

## NONTRIVIAL SOLUTIONS OF LINEAR FUNCTIONAL EQUATIONS: METHODS AND EXAMPLES

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**Abstract.** For a wide class of linear functional equations the solutions are generalized polynomials. The existence of non-trivial monomial terms of the solution strongly depends on the algebraic properties of some related families of parameters. As a continuation of the previous work [A. Varga, Cs. Vincze, G. Kiss, *Algebraic methods for the solution of linear functional equations*, Acta Math. Hungar.] we are going to present constructive algebraic methods of the solution in some special cases. Explicit examples will be also given.

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### 1. INTRODUCTION

Consider functional equation

$$\sum_{i=0}^n a_i f(b_i x + (1 - b_i)y) = 0 \quad (x, y \in I), \quad (1.1)$$

where  $I$  is a nonempty open real interval,  $0 < b_0 < b_1 < \dots < b_n < 1$  and  $a_0, a_1, \dots, a_n$  are given nonzero real numbers such that  $\sum_{i=0}^n a_i = 0$  (it is natural because of the substitution  $x = y$ ). According to the basic results [6] and [7] of the theory the solutions of equation (1.1) must be generalized polynomials of the form

$$f(x) = \sum_{k=1}^{n-1} A_k(x, \dots, x) + A_0 \quad (x \in I), \quad (1.2)$$

where  $A_k: \mathbb{R}^k \rightarrow \mathbb{R}$  is a symmetric  $k$ -additive function,  $A_0 \in \mathbb{R}$  for any  $k = 1, \dots, n-1$ . We have the following system of equations for the monomial term  $A_k$ :

$$(A_k) : \begin{cases} A_k(s, \dots, s, t) + \sum_{i=1}^{n-1} \alpha_i A_k(s, \dots, s, t\beta_i) = 0, \\ A_k(s, \dots, s, t, t) + \sum_{i=1}^{n-1} \alpha_i A_k(s, \dots, s, t\beta_i, t\beta_i) = 0, \\ \dots \\ \dots \\ \dots \\ A_k(t, \dots, t) + \sum_{i=1}^{n-1} \alpha_i A_k(t\beta_i, \dots, t\beta_i) = 0, \end{cases}$$

where  $s, t \in \mathbb{R}$ ,

$$\alpha_i := \frac{a_i}{a_n} \quad \text{and} \quad \beta_i := \frac{b_i - b_0}{b_n - b_0}$$

( $i = 1, \dots, n-1$ ); for details see [10] and [11]. The problem is to find necessary and sufficient conditions in terms of the parameters  $\alpha_1, \dots, \alpha_{n-1}$  and  $\beta_1, \dots, \beta_{n-1}$  or constructive methods to decide the existence of a non-trivial generalized polynomial solution. A descending tendency can be concluded: if equation (1.1) has a solution of degree  $k > 1$ , then it also has solutions of degree  $1, \dots, k-1$ , respectively. Therefore, the existence of the non-trivial solution depends on the equation

$$(A_1): A_1(t) + \sum_{i=1}^{n-1} \alpha_i A_1(t\beta_i) = 0.$$

Using some recent results [5], see also [2] and [4], spectral synthesis and spectral analysis hold in some related varieties of the functional equation. The spectral analysis allows us to choose a special kind of solution called *exponentials*. Since the varieties are considered over the multiplicative group of the extension of  $\mathbb{Q}$  with the parameters, the additivity and the exponential property result in field homomorphisms. These are characterized in the following important theorem.

**Theorem 1.1** ([2]). *There exists a not identically zero  $k$ -additive symmetric function satisfying  $(A_k)$  if and only if there exists a collection of injective field homomorphisms  $\delta_1, \dots, \delta_k: \mathbb{Q}(\beta_1, \dots, \beta_{n-1}) \rightarrow \mathbb{C}$  such that*

$$(F_k) : \begin{cases} 1 + \sum_{i=1}^{n-1} \alpha_i \delta_j(\beta_i) = 0 & (j = 1, \dots, k), \\ 1 + \sum_{i=1}^{n-1} \alpha_i \delta_{j_1}(\beta_i) \delta_{j_2}(\beta_i) = 0 & (j_1 \neq j_2, j_1, j_2 = 1, \dots, k), \\ \dots \\ \dots \\ \dots \\ 1 + \sum_{i=1}^{n-1} \alpha_i \delta_1(\beta_i) \dots \delta_k(\beta_i) = 0. \end{cases}$$

The importance of Theorem 1.1 is that  $(F_k)$  provides us with a system of equations to check the existence of generalized monomial solutions. The generalized monomial solutions cannot be given explicitly in general but we can check (at least theoretically)

the algebraic properties of the parameters to decide whether  $(F_k)$  can be satisfied or not. The aim of the paper is to present methods and examples. Table 1 shows the tendency under increasing the maximal degree of the solution:  $|F_k|$  denotes the number of equations of the system  $(F_k)$ ,  $|\delta_i(\beta_j)|$  is the number of elements  $\delta_i(\beta_j)$  and  $k = 1, \dots, n - 1$ .

**Table 1.**

$n$	$n - 1$	$1 \leq k \leq n - 1$	$ F_k  = 2^k - 1$	$ \delta_i(\beta_j)  = k(n - 1)$
2	1	1	1	1
3	2	1	1	2
		2	3	4
4	3	1	1	3
		2	3	6
		3	7	9

By Theorem 1.1, we have to solve the problem of existence of injective field homomorphisms<sup>1)</sup> satisfying  $(F_k)$ , especially,

$$(F_1) : 1 + \sum_{i=1}^{n-1} \alpha_i \delta_1(\beta_i) = 0.$$

We are going to present constructive algebraic methods of the solution in some special cases: the case of algebraic parameters (sufficient and necessary conditions, Gauss elimination and characteristic polynomials), biadditive solutions in case of  $n = 3$  (for solutions of degree one see [11]), explicit examples.

