

## POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS WITH NONLINEAR NONLOCAL BOUNDARY CONDITIONS

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**Abstract.** We consider the existence of at least three positive solutions of a nonlinear first order problem with a nonlinear nonlocal boundary condition given by

$$x'(t) = r(t)x(t) + \sum_{i=1}^m f_i(t, x(t)), \quad t \in [0, 1],$$
$$\lambda x(0) = x(1) + \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)), \quad \tau_j \in [0, 1],$$

where  $r : [0, 1] \rightarrow [0, \infty)$  is continuous; the nonlocal points satisfy  $0 \leq \tau_1 < \tau_2 < \dots < \tau_n \leq 1$ , the nonlinear function  $f_i$  and  $\tau_j$  are continuous mappings from  $[0, 1] \times [0, \infty) \rightarrow [0, \infty)$  for  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$  respectively, and  $\lambda > 0$  is a positive parameter.

**Keywords:** positive solutions, Leggett-Williams fixed point theorem, nonlinear boundary conditions.

**Mathematics Subject Classification:** 34B08, 34B18, 34B15, 34B10.

### 1. INTRODUCTION

Consider the first order boundary value problem with a nonlinear nonlocal boundary condition

$$x'(t) = r(t)x(t) + \sum_{i=1}^m f_i(t, x(t)), \quad t \in [0, 1], \tag{1.1}$$

$$\lambda x(0) = x(1) + \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)), \quad \tau_j \in [0, 1], \tag{1.2}$$

where  $r : [0, 1] \rightarrow [0, \infty)$  is continuous,  $f_i : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ ,  $\tau_j : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$  are continuous,  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$ , the nonlocal points satisfy  $0 \leq \tau_1 < \tau_2 < \dots < \tau_n \leq 1$ , and the scalar  $\lambda$  satisfies

$$\lambda > \exp\left(\int_0^1 r(\eta) d\eta\right). \quad (1.3)$$

The motivation of this present paper has come from a recent paper due to Anderson [1], who used the Leggett-Williams multiple fixed point theorem [8] to establish three positive solutions of the boundary value problem (1.1) and (1.2). For completeness, we state here the statement of the theorem.

Assume that the nonlinear functions  $\Lambda_j$  satisfy

$$0 \leq x\psi_j(t, x) \leq \Lambda_j(t, x) \leq x\Psi_j(t, x), \quad t \in [0, 1], \quad x \in [0, \infty) \quad (1.4)$$

for some positive continuous functions  $\psi_j, \Psi_j : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ . Set

$$\beta_j = \max_{[0,1] \times [0,c]} \Psi_j(t, x) \quad \text{and} \quad \alpha_j = \min_{[0,1] \times [0,d]} \psi_j(t, x) \quad (1.5)$$

for some real constants  $c$  and  $d$ . Then, using the Leggett-Williams multiple fixed point theorem, Anderson proved the following theorem.

**Theorem 1.1.** *Suppose that (1.4) holds and the scalar  $\lambda$  satisfies*

$$\lambda > \left(1 + \sum_{j=1}^m \beta_j\right) \exp\left(\int_0^1 r(\eta) d\eta\right) > 1. \quad (1.6)$$

Further, suppose that there exist constants  $0 < c_1 < c_2 < \lambda c_2 \leq c_4$  such that

$$(F_1) \quad f_i(t, x) \leq \frac{M c_4}{m} \quad \text{for } t \in [0, 1] \text{ and } x \in [0, c_4];$$

$$(F_2) \quad f_i(t, x) > \frac{N c_2}{m} \quad \text{for } t \in [0, 1] \text{ and } x \in [c_2, \lambda c_2];$$

$$(F_3) \quad f_i(t, x) < \frac{M c_1}{m} \quad \text{for } t \in [0, 1] \text{ and } x \in [0, c_1]$$

for  $i = 1, 2, \dots, m$ , where

$$M = \frac{1}{\int_0^1 G(1, s) ds} \left[ 1 - \frac{\exp\left(\int_0^1 r(\eta) d\eta\right) \sum_{j=1}^n \beta_j}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} \right] \quad (1.7)$$

and

$$N = \frac{1}{\int_0^1 G(0, s) ds} \left[ 1 - \frac{\sum_{j=1}^n \alpha_j}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} \right]. \quad (1.8)$$

Then the boundary value problem (1.1) and (1.2) has at least three positive solutions  $x_1, x_2$  and  $x_3$  satisfying  $\|x_1\| = x_1(1) < c_1$ ,  $c_2 < \psi(x_2) = x_2(0)$ ,  $\|x_3\| = x_3(1) > c_1$  with  $\psi(x_3) = x_3(0) < c_2$ .

