta(\tau)d\tau,$$

where $Z(t) = \beta(t)\tilde{Q}\theta(t) + \gamma(t)\theta(a)$ for $t \in [0,a]$.

For the above $\tilde{L} \in \mathbb{L}([0,a], \mathbb{R}_+)$, we denote by $C_{\kappa.r.\tilde{L}}(E_0 \cup E, \mathbb{R})$ the set of all $z \in C_\kappa(E_0 \cup E, \mathbb{R})$ such that $|z(t,x) - z(t,\bar{x})| \le \tilde{L}(t)\|x - \bar{x}\|$ on $E_0 \cup E$ and $|z(t,x)| \le r$ on $E_0 \cup E$.

Now we formulate a theorem on the local existence of weak solutions of (1.1), (1.2).

Theorem 3.1. *If Assumptions $H_0[f, F]$, $H[f, F]$, $H[\varphi]$ are satisfied and $\kappa \in \mathcal{X}$, then there is a generalized solution $\bar{z} \colon E_0 \cup E \to \mathbb{R}$ of (1.1), (1.2) and $\bar{z} \in C_{\kappa.r.\tilde{L}}(E_0 \cup E, \mathbb{R})$. If $\tilde{\kappa} \in X$ and $\tilde{z}$ is a generalized solution of equation (1.1) with the initial condition*

$$\tilde{z}(t,x) = \tilde{\kappa}(t,x) \quad \text{on } E_0,$$

then

$$\|\bar{z} - \tilde{z}\|_{(t,\mathbb{R})} \le \Big[\exp\Big\{\int_0^t l(\tau)d\tau\Big\} + \Lambda\int_0^t u(\tau)\exp\Big\{\int_\tau^t l(s)ds\Big\}d\tau\Big]\,\|\kappa - \tilde{\kappa}\|, \tag{3.3}$$

$t \in [0,a]$, where Λ is given by (2.5) and

$$l(\tau) = \beta(\tau) + \int_0^a \rho_0(s)ds(\beta(\tau)\tilde{Q}\tilde{L}(\tau)\theta(\tau) + \gamma(\tau)\tilde{L}(a)\theta(a)),$$

$$u(\tau) = \gamma(\tau) + a\int_0^a \rho_0(s)ds(\beta(\tau)\tilde{Q}\tilde{L}(\tau)\theta(\tau) + \gamma(\tau)\tilde{L}(a)\theta(a)).$$

Proof. We first prove that $z^{(m)} \in C_{\kappa.r.\tilde{L}}(E_0 \cup E, \mathbb{R})$. By virtue of the definition of $\tilde{L}$ we have $|z^{(0)}(t,x)| \leq r$, therefore $z^{(0)} \in C_{\kappa.r.\tilde{L}}(E_0 \cup E, \mathbb{R})$. Suppose that $z^{(m)} \in C_{\kappa.r.\tilde{L}}(E_0 \cup E, \mathbb{R})$ is given, $m \geq 0$. We will show that $z^{(m+1)}$ has the same property. It follows from Assumption $H[\,f, F\,]$ that

$$\begin{aligned}
&|z^{(m+1)}(t,x) - z^{(m+1)}(t,\bar{x})| \leq L_0 \|g[z](0,t,x) - g[z](0,t,\bar{x})\| \\
&\quad + \int_0^t \big\{\beta_0(\tau) + \beta(\tau)\tilde{L}(\tau)\tilde{Q} + \gamma(\tau)\tilde{L}(\tau)\big\} \|g[z](\tau,t,x) - g[z](\tau,t,\bar{x})\| d\tau \\
&\leq \tilde{L}(t)\|x - \bar{x}\|.
\end{aligned}$$

Moreover, if we assume that $\|z^{(m)}(t,x)\|_\infty \leq r$, then for $m+1$ we have

$$\begin{aligned}
\|z^{(m+1)}\|_{(t,\mathbb{R})} \leq \|\kappa\|_X &+ \int_0^t \beta(\tau)\|z^{(m)}\|_{(\tau,\mathbb{R})} d\tau \\
&+ \|z^{(m)}\|_{(a,\mathbb{R})} \int_0^t \gamma(\tau) d\tau + \int_0^t \gamma_0(\tau) d\tau \leq r.
\end{aligned}$$

The above relation implies $z^{(m)} \in C_{\kappa.r.\tilde{L}}(E_0 \cup E, \mathbb{R})$ for $m > 0$.