## 2. PRELIMINARY RESULTS AND BASIC CONCEPTS

The element  $\vec{\beta} := (\beta_1, \dots, \beta_m) \in \mathbb{R}^m$  or  $\mathbb{C}^m$  is called an algebraically dependent system if there exists a not identically zero  $p \in \mathbb{Q}[x_1, \dots, x_m]$  such that  $p(\vec{\beta}) = 0$ . Otherwise, we say that it is an algebraically independent system. The ideal

$$\mathcal{I}(\vec{\beta}) := \{p \in \mathbb{Q}[x_1, \dots, x_m] \mid p(\vec{\beta}) = 0\}$$

of the polynomial ring  $\mathbb{Q}[x_1, \dots, x_m]$  is called the defining ideal of  $\vec{\beta}$ . If  $\mathcal{I}(\vec{\beta}) = \mathcal{I}(\vec{\gamma})$ , then we say that  $\vec{\beta}$  and  $\vec{\gamma}$  are algebraically conjugated. It is known [10] that there exists a field isomorphism

$$\delta: \mathbb{Q}(\beta_1, \dots, \beta_m) \rightarrow \mathbb{Q}(\gamma_1, \dots, \gamma_m)$$

such that  $\delta(\beta_i) = \gamma_i$  ( $i = 1, 2, \dots, m$ ) if and only if  $\vec{\beta}$  and  $\vec{\gamma}$  are algebraically conjugated. In case of  $n = 2$  equation  $(F_1)$  reduces to  $1 + \alpha_1 \delta_1(\beta_1) = 0$  and the value of  $\delta_1(\beta_1)$

<sup>1)</sup> In what follows we shall frequently use that any injective homomorphism  $\delta: \mathbb{Q}(\beta_1, \dots, \beta_{n-1}) \rightarrow \mathbb{C}$  can be extended to an automorphism  $\delta: \mathbb{C} \rightarrow \mathbb{C}$ ; see [3].

can be directly expressed. Therefore, there exists a not identically zero solution iff  $\beta_1$  and  $-(1/\alpha_1)$  are algebraically conjugated. This is one of the first results due to Z. Daróczy [1] for the solvability of equation  $(A_1)$ . We also have some important results as a kind of generalization of Daróczy's theorem for  $n \geq 3$ .

**Theorem 2.1** ([9]). *Let  $n \geq 2$  be a given natural number. Suppose that the outer parameters  $\alpha_i$ 's ( $i = 1, \dots, n - 1$ ) form an algebraically independent system. There exists a not identically zero  $A_1: \mathbb{R} \rightarrow \mathbb{R}$  additive solution of equation  $(A_1)$  iff at least one of the inner parameters  $\beta_i$ 's is transcendental.*

The proof goes back to the construction of a field homomorphism  $\delta_1$  satisfying  $(F_1)$ . Using  $\delta_1^{-1}$  the role of the outer and the inner parameters in Theorem 2.1 can be interchanged; for the details see [9].

### 3. THE CASE OF ALGEBRAIC PARAMETERS

Another interesting pure case when all the inner (or the outer) parameters are algebraic numbers over the rationals. Then we have an eliminating technique due to A. Varga [8] to solve equation  $(A_1)$ , i.e. we can decide algorithmically the problem of the existence of non-trivial solutions.

#### 3.1. THE GAUSS ELIMINATION

The main steps of the algorithm are as follows: since all the parameters  $\beta_i$  ( $i = 1, \dots, n - 1$ ) are algebraic, the extension of the rationals with elements  $\beta_i$ 's can be written into the form  $\mathbb{Q}(\beta_1, \dots, \beta_{n-1}) = \mathbb{Q}(u)$  for some algebraic element  $u$  over the rationals. Let  $p(t) := t^{d+1} + \sum_{i=0}^d q_i t^i$  be the defining polynomial of  $u$ , i.e.  $p(u) = 0$  and consider the inner parameters as

$$\beta_i = \sum_{j=0}^d r_{ij} u^j \quad (i = 1, \dots, n - 1),$$

where  $r_{ij}$  are rational coefficients to express the parameters in terms of the basis  $u^d, \dots, u^1, 1$  of the vector space  $\mathbb{Q}(u)$  over  $\mathbb{Q}$ . Therefore, we can write equation  $(A_1)$  into the form

$$p_d A_1(u^d t) + \dots + p_1 A_1(ut) + p_0 A_1(t) = 0, \tag{3.1}$$

where  $p_j = \sum_{i=1}^{n-1} r_{ij} \alpha_i$ ,  $j = 0, \dots, d$ . Using the defining polynomial of  $u$  and the rational homogeneity of additive functions, subsequent substitutions of  $ut$  instead of  $t$  in equations of type (3.1) give a linear system of equations for the variables  $A_1(t), A_1(ut), \dots, A_1(u^d t)$ . The method can be formulated in terms of the outer parameters, too [8].