We consider a Banach space  $X = C[0, 1]$  endowed with the sup norm. Define a cone  $K$  on  $X$  by

$$K = \{x \in X; x \geq 0 \text{ and } x \text{ increasing}\}, \tag{1.9}$$

and a nonnegative concave continuous functional  $\psi$  on  $K$  by

$$\psi(x) = \min_{t \in [0,1]} x(t) = x(0), \quad x \in K. \tag{1.10}$$

Now, we state the main result of this paper.

**Theorem 1.2.** *Assume that (1.3) holds. Suppose that there exist three constants  $0 < c_1 < c_2 < \lambda c_2 \leq c_4$  such that*

$$(H_1) \quad \lambda \sum_{i=1}^m \int_0^1 \exp\left(-\int_0^s r(\eta) d\eta\right) f_i(s, x(s)) ds + \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)) \leq \frac{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right) c_4}{\exp\left(\int_0^1 r(\eta) d\eta\right)}$$

for  $\tau_j \in [0, 1], j = 1, 2, \dots, n$  and  $x \in [0, c_4]$ ;

$$(H_2) \quad \sum_{i=1}^m \int_0^1 \exp\left(\int_s^1 r(\eta) d\eta\right) f_i(s, x(s)) ds + \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)) > c_2 \left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)$$

for  $\tau_j \in [0, 1], j = 1, 2, \dots, n$  and  $x \in [c_2, \lambda c_2]$ ;

and

$$(H_3) \quad \lambda \sum_{i=1}^m \int_0^1 \exp\left(-\int_0^s r(\eta) d\eta\right) f_i(s, x(s)) ds + \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)) < \frac{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right) c_1}{\exp\left(\int_0^1 r(\eta) d\eta\right)}$$

for  $\tau_j \in [0, 1], j = 1, 2, \dots, n$  and  $x \in [0, c_4]$  hold. Then the boundary value problem (1.1) and (1.2) has at least three positive solutions  $x_1, x_2$  and  $x_3$  satisfying  $\|x_1\| = x_1(1) < c_1, c_2 < \psi(x_2) = x_2(0)$  and  $\|x_3\| = x_3(1) > c_1$  with  $\psi(x_3) = x_3(0) < c_4$ .

We, now show that the conditions of Theorem 1.1 imply the conditions of Theorem 1.2. In fact, (1.6) implies (1.3). First, suppose that (1.4) and  $(F_1)$  hold, that is,  $f_i(t, x) \leq \frac{Mc_4}{m}$  for  $t \in [0, 1]$  and  $x \in [0, c_4]$ . Then,

$$\begin{aligned} & \lambda \sum_{i=1}^m \int_0^1 \exp\left(-\int_0^s r(\eta) d\eta\right) f_i(s, x(s)) ds + \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)) \\ & \leq \lambda \frac{Mc_4}{m} \sum_{i=1}^m \int_0^1 \exp\left(-\int_0^s r(\eta) d\eta\right) ds + c_4 \sum_{j=1}^n \beta_j \\ & \leq \lambda M c_4 \int_0^1 \exp\left(-\int_0^s r(\eta) d\eta\right) ds + c_4 \sum_{j=1}^n \beta_j \\ & = c_4 \left[ \lambda M \int_0^1 \exp\left(-\int_0^s r(\eta) d\eta\right) ds + \sum_{j=1}^n \beta_j \right]. \end{aligned}$$

