AFFINE EXTENSIONS OF FUNCTIONS
WITH A CLOSED GRAPH

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Abstract. Let $A$ be a closed $G_δ$-subset of a normal space $X$. We prove that every function $f_0 : A \to \mathbb{R}$ with a closed graph can be extended to a function $f : X \to \mathbb{R}$ with a closed graph, too. This is a consequence of a more general result which gives an affine and constructive method of obtaining such extensions.

Keywords: real-valued functions with a closed graph, points of discontinuity, affine extensions of functions.

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1. INTRODUCTION

Let $\mathcal{C}(A)$ denote the set of all continuous functions on a nonempty subset $A$ of a Hausdorff space $X$. In this paper, every considered function is real. The set of all closed-graph functions on $X$ is denoted by $\mathcal{U}(X)$. Obviously $\mathcal{C}(X) \subset \mathcal{U}(X)$. This paper deals with the following general problem in the theory of real functions, which is inspired by the Tietze extension theorem:

(P) Let $A$ be a nonempty subset of a topological space $X$ and let $f_0 \in \mathbb{R}^A$ be a function with a certain property $(W)$. Can $f_0$ be extended to a function $f \in \mathbb{R}^X$ with the same property $(W)$?

It is well known that if $X$ is a metric space, and $A$ is a closed subset of $X$, the Tietze theorem can be significantly strengthened: In 1933 Borsuk [4] proved that there is a positive linear operator Ext from $\mathcal{C}(A)$ into $\mathcal{C}(X)$ such that $\text{Ext}(f_0)|_A = f_0$ for every $f_0 \in \mathcal{C}(A)$; furthermore, the restriction of Ext to the space $\mathcal{C}^b(A)$ of all bounded elements of $\mathcal{C}(A)$ is a positive isometry into $\mathcal{C}^b(X)$. Thus, the Borsuk’s operator Ext was the first example of a linear extension operator: its existence proved it is possible to extend two functions $f, g \in \mathcal{C}(A)$ in such a way that the extension of $f + g$ to an element of $\mathcal{C}(X)$ is the sum of extensions of $f$ and $g$, respectively (one should note...
that in 1951 Dugundji [7] generalized Borsuk’s theorem for continuous mappings into a locally convex linear space, instead of \( \mathbb{R} \), but in this paper we do not consider such kinds of extensions; we confine our studies only to real-valued functions.

The first results concerning the case of the Borsuk-Dugundji theorem for spaces of differentiable functions came from Merrien [11] and Bromberg [5], and for spaces of analytic mappings - from Aron and Berner [1]. In 2007, Fefferman [8] obtained a generalization of Merrien’s and Bromberg’s results. He proved that if \( C^m(E) \) denotes the space of restrictions to \( E \subset \mathbb{R}^n \) of \( m \)-differentiable functions \( f:\mathbb{R}^n \to \mathbb{R} \), then there is a linear and continuous operator \( T: C^m(E) \to C^m(\mathbb{R}^n) \) such that \( T(f|_E) = f \).

A natural question related to the above-mentioned results and problem (P) reads as follows: Does there exist a larger class of functions, including the class of continuous functions, where Tietze-type theorems hold true? This question has a few positive answers. A first result of this kind is due to Kuratowski [10]: in 1933 he obtained a Tietze-type result for functions of the first Baire class defined on \( G_\delta \)-subsets of a metric space, and not until 2005 Kalenda and Spurný [9] extended Kuratowski’s theorem for completely regular spaces. On the other hand, in 2010 we proved [12] that if \( X \) is a \( P \)-space (i.e., every \( G_\delta \)-subset of \( X \) is open) then \( \mathcal{C}(X) = \mathcal{U}(X) \), and thus (formally) for every closed subset \( A \) of \( X \), every \( f_0 \in \mathcal{U}(A) \) can be extended to \( f \in \mathcal{U}(X) \). This observation has led us to the conjecture that a Tietze-type theorem should hold for the class of closed graph functions defined on some subsets of a Hausdorff space \( X \). The conjecture is confirmed in our Theorem 3.2 below, where we show that there is a positively affine extension operator from \( \mathcal{U}(A) \) into \( \mathcal{U}(X) \), where \( A \) is a zero-subset of \( X \).