3.2. AN EXAMPLE

We can also use the substitution of the elements  $\beta_1^{i_1} \cdot \dots \cdot \beta_{n-1}^{i_{n-1}}$  in lexicographic order, where  $0 \leq i_1 \leq d_1, \dots, 0 \leq i_{n-1} \leq d_{n-1}$  and  $d_i + 1$  is the degree of  $\beta_i$  over  $\mathbb{Q}$ . They form a directly given generating system of  $\mathbb{Q}(\beta_1, \dots, \beta_{n-1})$  over  $\mathbb{Q}$  instead of  $1, u, \dots, u^d$ . The following example shows how the method is working in practice. Consider equation

$$\alpha_1 A_1(\sqrt{2}t) + \alpha_2 A_1(\sqrt{5}t) + \alpha_3 A_1(\sqrt{7}t) + A(t) = 0, \tag{3.2}$$

where the outer parameters are non-zero real numbers. Substituting  $\sqrt{2}t, \sqrt{5}t, \sqrt{7}t, \sqrt{2}\sqrt{5}t, \sqrt{2}\sqrt{7}t, \sqrt{5}\sqrt{7}t$  and  $\sqrt{2}\sqrt{5}\sqrt{7}t$  in the arguments of  $A_1$ , respectively, we have a linear system of functional equations

$$M \begin{pmatrix} A(t) \\ A(\sqrt{2}t) \\ A(\sqrt{5}t) \\ A(\sqrt{7}t) \\ A(\sqrt{2}\sqrt{5}t) \\ A(\sqrt{2}\sqrt{7}t) \\ A(\sqrt{5}\sqrt{7}t) \\ A(\sqrt{2}\sqrt{5}\sqrt{7}t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$M := M(\alpha_1, \alpha_2, \alpha_3)$$

and

$$M(x, y, z) := \begin{pmatrix} 1 & x & y & z & 0 & 0 & 0 & 0 \\ 2x & 1 & 0 & 0 & y & z & 0 & 0 \\ 5y & 0 & 1 & 0 & x & 0 & z & 0 \\ 7z & 0 & 0 & 1 & 0 & x & y & 0 \\ 0 & 5y & 2x & 0 & 1 & 0 & 0 & z \\ 0 & 7z & 0 & 2x & 0 & 1 & 0 & y \\ 0 & 0 & 7z & 5y & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 7z & 5y & 2x & 1 \end{pmatrix}.$$

The subdeterminant

$$\det \begin{pmatrix} z & 0 & 0 & 0 & 0 \\ 0 & y & z & 0 & 0 \\ 0 & x & 0 & z & 0 \\ 1 & 0 & x & y & 0 \\ 0 & 1 & 0 & 0 & z \end{pmatrix} = -2z^3xy$$

shows that the rank of  $M$  is at least 5. If the rank is exactly 5, then the vanishing of the larger subdeterminants<sup>2)</sup> implies that  $x^2 = \frac{1}{2}$ ,  $y^2 = 1$  and  $z^2 = \frac{5}{7}$ . Otherwise, the rank is at least 6. In what follows, we will reduce the number of the parameters in each possible case. If the rank of  $M$  is 5, then we have that

$$\pm \frac{1}{\sqrt{2}}A(\sqrt{2}t) \pm A(\sqrt{5}t) \pm \frac{\sqrt{5}}{\sqrt{7}}A(\sqrt{7}t) + A(t) = 0$$

and, consequently,

$$\pm \frac{1}{\sqrt{2}}A(\sqrt{2}t) \pm A((\sqrt{5} + 1)t) \pm \frac{\sqrt{5}}{\sqrt{7}}A(\sqrt{7}t) = 0.$$

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<sup>2)</sup> If the rank is less than 6, then the vanishing of the subdeterminant

$$\det \begin{pmatrix} y & z & 0 & 0 & 0 & 0 \\ 0 & 0 & y & z & 0 & 0 \\ 1 & 0 & x & 0 & z & 0 \\ 0 & 1 & 0 & x & y & 0 \\ 2x & 0 & 1 & 0 & 0 & z \\ 0 & 2x & 0 & 1 & 0 & y \end{pmatrix} = 4z^2y^2(1 - 2x^2)$$

implies that  $x^2 = \frac{1}{2}$ . The vanishing of the subdeterminant

$$\det \begin{pmatrix} x & y & z & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & y & z & 0 & 0 \\ 0 & 1 & 0 & x & 0 & z & 0 \\ 0 & 0 & 1 & x & 0 & y & 0 \\ 5y & 2x & 0 & 1 & 0 & 0 & z \\ 7z & 0 & 2x & 0 & 1 & 0 & y \\ 0 & 7z & 5y & 0 & 0 & 1 & x \end{pmatrix} = -8x^5yz + 56z^3yx^3 - 8zyx^3$$

$$+ 40zy^3x^3 - 98xyz^5 + 140xz^3y^3 - 28xz^3y + 6zyx - 50xyz^5 - 20xyz^3$$

$$= xyz(-8x^4 + 56z^2x^2 - 8x^2 + 40y^2x^2 - 98z^4 + 140z^2y^2 - 28z^2 + 6 - 50y^4 - 20y^2)$$

and  $x^2 = \frac{1}{2}$  give that

$$0 = -98z^4 + 140z^2y^2 - 50y^4 = -2(7z^2 - 5y^2)^2$$

and, consequently,  $y^2 = \frac{7}{5}z^2$ . Since

$$\det M(x, y, z) = 1 + 16x^8 + 560z^2y^2x^4 + 2401z^8 + 392z^4x^2 - 500y^6 - 28z^2$$

$$+ 112z^2x^4 + 24x^4 - 6860z^6y^2 + 7350z^4y^4 - 3500z^2y^6 - 2800z^2y^2x^2 + 625y^8$$

$$+ 1960z^4y^2x^2 + 1400z^2y^4x^2 - 160x^6y^2 - 224x^6z^2 + 600x^4y^4 + 80x^4y^2$$

$$+ 1176x^4z^4 - 1000x^2y^6 + 200x^2y^4 - 2744x^2z^6 + 980z^4y^2 - 1372z^6$$

$$+ 700z^2y^4 - 32x^6 + 150y^4 + 40x^2y^2 + 294z^4 - 20y^2 - 8x^2 + 56z^2x^2 + 140z^2y^2,$$

we have, by the vanishing of the determinant,  $x^2 = \frac{1}{2}$  and  $y^2 = \frac{7}{5}z^2$ , that

$$0 = -294z^4 + \frac{14406}{25}z^8.$$

Therefore,  $z^2 = \frac{5}{7}$  and  $y^2 = 1$ .