Using the value of  $M$ , which is given in Theorem 1.1, the above inequality gives

$$\begin{aligned}
& \lambda \sum_{i=1}^m \int_0^1 \exp \left( - \int_0^s r(\eta) d\eta \right) f_i(s, x(s)) ds + \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)) \\
&= c_4 \left[ \lambda \int_0^1 \exp \left( - \int_0^s r(\eta) d\eta \right) ds \frac{1}{\int_0^1 G(1, s) ds} \left( 1 - \frac{\exp \left( \int_0^1 r(\eta) d\eta \right) \sum_{j=1}^n \beta_j}{\lambda - \exp \left( \int_0^1 r(\eta) d\eta \right)} \right) + \sum_{j=1}^n \beta_j \right] \\
&= c_4 \left[ \lambda \int_0^1 \exp \left( - \int_0^s r(\eta) d\eta \right) ds \frac{\left( \lambda - \exp \left( \int_0^1 r(\eta) d\eta \right) \right)}{\lambda \int_0^1 \exp \left( \int_s^1 r(\eta) d\eta \right) ds} \right. \\
&\quad \left. \times \left( 1 - \frac{\exp \left( \int_0^1 r(\eta) d\eta \right) \sum_{j=1}^n \beta_j}{\lambda - \exp \left( \int_0^1 r(\eta) d\eta \right)} \right) + \sum_{j=1}^n \beta_j \right] \\
&= c_4 \left[ \frac{\int_0^1 \exp \left( - \int_0^s r(\eta) d\eta \right) ds}{\int_0^1 \exp \left( \int_s^1 r(\eta) d\eta \right) ds} \left( \lambda - \exp \left( \int_0^1 r(\eta) d\eta \right) - \exp \left( \int_0^1 r(\eta) d\eta \right) \sum_{j=1}^n \beta_j \right) + \sum_{j=1}^n \beta_j \right] \\
&= c_4 \left[ \frac{\int_0^1 \exp \left( - \int_0^s r(\eta) d\eta \right) ds \exp \left( \int_0^1 r(\eta) d\eta \right)}{\int_0^1 \exp \left( \int_s^1 r(\eta) d\eta \right) ds} \left( \frac{\lambda - \exp \left( \int_0^1 r(\eta) d\eta \right)}{\exp \left( \int_0^1 r(\eta) d\eta \right)} - \sum_{j=1}^n \beta_j \right) + \sum_{j=1}^n \beta_j \right] \\
&= \frac{\lambda - \exp \left( \int_0^1 r(\eta) d\eta \right)}{\exp \left( \int_0^1 r(\eta) d\eta \right)} c_4,
\end{aligned}$$

which implies that  $(H_1)$  holds. In a similar way, we can show that if (1.4) and  $(F_3)$  hold, then  $(H_3)$  holds.

Finally, suppose that  $(F_2)$  holds. In addition, suppose that (1.4) holds. Then for  $c_2 \leq x \leq \lambda c_2$ ,  $t \in [0, 1]$ , using the value of  $N$ , given in Theorem 1.1, we obtain

$$\begin{aligned}
 & \sum_{i=1}^m \int_0^1 \exp \left( \int_s^1 r(\eta) d\eta \right) f_i(s, x(s)) ds + \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)) \\
 & > \frac{Nc_2}{m} \sum_{i=1}^m \int_0^1 \exp \left( \int_s^1 r(\eta) d\eta \right) ds + c_2 \sum_{j=1}^n \alpha_j \\
 & = c_2 \left[ N \int_0^1 \exp \left( \int_s^1 r(\eta) d\eta \right) ds + \sum_{j=1}^n \alpha_j \right] \\
 & = c_2 \left[ \int_0^1 \exp \left( \int_s^1 r(\eta) d\eta \right) ds \frac{1}{\int_0^1 G(0, s) ds} \left( 1 - \frac{\sum_{j=1}^n \alpha_j}{\alpha - \exp \left( \int_0^1 r(\eta) d\eta \right)} \right) + \sum_{j=1}^n \alpha_j \right] \\
 & = c_2 \left[ \frac{\int_0^1 \exp \left( \int_s^1 r(\eta) d\eta \right) ds \left( \lambda - \exp \left( \int_0^1 r(\eta) d\eta \right) \right)}{\int_0^1 \exp \left( \int_s^0 r(\eta) d\eta \right) \exp \left( \int_0^1 r(\eta) d\eta \right) ds} \left( 1 - \frac{\sum_{j=1}^n \alpha_j}{\lambda - \exp \left( \int_0^1 r(\eta) d\eta \right)} \right) \right. \\
 & \quad \left. + \sum_{j=1}^n \alpha_j \right] \\
 & = c_2 \left[ \lambda - \exp \left( \int_0^1 r(\eta) d\eta \right) - \sum_{j=1}^n \alpha_j + \sum_{j=1}^n \alpha_j \right] \\
 & = c_2 \left( \lambda - \exp \left( \int_0^1 r(\eta) d\eta \right) \right).
 \end{aligned}$$