2. NOTATIONS AND DEFINITIONS

For every subset \( A \subset X \), let \( \text{cl}(A) \), \( \text{int}(A) \) and \( \text{bd}(A) \) denote the closure, interior and boundary of \( A \), respectively. The spaces \( \mathbb{R} \) and \( X \times \mathbb{R} \) are considered with their standard topologies. A function \( f: X \to \mathbb{R} \) is piecewise continuous if there are nonempty closed sets \( X_n \subset X \), \( n \in \mathbb{N} \) such that \( X = \bigcup_{n=0}^{\infty} X_n \) and the restriction \( f|_{X_n} \) is continuous for each \( n \in \mathbb{N} \). For every function \( f: X \to \mathbb{R} \), the symbol \( G(f) \) denotes the graph of \( f \), and the symbols \( C(f) \) and \( D(f) \) (\( = X \setminus C(f) \)) denote the sets of continuity and discontinuity points of \( f \), respectively. We say that \( f: X \to \mathbb{R} \) is a function with a closed graph, if \( G(f) \) is a closed subset of \( X \times \mathbb{R} \). The symbol \( \mathcal{U}^+(X) \) stands for the set of all non-negative elements of \( \mathcal{U}(X) \).

In 1985, Doboš [6] proved that the sum of two non-negative functions with a closed graph is a function with a closed graph. Since \( 0 \in \mathcal{U}^+(X) \), we have

\[
\mathcal{U}^+(X) + \mathcal{U}^+(X) = \mathcal{U}^+(X). \tag{2.1}
\]

Notice, however, that \( \mathcal{U}^+(X) - \mathcal{U}^+(X) \neq \mathcal{U}(X) \), i.e. there is an example of a space \( X \) and functions \( f, g \in \mathcal{U}^+(X) \) such that \( f - g \notin \mathcal{U}(X) \) (see [6, p. 9]).
**Definition 2.1.** Let $L_1, L_2$ be two cones in linear spaces $E_1, E_2$, respectively (i.e. $L_i + L_i \subseteq L_i$, $aL_i \subseteq L_i$, $i = 1, 2$, for every $a \in \mathbb{R}^+$, and $L_i \cap (-L_i) = \{0\}$). We say that a mapping $T : L_1 \to L_2$ is positively affine if, for any elements $x, y \in L_1$ and $a, b \in \mathbb{R}^+$ such that $a + b = 1$, we have $T(ax + by) = aT(x) + bT(y)$.

### 3. MAIN THEOREM

Let $X$ be a topological space, let $A$ be a nonempty zero-set (i.e. $A = \{g = 0\} := g^{-1}(0)$ for some $g \in \mathcal{C}(X)$), and let $f_0 : A \to \mathbb{R}$ be a function with a closed graph. The symbol $f_{(A, g)}$ denotes a real function defined on $X$ of the form

$$f_{(A, g)}(x) = \begin{cases} f_0(x), & x \in A, \\ \frac{1}{g(x)}, & x \notin A. \end{cases} \quad (3.1)$$

To simplify notations, for $A$ and $g$ fixed, we write $f$ instead of $f_{(A, g)}$. The symbol $\operatorname{Ext}_{(A, g)}$ denotes a mapping $\mathbb{R}^A \to \mathbb{R}^X$ defined by the formula

$$\operatorname{Ext}_{(A, g)}(f_0) = f.$$

**Remark 3.1.** From the above definitions it follows that if $A = g_1^{-1}(0) = g_2^{-1}(0)$ and $g_1 \neq g_2$, then $f_{(A, g_1)} \neq f_{(A, g_2)}$, and hence $\operatorname{Ext}_{(A, g_1)}(f) \neq \operatorname{Ext}_{(A, g_2)}(f)$ for every $f \in \mathbb{R}^A$.

The main result of this paper reads as follows.

**Theorem 3.2.** Let $X$ be a topological Hausdorff space, let $A$ be a nonempty zero-subset of $X$, and let $f_0 : A \to \mathbb{R}$ be a map with a closed graph. Then

(a) there is a function $f : X \to \mathbb{R}$ with a closed graph such that $f|_A = f_0$, and

(b) the set $D(f)$, of points of discontinuity of $f$, is of the form

$$D(f) = D(f_0) \cup \overline{bd} A. \quad (3.2)$$

More exactly, for every fixed function $g \in \mathcal{C}(X)$ such that $A = g^{-1}(0)$, the operator $\operatorname{Ext}_{(A, g)}$ defined above maps $\mathcal{U}(A)$ into $\mathcal{U}(X)$ and is positively affine.

One should note that from formula (2) it follows that the resulting function $f$ is unbounded and discontinuous, in general, unless the set $A$ is closed and open.