The substitution of  $\frac{t}{1+\sqrt{5}}$  gives the reduced equation

$$\pm \frac{1}{\sqrt{2}} A\left(\frac{\sqrt{2}}{1+\sqrt{5}}t\right) \pm \frac{\sqrt{5}}{\sqrt{7}} A\left(\frac{\sqrt{7}}{1+\sqrt{5}}t\right) \pm A(t) = 0.$$

In the case of rank  $M \geq 6$  the null-space is of dimension at most 2 and

$$\begin{pmatrix} A(t) \\ A(\sqrt{2}t) \\ A(\sqrt{5}t) \\ A(\sqrt{7}t) \\ A(\sqrt{2}\sqrt{5}t) \\ A(\sqrt{2}\sqrt{7}t) \\ A(\sqrt{5}\sqrt{7}t) \\ A(\sqrt{2}\sqrt{5}\sqrt{7}t) \end{pmatrix} = \lambda(t) \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{pmatrix} + \mu(t) \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \end{pmatrix}$$

for some additive functions  $\lambda$  and  $\mu$ . Using subdeterminants of type  $3 \times 3$  we can reduce the number of the parameters again: for example

$$0 = (v_1w_2 - v_2w_1)A(t) - (v_0w_2 - v_2w_0)A(\sqrt{2}t) + (v_0w_1 - v_1w_0)A(\sqrt{5}t).$$

The algorithm can be obviously repeated after the reduction.

### 3.3. THE CHARACTERISTIC POLYNOMIALS

In what follows we present an alternative approach to the case of algebraic parameters. Using the notations of Subsection 3.1 we can write equation  $(F_1)$  into the form

$$1 + \sum_{j=0}^d \left( \sum_{i=1}^{n-1} \alpha_i r_{ij} \right) s^j = 0, \tag{3.3}$$

where  $\delta_1(u) = s$  is one of the algebraically conjugated elements to  $u$ .

A possible way of solving the problem is to check the vanishing of the factor (3.3) for any choice of the conjugate elements to  $u$ . To avoid the difficulties of computing the conjugate elements (the roots of the defining polynomial  $p$ ) we can introduce the so-called characteristic polynomial as follows: consider the product

$$\mathcal{P}_1(\alpha_1, \dots, \alpha_{n-1}) := \prod_{h=1}^{d+1} \left( 1 + \sum_{j=0}^d \left( \sum_{i=1}^{n-1} \alpha_i r_{ij} \right) s_h^j \right), \tag{3.4}$$

where  $s_1, \dots, s_{d+1}$  are the roots of the defining polynomial  $p$ . Since (3.4) is symmetric in the variables  $s_h$ 's we have that it is a polynomial of the elementary symmetric polynomials

$$s_1 + \dots + s_{d+1}, \dots, s_1 \cdots \cdots s_{d+1}.$$

On the other hand they give the (rational) coefficients of  $p$ . Therefore,  $\mathcal{P}_1$  is a polynomial with rational coefficients of the variables  $\alpha_1, \dots, \alpha_{n-1}$ .

**Theorem 3.1.** *Suppose that the inner parameters are algebraic; we have a non-trivial solution of equation (1.1) iff  $\mathcal{P}_1(\alpha_1, \dots, \alpha_{n-1}) = 0$ .*

*Proof.* The only if part is obvious from the construction of  $\mathcal{P}_1$ . Conversely, suppose that  $\mathcal{P}_1(\alpha_1, \dots, \alpha_{n-1}) = 0$ . Then at least one of the factors of the form (3.3) must be zero. The field homomorphism defined as  $\delta_1(u) := s$  satisfies equation  $(F_1)$ .  $\square$

It is natural to modify the construction for the case of algebraic outer parameters. We can also introduce polynomials  $\mathcal{P}_1, \dots, \mathcal{P}_{n-1}$  related to the higher order terms by using the system  $(F_k)$  in Theorem 1.1 for any  $k = 1, \dots, n - 1$ :

$$\mathcal{P}_k(\alpha_1, \dots, \alpha_{n-1}) := \prod_{h_1=1}^{d+1} \prod_{h_2=1}^{d+1} \dots \prod_{h_k=1}^{d+1} \left( 1 + \sum_{i=1}^{n-1} \alpha_i \left( \left( \sum_{j_1=0}^d r_{ij_1} s_{h_1}^{j_1} \right) \cdot \dots \cdot \left( \sum_{j_k=0}^d r_{ij_k} s_{h_k}^{j_k} \right) \right) \right).$$

**Definition 3.2.** The polynomials  $\mathcal{P}_1, \dots, \mathcal{P}_{n-1} \in \mathbb{Q}[x_1, \dots, x_{n-1}]$  are called the characteristic polynomials of the functional equation.