This shows that condition (1.4) and  $(F_2)$  implies condition  $(H_2)$ . Thus, Theorem 1.2 provides a better condition than the condition given in Theorem 1.1.

Leggett-Williams multiple fixed point theorem has played an important role in establishing multiple positive periodic solutions of functional differential equations. For example, one may refer to [2, 4–7, 9–15] for first order ordinary and delay differential equations and refer to [3] for higher order equations. Once the problem is transformed into an equivalent integral operator, then it becomes easy to study the existence of fixed points of the operator using the Leggett-Williams multiple fixed point theorem, which is equivalent to establish the existence of positive solutions of the concerned problem. In a recent work, Padhi *et al.* [11], provide some simple way of applying the Leggett-Williams multiple fixed point theorem to a system of first order functional differential equations. In this paper, we have applied the same technique, and obtained Theorem 1.2. The detailed proof of Theorem 1.2 is given in Section 3.

## 2. PRELIMINARIES

Let  $X$  be a Banach space,  $K$  be a cone in  $X$ , and  $\psi$  be a nonnegative continuous functional on  $K$ . Further, let  $a, b, c > 0$  be constants. Define

$$K_a = \{x \in K; \|x\| < a\}$$

and

$$K(\psi, b, c) = \{x \in K; \psi(x) \geq b, \|x\| \leq c\}.$$

**Theorem 2.1** (Leggett-Williams multiple fixed point theorem, [8]). *Let  $X = (X, \|\cdot\|)$  be a Banach space and  $K \subset X$  be a cone, and  $c_4 > 0$  be a constant. Suppose that there exists a concave nonnegative continuous function  $\psi$  on  $K$  with  $\psi(x) \leq \|x\|$  for  $x \in \overline{K}_{c_4}$  and let  $A : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$  be a continuous compact map. Assume that there are numbers  $c_1, c_2$  and  $c_3$  with  $0 < c_1 < c_2 < c_3 \leq c_4$  such that*

- (i)  $\{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \emptyset$  and  $\psi(Ax) > c_2$  for all  $x \in K(\psi, c_2, c_3)$ ;
- (ii)  $\|Ax\| < c_1$  for all  $x \in K_{c_1}$ ;
- (iii)  $\psi(Ax) > c_2$  for all  $x \in K(\psi, c_2, c_4)$  with  $\|Ax\| > c_3$ .

*Then  $A$  has at least three fixed points  $x_1, x_2$  and  $x_3$  in  $\overline{K}_{c_4}$ . Furthermore, we have  $x_1 \in \overline{K}_{c_1}$ ,  $x_2 \in \{x \in K(\psi, c_2, c_4) : \psi(x) > c_2\}$ , and  $x_3 \in \overline{K}_{c_4} \setminus \{K(\psi, c_2, c_4) \cup \overline{K}_{c_1}\}$ .*

## 3. PROOF OF THEOREM 1.2

Consider an operator  $A : K \rightarrow X$  by

$$Ax(t) = \sum_{i=1}^m \int_0^1 G(t, s) f_i(s, x(s)) ds + \frac{\exp\left(\int_0^t r(\eta) d\eta\right) \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)}, \quad (3.1)$$

where  $G(t, s)$  is the Green's Kernel, given by

$$G(t, s) = \frac{\exp\left(\int_s^t r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} \times \begin{cases} \lambda & \text{if } 0 \leq s \leq t \leq 1, \\ \exp\left(\int_0^1 r(\eta) d\eta\right) & \text{if } 0 \leq t \leq s \leq 1. \end{cases} \quad (3.2)$$