**Proof.** We shall prove first that the mapping $f = f_{(A, g)}$ defined by formula (3.1) has a closed graph. Let $(x_δ)$ be a Moore-Smith (MS) sequence such that $x_δ \to x$ and $f(x_δ) \to t$.

If $x \notin A$, the continuity of $g$ implies that $t = \frac{1}{g(x)} = f(x)$.

For $x \in A$, we consider the following two cases:

(i) $x \in \text{int } A \neq \emptyset$,

(ii) $x \in A \setminus \text{int } A$. 
In case (i), the nonempty set \( \text{int } A \) is open, thus there is \( \alpha_0 \) such that \( x_{\alpha} \in \text{int } A \) for every \( \alpha > \alpha_0 \). Therefore \( f(x_{\alpha}) = f_0(x_{\alpha}) \to t \) and \( t = f_0(x) = f(x) \) because \( f_0 \) has a closed graph.

In case (ii), we have \( f(x) = f_0(x) \) and \( g(x) = 0 \). We claim there is \( \beta \) such that, for every \( \alpha > \beta \), we have \( x_{\alpha} \in A \). Indeed, otherwise, for every index \( \beta \) there would be an index \( \alpha_\beta > \beta \) such that \( x_{\alpha_\beta} = y_\beta \in X \setminus A \). Then

\[
    f(y_\beta) = \frac{1}{g(y_\beta)} \to t \neq 0
\]

(the case \( t = 0 \) is impossible, because then we would have \( |g(y_\beta)| \to \infty \) with \( y_\beta \to x \), which contradicts the continuity of \( g \) at \( x \)). Hence

\[
    g(y_\beta) \to \frac{1}{t} \in (0, \infty). \tag{3.3}
\]

On the other hand, the continuity of \( g \) implies that \( g(y_\beta) \to g(x) = 0 \), which contradicts (3.3). Thus, there is an element \( \beta \) such that, for any index \( \alpha > \beta \), we have \( f(x_{\alpha}) = f_0(x_{\alpha}) \to t \). Now the closedness of the graph of \( f_0 \) implies that \( t = f_0(x) = f(x) \). We thus have showed that \( f \) has a closed graph, as claimed.

Now we shall prove equality (3.2); equivalently,

\[
    D(f) = (X \setminus C(f_0)) \cup \left( A \cap (X \setminus \text{int } A) \right). \tag{3.4}
\]

Let us fix \( x \in D(f) \). Suppose, by way of contradiction, that \( x \notin D(f_0) \cup \text{bd } A \). Then, by (3.4), we have \( x \in C(f_0) \cap [(X \setminus A) \cup \text{int } A] \), whence \( x \in C(f_0) \) and \( x \in (X \setminus A) \cup \text{int } A \). If \( x \in X \setminus A \), we have \( f(x) = \frac{1}{g(x)} \), whence \( x \in C(g) \subset C(f) \), and if \( x \in \text{int } A \neq \emptyset \), we have \( f(x) = f_0(x) \), and hence \( x \in C(f_{\text{int } A}) \subset C(f) \). In both the cases we thus have \( x \in C(f) \), contrary to our hypothesis. We thus have shown that

\[
    D(f) \subset D(f_0) \cup \text{bd } A. \tag{3.5}
\]

For the proof of the reversed inclusion to (3.5), let us fix \( x \in D(f_0) \cup \text{bd } A \). Assume first that \( x \in D(f_0) \). Since each point of the discontinuity of \( f_0 \) is a point of the discontinuity of \( f \), we obtain \( x \in D(f) \). Moreover, if \( x \in \text{bd } A = A \cap (X \setminus \text{int } A) \), there is an MS-sequence \( (x_\delta) \subset X \setminus A \) convergent to \( x \). By the continuity of \( g \), we obtain \( \frac{1}{f(x_\delta)} = g(x_\delta) \to 0 \). Therefore \( |f(x_{\alpha})| \to \infty \), whence \( x \in D(f) \). We thus have shown that if \( x \in D(f_0) \cup \text{bd } A \) then \( x \in D(f) \), i.e.,

\[
    D(f_0) \cup \text{bd } A \subset D(f). \tag{3.6}
\]

Combining inclusions (3.5) and (3.6), we obtain (3.2). Obviously, \( \text{Ext}_{(A, \varnothing)} \) is positively affine. The proof is complete.

The following corollary is an immediate consequence of Theorem 3.2.