Now we prove that the sequence $\{z^{(m)}\}$ is uniformly convergent on $E_0 \cup E$. Define function $\tilde{\gamma}_0 \in \mathbb{L}([0,a], \mathbb{R}_+)$ by the relation

$$\|z^{(0)} - \mathcal{F}[z^{(0)}]\|_{(t,\mathbb{R})} \leq \int_0^t \tilde{\gamma}_0(\tau) d\tau, \quad t \in [0,a].$$

Define the sequence $\{\omega^{(k)}\}$ as follows: $\{\omega^{(0)}\}$ is a solution of the Cauchy problem

$$\omega'(t) = \beta(t)\omega(t) + \omega(a)\gamma(t) + \tilde{\gamma}_0(t), \quad \omega(0) = 0. \tag{3.4}$$

If $\omega^{(m)}$ is a known function, then $\omega^{(m+1)}(t)$ is given by

$$\omega^{(m+1)}(t) = \int_0^t \beta(\tau)\omega^{(m)}(\tau) d\tau + \omega^{(m)}(a) \int_0^t \gamma(\tau) d\tau.$$

From condition 6) of Assumption $H_0[f,F]$ it follows that there is exactly one solutions to the problem (3.4).

It can be easily seen that

$$0 \leq \omega^{(m+1)}(t) \leq \omega^{(m)}(t) \quad \text{for } t \in [0,a] \text{ and } m \geq 0.$$

Hence, there is $\tilde{\omega}(t) = \lim_{m\to\infty} \omega^{(m)}(t)$ on $[0,a]$ uniformly. From condition 6) of Assumption $H_0[f,F]$ we have $\tilde{\omega}(t) = 0$.

It is easy to show that $\{z^{(m)}\}$ is a Cauchy sequence in $C_{\kappa.r.\tilde{L}}(E_0 \cup E, \mathbb{R})$, i.e.

$$\|z^{(m+p)} - z^{(m)}\|_{(t,\mathbb{R})} \leq \omega^{(m)}(t), \quad t \in [0,a],\ m,p \geq 0.$$

Furthermore, we have that

$$\bar{z}(t,x) = \mathcal{F}[\bar{z}](t,x), \quad (t,x) \in E. \tag{3.5}$$

For a given $(t,x) \in E$ let us put $y = g[\bar{z}](0,t,x)$. It follows that $g[\bar{z}](\tau,t,x) = g[\bar{z}](\tau,0,y)$. We conclude from (3.5) that

$$\begin{aligned}\bar{z}(t, g[\bar{z}](t,0,y)) &= \kappa(0,y) \\ &\quad + \int_0^t F\big(\tau, g[\bar{z}](\tau,0,y), \bar{z}_{\varphi(\tau,g[\bar{z}](\tau,0,y))}, \bar{z}_{[\tau,g[\bar{z}](\tau,0,y)]}\big)d\tau.\end{aligned} \tag{3.6}$$

The relations $y = g[\bar{z}](0,t,x)$ and $x = g[\bar{z}](t,0,y)$ are equivalent. By differentiating (3.6) with respect to t and by putting again $x = g[\bar{z}](t,0,y)$ we obtain that $\bar{z}$ satisfies (1.1) on E.

Now we prove inequality (3.3). First we take

$$\|\bar{z} - \tilde{z}\|_{(t,\mathbb{R}^k)} \leq \|\kappa - \tilde{\kappa}\|_X + \int_0^t [l(\tau)\|\bar{z} - \tilde{z}\|_{(\tau,\mathbb{R})} + u(\tau)\|\bar{z} - \tilde{z}\|_{(a,\mathbb{R})}]d\tau.$$

The inequality (3.3) follows directly from Lemma 2.3. This completes the proof. □

4. DIFFERENTIABILITY OF SOLUTIONS WITH RESPECT TO INITIAL FUNCTIONS

We denote by $CL(A,\mathbb{R})$ and $CL(B,\mathbb{R})$ the class of linear continuous functionals defined on $C(A,\mathbb{R})$ and $C(B,\mathbb{R})$ respectively. The norms in $CL(A,\mathbb{R})$ and $CL(B,\mathbb{R})$ generated by the maximum norms in $C(A,\mathbb{R})$ and $C(B,\mathbb{R})$ will be denoted by $\|\cdot\|_{A\star}$ and $\|\cdot\|_{B\star}$, respectively.