Conditions  $\mathcal{P}_1(\alpha_1, \dots, \alpha_{n-1}) = 0, \dots, \mathcal{P}_k(\alpha_1, \dots, \alpha_{n-1}) = 0$  are obviously necessary for the existence of generalized polynomial solutions of degree  $k$ . Unfortunately they are not automatically sufficient conditions because the vanishing factors may contain different systems of conjugates. Alternatively we can check the vanishing of the factors for any choice of the system of conjugate elements to  $u$ . The advantage of using the characteristic polynomials is that they do not involve explicitly the algebraic conjugates of  $u$ . The symmetrization process integrates them into the coefficients of the defining polynomial.

**Remark 3.3.** Using the Gauss elimination method 3.1 or the characteristic polynomial method we can find every isomorphism solution because algebraic parameters provide us with a finite dimensional vector space environment for the Gauss elimination or finitely many algebraic conjugates as possible values of the field isomorphisms.

#### 4. SOLUTIONS OF ORDER 2 IN CASE OF $n = 3$

In case of  $n = 3$  the maximal degree of the solutions of equation (1.1) is 2. We are going to present algebraic methods to decide the existence of non-trivial solutions of maximal degree.

**Theorem 4.1** ([11]). *Let the reals  $a_i \neq 0, b_i$  ( $i = 0, 1, 2, 3$ ) be given such that*

$$a_0 + a_1 + a_2 + a_3 = 0 \quad \text{and} \quad 0 < b_0 < b_1 < b_2 < b_3 < 1.$$

*There exist a not identically zero symmetric biadditive function such that its diagonalization is a solution of equation*

$$\sum_{i=0}^3 a_i f(b_i x + (1 - b_i)y) = 0 \quad (x, y \in I) \tag{4.1}$$



if and only if there exists  $\mu \in \mathbb{C}$  such that

- $(\beta_1, \beta_2)$  and  $\left(\frac{\beta_1}{\mu(\beta_1-1)+1}, \frac{\beta_2}{\mu(\beta_2-1)+1}\right)$  are algebraically conjugated,
- $(\alpha_1, \alpha_2)$  and  $\left(\frac{1-\beta_2}{\beta_2-\beta_1} \cdot \frac{\mu(\beta_1-1)+1}{\beta_1}, \frac{\beta_1-1}{\beta_2-\beta_1} \cdot \frac{\mu(\beta_2-1)+1}{\beta_2}\right)$  are algebraically conjugated,

where  $\alpha_i := \frac{a_i}{a_3}$  and  $\beta_i := \frac{b_i-b_0}{b_3-b_0}$  ( $i = 1, 2$ ).

An important intermediate result in the proof of Theorem 4.1 is that

$$\lambda := \frac{\beta_1(1 - \beta_2)}{\beta_2 - \beta_1}$$

is invariant in the sense that  $\delta_1(\lambda) = \delta_2(\lambda)$ , where the homomorphisms  $\delta_1$  and  $\delta_2$  satisfy  $(F_2)$  under  $n = 3$ :

$$(F_2) : \begin{cases} 1 + \alpha_1\delta_1(\beta_1) + \alpha_2\delta_1(\beta_2) = 0, \\ 1 + \alpha_1\delta_2(\beta_1) + \alpha_2\delta_2(\beta_2) = 0, \\ 1 + \alpha_1\delta_1(\beta_1)\delta_2(\beta_1) + \alpha_2\delta_1(\beta_2)\delta_2(\beta_2) = 0 \end{cases}$$

(for the details<sup>3)</sup> see [11]).

In what follows we are going to eliminate the free parameter  $\mu$  from the conditions of Theorem 4.1. The investigation is based on the algebraic properties of  $\alpha_1, \alpha_2$  or  $\beta_1, \beta_2$  as follows:

- I.  $\vec{\alpha} = (\alpha_1, \alpha_2)$  is an algebraically dependent system,
- II.  $\vec{\beta} = (\beta_1, \beta_2)$  is an algebraically dependent system,
- III. both  $\vec{\alpha} = (\alpha_1, \alpha_2)$  and  $\vec{\beta} = (\beta_1, \beta_2)$  are algebraically independent.

Case I. Suppose that  $\vec{\alpha} = (\alpha_1, \alpha_2)$  is an algebraically dependent system and let  $p \in \mathcal{I}(\alpha_1, \alpha_2)$  be a non-zero polynomial such that

$$\sum_{i,j} p_{ij} \alpha_1^i \alpha_2^j = 0.$$

By Theorem 4.1,

$$\sum_{i,j} p_{ij} \left(\frac{1 - \beta_2}{\beta_2 - \beta_1} \cdot \frac{\mu(\beta_1 - 1) + 1}{\beta_1}\right)^i \left(\frac{\beta_1 - 1}{\beta_2 - \beta_1} \cdot \frac{\mu(\beta_2 - 1) + 1}{\beta_2}\right)^j = 0. \tag{4.2}$$

<sup>3)</sup> The invariance of  $\lambda$  is equivalent to the vanishing of the determinant

$$\det \begin{pmatrix} 1 & \delta_1(\beta_1) & \delta_1(\beta_2) \\ 1 & \delta_2(\beta_1) & \delta_2(\beta_2) \\ 1 & \delta_1(\beta_1)\delta_2(\beta_1) & \delta_1(\beta_2)\delta_2(\beta_2) \end{pmatrix}.$$

In terms of  $\varphi := \delta_2^{-1} \circ \delta_1$  we have that  $\varphi(\lambda) = \lambda$ . Especially  $\lambda$  is the cross ratio of the elements  $\beta_1, 1, 0, \beta_2$ . Therefore, there exists a Möbius transformation on the complex plane such that  $M(\beta_1) = \varphi(\beta_1), M(1) = 1, M(0) = 0$  and  $M(\beta_2) = \varphi(\beta_2)$ . Using that  $M(1) = 1$  and  $M(0) = 0$  the general form of a Möbius transformation can be reduced to

$$M(z) = \frac{z}{\mu(z - 1) + 1}$$

which is the geometric meaning of the parameter  $\mu$  in Theorem 4.1.