**Corollary 3.3.** Let \( A \) be a closed and \( G_\delta \) (closed, respectively) subset of a normal (perfectly normal, respectively) space \( X \). Then there is a positively affine extension operator \( \text{Ext} : \mathcal{U}(A) \to \mathcal{U}(X) \).
Affine extensions of functions with a closed graph

Notice that the Tietze theorem asserts that if \( A \) is a closed subset of a normal space \( X \), then the restriction from \( C(X) \) to \( C(A) \) is surjective. From Theorem 3.2 we obtain a similar result.

**Corollary 3.4.** Let \( X \) be a topological Hausdorff space, and let \( A \) be a zero-set. Then the restriction operator \( r_A : \mathcal{U}(X) \to \mathcal{U}(A) \) (given by \( r_A(f) = f\mid_A \)) is a surjection.

In two examples below we show that the requirement in Corollary 3.3, “\( A \) to be a closed subset of \( X \)” cannot be replaced by the weaker condition: “\( A \) to be an \( F_\sigma \)-set.” We do not know, however, if the hypothesis of Theorem 1 about \( A \) is essential, i.e., we cannot indicate a closed and non-zero-subset \( A \) of a Hausdorff space \( X \) such that some \( f_0 \in \mathcal{U}(A) \) cannot be extended to an element of \( \mathcal{U}(X) \).

In Example 3.5 we address an “extremely bad” case: there is a nonempty \( F_\sigma \)-subset \( A \) of a metric space \( X \) and \( f \in \mathcal{U}(A) \) such that, for every subset \( B \) of \( A \) such that \( \text{int}(\text{cl}(B)) \neq \emptyset \), the restriction \( f\mid_B \) cannot be extended to an element of \( \mathcal{U}(\text{cl}(B)) \).

**Example 3.5.** Let \( X = [0, 1] \) be the unit interval with the standard topology. Set \( A = (0, 1) \cap \mathbb{Q} \subset X \), and let \( B \) be any fixed subset of \( A \) such that \( \text{int}(\text{cl}(B)) \neq \emptyset \). Let \( f : A \to \mathbb{R} \) be a function defined as \( f\left(\frac{m}{n}\right) = n \) with \( m, n \) positive integers and \( \frac{m}{n} \) irreducible. Then \( f \) is a function with a closed graph which is discontinuous at every point of \( A \) (due to the fact, that the number of irreducible fractions in \( A \) with a given denominator is finite). Since \( \text{int}(\text{cl}(B)) \neq \emptyset \), there are real numbers \( 0 < a < b < 1 \) such that \( [a, b] \subset \text{cl}(B) \). Suppose that \( f_B := f\mid_B \) can be extended to \( \overline{f}_B \in \mathcal{U}(\text{cl}(B)) \). Then (see [3, Lemma 2.2]) \( \overline{f}_B \) is piecewise continuous, and thus there is a sequence \( (B_n) \) of closed subsets of \( [a, b] \) such that \( [a, b] = \bigcup_{n=1}^\infty B_n \) and the restriction \( \overline{f}_B\mid_{B_n} \) is continuous for each \( n \in \mathbb{N} \). Then, by the Baire property, there is a number \( n_0 \in \mathbb{N} \) such that \( \text{int}(B_{n_0}) \neq \emptyset \). Hence there is a nonempty interval \( (c, d) \) contained in \( B_{n_0} \). Thus, by the continuity of the restrictions \( \overline{f}_B\mid_{B_n} \), every rational number \( \xi \in (c, d) \) would be the point of continuity of \( \overline{f}_B \), and thus the point of continuity of \( f_B = f\mid_B \), but this contradicts the discontinuity of \( f \).

In the next example we show that the hypothesis in Corollary 3.3: “\( A \) is closed” cannot be replaced by “\( A \) is open \( F_\sigma \).” But now, in contrast to Example 3.5, there are subsets \( B \subset A \) such that \( \text{int}(B) \neq \emptyset \) and \( f\mid_B \) has an extension to an element of \( \mathcal{U}(\text{cl}(B)) \).

**Example 3.6.** Let \( X = \mathbb{R} \) and \( A = (0, \infty) \). Thus \( A \) is an open and \( F_\sigma \) subset of \( X \). Let \( f_0 : (0, \infty) \to \mathbb{R} \) be a map given by the formula \( f_0(x) = \sin \frac{1}{x} \). The function \( f_0 \) is of course continuous at every point \( x \in A \), whence \( f_0 \in \mathcal{U}(A) \). However, the function \( f_0 \) cannot be extended to any function \( f : [0, \infty) \to \mathbb{R} \) with a closed graph because \( \text{cl}G(f_0) \supset \{0\} \times [-1, 1] \).

REFERENCES