Assumption $H_*[f, F]$. The functions $f\colon E \to \mathbb{R}^n$, $F\colon E \times C(A,\mathbb{R}) \times C(B,\mathbb{R}) \to \mathbb{R}$ satisfy the following conditions:

1) there exist the derivatives

$$\partial_x f = \big[\,\partial_{x_j} f_i\,\big]_{i,j=1,\ldots,n}$$

on Ω and $\partial_x f_i(\,\cdot, x, v, w)\colon I[x] \to \mathbb{R}^n$ is integrable for $(x,v,w) \in [-b,b] \times C(A,\mathbb{R}) \times C(B,\mathbb{R})$ and $\partial_x f(t,\cdot)\colon S_t \times C(A,\mathbb{R}) \times C(B,\mathbb{R}) \to \mathbb{R}^n$ is continuous for almost all $t \in I[x]$,

2) the derivatives $\partial_x F = (\partial_{x_1} F, \ldots, \partial_{x_2} F)$ exist on Ω and $\partial_x F(\,\cdot, x, v, w)\colon I[x] \to \mathbb{R}^n$ is integrable for $(x,v,w) \in [-b,b] \times C(A,\mathbb{R}) \times C(B,\mathbb{R})$ and $\partial_x F(t,\cdot)\colon S_t \times C(A,\mathbb{R}) \times C(B,\mathbb{R}) \to \mathbb{R}^n$ is continuous for almost all $t \in I[x]$,

3) the Fréchet derivatives $\partial_v f(t,x,v,w) = (\partial_v f_1(t,x,v,w),\dots,\partial_v f_n(t,x,v,w))$, $\partial_w f(t,x,v,w) = (\partial_w f_1(t,x,v,w),\dots,\partial_w f_n(t,x,v,w))$ exist for $(t,x,v,w)\in\Omega$ and $\partial_v f_i(t,x,v,w)\in CL(A,\mathbb{R})$, $\partial_w f_i(t,x,v,w)\in CL(B,\mathbb{R})$ for $i=1,\dots,n$, $(t,x,v,w)\in\Omega$,
4) the functions $\partial_v f_i(\,\cdot,x,v,w)\tilde v\colon I[x]\to\mathbb{R}$ are integrable for $(x,v,w)\in[-b,b]\times C(A,\mathbb{R})\times C(B,\mathbb{R})$, $\tilde v\in C(A,\mathbb{R})$ and $\partial_w f_i(\,\cdot,x,v,w)\tilde w\colon I[x]\to\mathbb{R}^n$ is integrable for $(x,v,w)\in[-b,b]\times C(A,\mathbb{R})\times C(B,\mathbb{R})$, $\tilde w\in C(B,\mathbb{R})$, $i=1,\dots,n$,
5) the Fréchet derivatives $\partial_v F(t,x,v,w)$, $\partial_w F(t,x,v,w)$ exist for $(t,x,v,w)\in\Omega$ and $\partial_v F(t,x,v,w)\in CL(A,\mathbb{R})$, $\partial_w F(t,x,v,w)\in CL(B,\mathbb{R})$ for $(t,x,v,w)\in\Omega$,
6) the functions $\partial_v F(\,\cdot,x,v,w)\tilde v\colon I[x]\to\mathbb{R}$ is integrable for $(x,v,w)\in[-b,b]\times C(A,\mathbb{R})\times C(B,\mathbb{R})$, $\tilde v\in C(A,\mathbb{R})$ and $\partial_w F(\,\cdot,x,v,w)\tilde w\colon I[x]\to\mathbb{R}^n$ is integrable for $(x,v,w)\in[-b,b]\times C(A,\mathbb{R})\times C(B,\mathbb{R})$, $\tilde w\in C(B,\mathbb{R})$,
7) there are $\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta,\gamma\in\mathbb{L}([0,a],\mathbb{R}_+)$ such that
$$\|\partial_x f_i(P)\|\le\alpha_0(t),\ \|\partial_v f_i(P)\|_{A\star}\le\alpha_1(t),\ \|\partial_w f_i(P)\|_{B\star}\le\alpha_2(t),$$
$$\|\partial_x F(P)\|\le\beta_0(t),\ \|\partial_v F(P)\|_{A\star}\le\beta(t),\ \|\partial_w F(P)\|_{B\star}\le\gamma(t)$$
for $P=(t,x,v,w)\in\Omega$,
8) there is $L\in\mathbb{L}([0,a],\mathbb{R}_+)$ such that
$$\|\partial_x F(t,x,v,w)-\partial_x F(t,\tilde x,\tilde v,\tilde w)\|\le L(t)(\|x-\tilde x\|+\|v-\tilde v\|_A-\|w-\tilde w\|_B),$$
$$\|\partial_v F(t,x,v,w)-\partial_v F(t,\tilde x,\tilde v,\tilde w)\|_{A\star}\le L(t)(\|x-\tilde x\|+\|v-\tilde v\|_A-\|w-\tilde w\|_B),$$
$$\|\partial_w F(t,x,v,w)-\partial_w F(t,\tilde x,\tilde v,\tilde w)\|_{B\star}\le L(t)(\|x-\tilde x\|+\|v-\tilde v\|_A-\|w-\tilde w\|_B)$$
for $(t,x)\in E$, $v,\tilde v\in C(A,\mathbb{R})$, $w,\tilde w\in C(B,\mathbb{R})$.