If the polynomial

$$r(t) := \sum_{i,j} p_{ij} \left( \frac{1 - \beta_2}{\beta_2 - \beta_1} \cdot \frac{t(\beta_1 - 1) + 1}{\beta_1} \right)^i \left( \frac{\beta_1 - 1}{\beta_2 - \beta_1} \cdot \frac{t(\beta_2 - 1) + 1}{\beta_2} \right)^j \tag{4.3}$$

is not identically zero, then the parameter  $\mu$  can be determined as one of the finitely many roots. Otherwise,  $\sum_{i,j} p_{ij} x^i(t) y^j(t) = 0$ , where

$$x(t) := \frac{1 - \beta_2}{\beta_2 - \beta_1} \cdot \frac{t(\beta_1 - 1) + 1}{\beta_1} \quad \text{and} \quad y(t) := \frac{\beta_1 - 1}{\beta_2 - \beta_1} \cdot \frac{t(\beta_2 - 1) + 1}{\beta_2}$$

give the parametrization of the line  $y = mx + b$ . The coordinates of the directional vector can be given as

$$x'(t) = \frac{1 - \beta_2}{\beta_2 - \beta_1} \cdot \frac{\beta_1 - 1}{\beta_1} \quad \text{and} \quad y'(t) = \frac{\beta_1 - 1}{\beta_2 - \beta_1} \cdot \frac{\beta_2 - 1}{\beta_2}$$

and, consequently, the slope is  $m = -\beta_1/\beta_2$ . Using the substitutions

$$x(0) = \frac{1 - \beta_2}{\beta_2 - \beta_1} \cdot \frac{1}{\beta_1} \quad \text{and} \quad y(0) = \frac{\beta_1 - 1}{\beta_2 - \beta_1} \cdot \frac{1}{\beta_2}$$

we have that the equation of the line is  $y = -(\beta_1/\beta_2)x - (1/\beta_2)$ . Since it is an algebraic “hyperplane” ([9]), both  $\beta_1/\beta_2$  and  $1/\beta_2$  are algebraic numbers. The algebraic numbers form a field which means that both  $\beta_1$  and  $\beta_2$  are algebraic. Therefore, the possible values of the parameter  $\mu$  can be determined from the equations

$$\omega = \frac{\beta_1}{\mu(\beta_1 - 1) + 1} \quad \text{and} \quad \eta = \frac{\beta_2}{\mu(\beta_2 - 1) + 1},$$

where  $\omega$  and  $\eta$  can be chosen from the finitely many algebraically conjugated elements to  $\beta_1$  and  $\beta_2$ , respectively.

*Case II.* Suppose that  $\vec{\beta} = (\beta_1, \beta_2)$  is an algebraically dependent system. It can be easily seen that the system  $(\beta_1^{-1}, \beta_2^{-1})$  is also algebraically dependent. Let  $p \in \mathcal{I}(\beta_1^{-1}, \beta_2^{-1})$  be a non-zero polynomial such that

$$\sum_{i,j} p_{ij} \left( \frac{1}{\beta_1} \right)^i \left( \frac{1}{\beta_2} \right)^j = 0.$$

By Theorem 4.1,

$$\sum_{i,j} p_{ij} \left( \frac{\mu(\beta_1 - 1) + 1}{\beta_1} \right)^i \left( \frac{\mu(\beta_2 - 1) + 1}{\beta_2} \right)^j = 0. \tag{4.4}$$

If the polynomial

$$s(t) := \sum_{i,j} p_{ij} \left( \frac{t(\beta_1 - 1) + 1}{\beta_1} \right)^i \left( \frac{t(\beta_2 - 1) + 1}{\beta_2} \right)^j \tag{4.5}$$

is not identically zero, then the parameter  $\mu$  can be determined as one of the finitely many roots – note that  $t = 0$  is automatically one of the roots of polynomial (4.5). Otherwise,  $\sum_{i,j} p_{ij}x^i(t)y^j(t) = 0$ , where

$$x(t) := \frac{t(\beta_1 - 1) + 1}{\beta_1} \quad \text{and} \quad y(t) := \frac{t(\beta_2 - 1) + 1}{\beta_2}$$

give the parametrization of the line  $y = mx + b$ . The coordinates of the directional vector can be given as

$$x'(t) = \frac{\beta_1 - 1}{\beta_1} \quad \text{and} \quad y'(t) = \frac{\beta_2 - 1}{\beta_2}$$

and, consequently, the slope is

$$m = \frac{\beta_2 - 1}{\beta_1 - 1} \cdot \frac{\beta_1}{\beta_2}.$$

Using the substitutions  $x(0) = 1/\beta_1$  and  $y(0) = 1/\beta_2$  we have that the equation of the line is

$$y = \frac{\beta_2 - 1}{\beta_1 - 1} \cdot \frac{\beta_1}{\beta_2} x + \frac{\beta_1 - \beta_2}{\beta_2(\beta_1 - 1)}.$$

Since it is an algebraic “hyperplane” [9], both

$$\frac{\beta_2 - 1}{\beta_1 - 1} \cdot \frac{\beta_1}{\beta_2} \quad \text{and} \quad \frac{\beta_1 - \beta_2}{\beta_2(\beta_1 - 1)}$$

are algebraic numbers. The algebraic numbers form a field which means that the invariant parameter

$$\lambda := \frac{\beta_1(1 - \beta_2)}{\beta_2 - \beta_1}$$

is also algebraic and, consequently,

$$\delta_i \left( \frac{\beta_1(1 - \beta_2)}{\beta_2 - \beta_1} \right) = \omega \quad (i = 1, 2), \tag{4.6}$$

where  $\omega$  belong to the set of the finitely many algebraically conjugated elements to the invariant parameter. In what follows we determine the values of  $\delta_i(\beta_j)$ ’s in terms of  $\omega$ . From equation (4.6)

$$\delta_i(\beta_2) = \delta_i(\beta_1) \frac{\omega + 1}{\delta_i(\beta_1) + \omega}.$$

According to  $(F_2)$

$$1 + \alpha_1 \delta_i(\beta_1) + \alpha_2 \delta_i(\beta_1) \frac{\omega + 1}{\delta_i(\beta_1) + \omega} = 0$$

which means that  $\delta_i(\beta_1)$  ( $i = 1, 2$ ) must be the root of the quadratic equation

$$\alpha_1 x^2 + (\alpha_2 + (\alpha_1 + \alpha_2)\omega + 1)x + \omega = 0. \tag{4.7}$$

In a similar way, equation (4.6) says that

$$\delta_i(\beta_1) = \delta_i(\beta_2) \frac{\omega}{\omega + 1 - \delta_i(\beta_2)}$$

and, by  $(F_2)$ ,

$$1 + \alpha_1 \delta_i(\beta_2) \frac{\omega}{\omega + 1 - \delta_i(\beta_2)} + \alpha_2 \delta_i(\beta_2) = 0 \quad (i = 1, 2).$$

Therefore,  $\delta_i(\beta_2)$  ( $i = 1, 2$ ) must be the root of the quadratic equation

$$\alpha_2 x^2 - (\alpha_2 + (\alpha_1 + \alpha_2)\omega - 1)x - (\omega + 1) = 0. \tag{4.8}$$

Let  $z_1, z_2$  and  $w_1, w_2$  be the roots of equations (4.7) and (4.8), respectively. We can conclude the following statement: there exists a non-trivial biadditive term in the solution of equation (4.1) if and only if there exist not necessarily different pairs  $(z_k, w_l)$  and  $(z_r, w_s)$  such that they are algebraically conjugated to  $\vec{\beta} = (\beta_1, \beta_2)$  and  $1 + \alpha_1 z_k z_r + \alpha_2 w_l w_s = 0$  for some  $k, l, r, s \in \{1, 2\}$ . For a given  $\omega$  (the conjugate of the invariant parameter) the number of the possible cases to check is  $2^4$  in general because we can choose the roots independently with repetitions.

*Case III.* Suppose that both  $\vec{\alpha} = (\alpha_1, \alpha_2)$  and  $\vec{\beta} = (\beta_1, \beta_2)$  are algebraically independent and choose an element  $\mu$  which is transcendent over the field  $\mathbb{Q}(\beta_1, \beta_2)$ . We are going to prove that

$$(\beta_1, \beta_2) \quad \text{and} \quad \left( \frac{\beta_1}{\mu(\beta_1 - 1) + 1}, \frac{\beta_2}{\mu(\beta_2 - 1) + 1} \right) \quad \text{are algebraically conjugated,}$$

i.e.

$$\frac{\beta_1}{\mu(\beta_1 - 1) + 1} \quad \text{and} \quad \frac{\beta_2}{\mu(\beta_2 - 1) + 1}$$

are algebraically independent. Let  $p \in \mathbb{Q}[x_1, x_2]$  be an arbitrary polynomial and suppose that

$$\sum_{i,j} p_{ij} \left( \frac{\mu(\beta_1 - 1) + 1}{\beta_1} \right)^i \left( \frac{\mu(\beta_2 - 1) + 1}{\beta_2} \right)^j = 0.$$

Since  $\mu$  is transcendent over the extension  $\mathbb{Q}(\beta_1, \beta_2)$  of the rationals, the polynomial  $s(t)$  defined in (4.5) must be identically zero because  $s(\mu) = 0$ . This means that the constant term must be also zero:

$$s(0) = \sum_{i,j} p_{ij} \left( \frac{1}{\beta_1} \right)^i \left( \frac{1}{\beta_2} \right)^j = 0.$$

If  $k := \max\{i \mid p_{ij} \neq 0\}$  and  $l := \max\{j \mid p_{ij} \neq 0\}$ , then

$$0 = \beta_1^k \beta_2^l \sum_{i,j} p_{ij} \left( \frac{1}{\beta_1} \right)^i \left( \frac{1}{\beta_2} \right)^j = \sum_{i,j} p_{ij} \beta_1^{k-i} \beta_2^{l-j}$$

which means that  $p_{ij} = 0$  for any indices  $i$  and  $j$  because  $\vec{\beta} = (\beta_1, \beta_2)$  is algebraically independent. In a similar way, we can check the condition

$$(\alpha_1, \alpha_2) \text{ and } \left( \frac{1-\beta_2}{\beta_2-\beta_1} \cdot \frac{\mu(\beta_1-1)+1}{\beta_1}, \frac{\beta_1-1}{\beta_2-\beta_1} \cdot \frac{\mu(\beta_2-1)+1}{\beta_2} \right) \text{ are algebraically conjugated,}$$

i.e.