Suppose that Assumptions $H_0[f,F]$, $H[f,F]$, $H_*[f,F]$, $H[\varphi]$, $H[\tilde L]$ are satisfied and $\kappa\in\mathcal{X}$.

The next theorem states that for each $\kappa\in\mathcal{X}$ there exists the Fréchet derivative of the solution to the problem (1.1), (1.2) with respect to the initial function.

We will denote by $z(\,\cdot\,;\kappa)$ the solution of (1.1) with the initial condition $z(t,x)=\kappa(t,x)$ on E_0.

Theorem 4.1. *If Assumptions $H_0[f,F]$, $H[f,F]$, $H_*[f,F]$ and $H[\varphi]$ are satisfied, then for each $\kappa\in\mathcal{X}$ there exists the Fréchet derivative of the solution to the problem* (1.1), (1.2) *with respect to initial functions. Moreover, if $\kappa,\chi\in\mathcal{X}$ and z_* is the Fréchet derivative of solution, then z_* is a solution of the equation*
$$z=\Lambda[z], \tag{4.1}$$

where on E

$$\begin{aligned}\Lambda[z](t,x) &= \partial_x\chi(0,g[z(\,\cdot\,;\kappa)](0,t,x))\partial_z g[z(\,\cdot\,;\kappa)](0,t,x)\,z\\ &\quad+\int_0^t \partial_x F(P[z(\,\cdot\,;\kappa)](\tau,t,x))\partial_z g[z(\,\cdot\,;\kappa)](\tau,t,x)\,z\,d\tau\\ &\quad+\int_0^t \partial_v F(P[z(\,\cdot\,;\kappa)](\tau,t,x))z_{\varphi(\tau,g[z(\,\cdot\,;\kappa)](\tau,t,x))}d\tau\\ &\quad+\int_0^t \partial_v F(P[z(\,\cdot\,;\kappa)](\tau,t,x))\partial_x z(\,\cdot\,;\kappa)_{\varphi(\tau,g[z(\,\cdot\,;\kappa)](\tau,t,x))}\\ &\qquad\quad \partial_x\varphi(\tau,g[z(\,\cdot\,;\kappa)](\tau,t,x))\partial_z g[z(\,\cdot\,;\kappa)](\tau,t,x)\,z\,d\tau\\ &\quad+\int_0^t \partial_w F(P[z(\,\cdot\,;\kappa)](\tau,t,x))z_{[\tau,g[z(\,\cdot\,;\kappa)](\tau,t,x)]}d\tau\\ &\quad+\int_0^t \partial_w F(P[z(\,\cdot\,;\kappa)](\tau,t,x))\partial_x z(\,\cdot\,;\kappa)_{[\tau,g[z(\,\cdot\,;\kappa)](\tau,t,x)]}\\ &\qquad\quad \partial_z g[z(\,\cdot\,;\kappa)](\tau,t,x)\,z\,d\tau.\end{aligned}$$

Proof. First we calculate differential quotients for the given functions

$$\Delta_s(t,x)=\frac{1}{s}[z(t,x;\kappa+s\chi)-z(t,x;\kappa)] \quad \text{on } E,$$

where $s\in\mathbb{R}$, $s\neq 0$. Using the mean value theorem we will obtain formula similar to (4.1) with the functions given in the mean points, and the differential quotient Δ_s instead of z_*. The outcome of these calculations will be denoted by $\Lambda_s[z](t,x)$.