$$\frac{1-\beta_2}{\beta_2-\beta_1} \cdot \frac{\mu(\beta_1-1)+1}{\beta_1} \text{ and } \frac{\beta_1-1}{\beta_2-\beta_1} \cdot \frac{\mu(\beta_2-1)+1}{\beta_2}$$

are algebraically independent. Let  $p \in \mathbb{Q}[x_1, x_2]$  be an arbitrary polynomial and suppose that

$$\sum_{i,j} p_{ij} \left( \frac{1-\beta_2}{\beta_2-\beta_1} \cdot \frac{\mu(\beta_1-1)+1}{\beta_1} \right)^i \left( \frac{\beta_1-1}{\beta_2-\beta_1} \cdot \frac{\mu(\beta_2-1)+1}{\beta_2} \right)^j = 0. \tag{4.9}$$

Since  $\mu$  is transcendent over the extension  $\mathbb{Q}(\beta_1, \beta_2)$  of the rationals, the polynomial  $r(t)$  defined in (4.3) must be identically zero because  $r(\mu) = 0$ . If  $p$  is a non-zero polynomial, then both  $\beta_1$  and  $\beta_2$  are algebraic numbers as we have seen in Case I. This is a contradiction. Therefore,  $p_{ij} = 0$  for any indices  $i$  and  $j$ .

**Corollary 4.2.** *If both  $\vec{\alpha} = (\alpha_1, \alpha_2)$  and  $\vec{\beta} = (\beta_1, \beta_2)$  are algebraically independent, then equation (4.1) has a not identically zero second order monomial solution.*

### 5. AN EXAMPLE

Consider functional equation (4.1) under the choice

$$\alpha_1 = \frac{1}{\sqrt{2}}, \quad \beta_1 = e, \quad \alpha_2 = \pi, \quad \beta_2 = \frac{2-e^2}{e+6}.$$

In this case both of the pairs  $\vec{\alpha} = (\alpha_1, \alpha_2)$  and  $\vec{\beta} = (\beta_1, \beta_2)$  are algebraically dependent because

$$\alpha_1^2 - \frac{1}{2} = 0 \quad \text{and} \quad \beta_1\beta_2 + \beta_1^2 + 6\beta_2 - 2 = 0. \tag{5.1}$$

#### 5.1. THE EXISTENCE OF THE BIADDITIVE TERM

To check the existence of a non-zero biadditive term in the solution we follow the steps of Section 4, Cases I and II, respectively. By (5.1),

$$r(t) := \left( \frac{1-\beta_2}{\beta_2-\beta_1} \cdot \frac{t(\beta_1-1)+1}{\beta_1} \right)^2 - \frac{1}{2}$$

and

$$\frac{1}{\beta_1} + \frac{1}{\beta_2} + 6 \left( \frac{1}{\beta_1} \right)^2 - 2 \left( \frac{1}{\beta_1} \right)^2 \frac{1}{\beta_2} = 0$$

implies that

$$s(t) := \frac{t(\beta_1 - 1) + 1}{\beta_1} + \frac{t(\beta_2 - 1) + 1}{\beta_2} + 6 \left( \frac{t(\beta_1 - 1) + 1}{\beta_1} \right)^2 - 2 \left( \frac{t(\beta_1 - 1) + 1}{\beta_1} \right)^2 \frac{t(\beta_2 - 1) + 1}{\beta_2}.$$

The possible values of the parameter  $\mu$  belong to the common zero's of polynomials  $r(t)$  and  $s(t)$ . The Figure 1<sup>4)</sup> created by MAPLE shows that they have no common roots and, by Theorem 4.1, the solutions are of degree at most one.

```
> R:=plot(100*r(t),t=-3.5..2.5,color=black, style=point):
> S:=plot(s(t),t=-3.5..2.5,color=green, style=point):
> display({R,S});
```

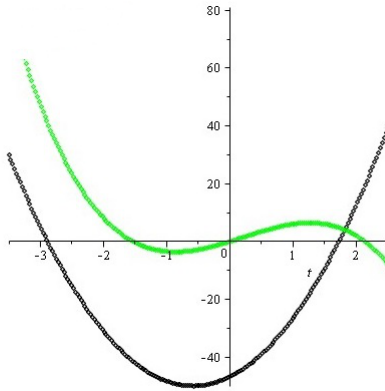


Fig. 1. The biadditive term

### 5.2. THE EXISTENCE OF THE ADDITIVE TERM

To check the existence of non-zero additive terms in the solution we have to find an injective field homomorphism  $\delta_1: \mathbb{Q}(\beta_1, \beta_2) \rightarrow \mathbb{C}$  satisfying  $(F_1)$ :

$$1 + \alpha_1 \delta_1(\beta_1) + \alpha_2 \delta_1(\beta_2) = 0, \quad \text{i.e.} \quad 1 + \frac{1}{\sqrt{2}} \delta_1(e) + \pi \delta_1 \left( \frac{2 - e^2}{e + 6} \right) = 0.$$

Taking  $x := \delta_1(e)$  we have the quadratic equation

$$0 = \left( \frac{1}{\sqrt{2}} - \pi \right) x^2 + (3\sqrt{2} + 1)x + (6 + 2\pi)$$

<sup>4)</sup> From technical reasons we use a proportional term for  $r(t)$  to illustrate the functions in a common coordinate system. The zero's are obviously unchanged.

and, consequently,

$$x_1 = -\sqrt{2} \quad \text{or} \quad x_2 = \frac{6 + 2\pi}{-1 + \sqrt{2}\pi}.$$

To provide the existence of  $\delta_1$  we have to show that  $x_2$  is transcendent. Since the algebraic numbers form a field it follows that the transcendency of  $x_2$  and  $\pi$  is equivalent. Therefore, we can give  $\delta_1$  as

$$\delta_1(e) := \frac{6 + 2\pi}{-1 + \sqrt{2}\pi}$$

satisfying  $(F_1)$  which means that there exist non-zero additive solutions.

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