We conclude from Assumption $H_*[\,f,F\,]$ that the function Δ_s satisfies the integral functional equation

$$z=\Lambda_s[z].$$

It is easily seen that there exists exactly one solution $z_*\in C_\chi(E_0\cup E,\mathbb{R})$, of equation (4.1). We thus get the formula for $(z_*-\Delta_s)(t,x)$ on E. It follows from the above relations and from conditions 5), 6) of Assumption $H_*[\,f,F\,]$ that there is $\tilde{L}_0\in\mathbb{L}([0,a],\mathbb{R}_+)$ such that

$$\begin{aligned}&\|z_*-\Delta_s\|_{(t,\mathbb{R})}\\ &\le\int_0^t \tilde{L}_0(\tau)\big(\|z(\,\cdot\,;\kappa+s\chi)-z(\,\cdot\,;\kappa)\|_{(\tau,\mathbb{R})}+\|z(\,\cdot\,;\kappa+s\chi)-z(\,\cdot\,;\kappa)\|_{(a,\mathbb{R})}\big)d\tau\\ &\quad+\int_0^t\beta(\tau)\|z_*-\Delta_s\|_{(\tau,\mathbb{R})}d\tau+\int_0^t\gamma(\tau)\|z_*-\Delta_s\|_{(a,\mathbb{R})}d\tau,\quad t\in[0,a].\end{aligned}$$

From Lemma 2.3 we obtain

$$\begin{aligned}\|z_* - \Delta_s\|_{(t,\mathbb{R})} &\le \|z_* - \Delta_s\|_{(a,\mathbb{R})} \int_0^t \gamma(\tau) \exp\left\{ \int_\tau^t \beta(\xi) d\xi \right\} d\tau \\ &+ \exp\left\{ \int_0^t \beta(\tau) d\tau \right\} \int_0^t \tilde{L}_0(\tau) \big(\|z(\,\cdot\,;\kappa + s\chi) - z(\,\cdot\,;\kappa)\|_{(\tau,\mathbb{R})} \\ &\qquad + \|z(\,\cdot\,;\kappa + s\chi) - z(\,\cdot\,;\kappa)\|_{(a,\mathbb{R})} \big) d\tau.\end{aligned} \tag{4.2}$$

We conclude from Theorem 3.1 that there is a function $\Gamma(t)$ such that

$$\|z(\,\cdot\,;\kappa + s\chi) - z(\,\cdot\,;\kappa)\|_{(t,\mathbb{R})} \le \Gamma(t)\, |s|\, \|\chi\|_{(0,\mathbb{R})}$$

and

$$\|z(\,\cdot\,;\kappa + s\chi) - z(\,\cdot\,;\kappa)\|_{(a,\mathbb{R})} \le \Gamma(a)\, |s|\, \|\chi\|_{(0,\mathbb{R})}.$$

Then

$$\|z_* - \Delta_s\|_{(a,\mathbb{R})} \le \Lambda \int_0^a \tilde{L}_0(\tau)(\Gamma(\tau) + \Gamma(a)) d\tau\, |s|\, \|\chi\|_{(0,\mathbb{R})}. \tag{4.3}$$

In conclusion,

$$\|z_* - \Delta_s\|_{(t,\mathbb{R})} \le \Gamma(t) \int_0^a \tilde{L}_0(\tau)[\Gamma(\tau) + \Gamma(a)] d\tau\, |s|\, \|\chi\|_{(0,\mathbb{R})}.$$

From (4.2) and (4.3) we obtain that there exist, $\lim_{s\to 0} \Delta_s$ and

$$\lim_{s\to 0} \Delta_s(t,x) = z_*(t,x) \quad \text{uniformly on } E.$$

This proves the theorem. □

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Elżbieta Puźniakowska-Gałuch
Elzbieta.Galuch@mat.ug.edu.pl

University of Gdańsk
Institute of Mathematics
Wit Stwosz Street 57
80-952 Gdańsk, Poland

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