The operator  $A$  in (3.1) can be rewritten as

$$\begin{aligned}
 Ax(t) = & \sum_{i=1}^m \int_0^t \frac{\lambda \exp\left(\int_s^t r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_s^t r(\eta) d\eta\right)\right)} f_i(s, x(s)) ds \\
 & + \sum_{i=1}^m \int_t^1 \frac{\exp\left(\int_0^1 r(\eta) d\eta\right) \exp\left(\int_s^t r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} f_i(s, x(s)) ds \quad (3.3) \\
 & + \frac{\exp\left(\int_0^t r(\eta) d\eta\right) \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)}.
 \end{aligned}$$

Let  $x \in K$ . Then the fact that  $f_i, i = 1, 2, \dots, m$  and  $\Lambda_j, j = 1, 2, \dots, n$  are positive imply that  $Ax \geq 0$  for all  $t \in [0, 1]$ . We claim that fixed points of the operator  $A$  are the solutions of the boundary value problem (1.1) and (1.2). In fact, if  $x = Ax$ , then from (3.3), we have

$$\begin{aligned}
 \lambda x(0) - x(1) = & \sum_{i=1}^m \int_0^1 \frac{\lambda \exp\left(\int_s^1 r(s) ds\right)}{\left(\lambda - \exp\left(\int_0^1 r(s) ds\right)\right)} f_i(s, x(s)) ds \\
 & + \lambda \frac{\sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} \\
 & - \sum_{i=1}^m \int_0^1 \frac{\lambda \exp\left(\int_s^1 r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} f_i(s, x(s)) ds \\
 & - \frac{\exp\left(\int_0^1 r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)), \quad \tau_j \in [0, 1], j = 1, 2, \dots, n.
 \end{aligned}$$

Hence, the boundary condition (1.2) is satisfied. Again differentiating (1.9) with respect to  $t$ , with  $Ax = x$ , we obtain

$$\begin{aligned}
 x'(t) &= \sum_{i=1}^m \int_0^t \frac{\lambda r(t) \exp\left(\int_s^t r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} f_i(s, x(s)) ds \\
 &+ \sum_{i=1}^m \frac{\lambda}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} f_i(t, x(t)) \\
 &+ \sum_{i=1}^m \int_t^1 \frac{r(t) \exp\left(\int_0^1 r(\eta) d\eta\right) \exp\left(\int_s^t r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} f_i(s, x(s)) ds \\
 &- \sum_{i=1}^m \frac{\exp\left(\int_0^1 r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} f_i(t, x(t)) \\
 &+ \frac{\exp\left(\int_0^t r(\eta) d\eta\right) r(t)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)) \\
 &= r(t) \left[ \sum_{i=1}^m \int_0^t \lambda \frac{\exp\left(\int_s^t r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} f_i(s, x(s)) ds \right] \\
 &+ r(t) \left[ \sum_{i=1}^m \int_t^1 \frac{\exp\left(\int_0^1 r(\eta) d\eta\right) \exp\left(\int_s^t r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} f_i(s, x(s)) ds \right] \\
 &+ r(t) \left[ \frac{\exp\left(\int_0^t r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)) \right] \\
 &+ \sum_{i=1}^m f_i(t, x(t)) \\
 &= r(t)x(t) + \sum_{i=1}^m f_i(t, x(t)),
 \end{aligned}$$

which shows that  $x(t)$  satisfies (1.1). Moreover,

$$x'(t) = (Ax)'(t) = r(t)x(t) + \sum_{i=1}^m f_i(t, x(t)), \quad t \in [0, 1]$$

implies that  $x$  is increasing and  $A : K \rightarrow K$ . Further, one may verify that  $A$  is completely continuous. We, now show that all conditions of Theorem 2.1 are satisfied. In order to use Theorem 2.1, we use  $\lambda c_2$  in place of  $c_3$ . For  $x \in K$ , we have  $x(0) = \psi(x) \leq \|x\| = x(1)$ , since  $x \in K$ . Let  $x \in \overline{K}_{c_4}$ , that is,  $\|x\| \leq c_4$ . Then we obtain, using  $(H_1)$ ,

$$\begin{aligned} \|Ax\| &= Ax(1) \\ &= \sum_{i=1}^m \int_0^1 G(1, s) f_i(s, x(s)) ds + \frac{\exp\left(\int_0^1 r(\eta) d\eta\right) \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} \\ &= \frac{\exp\left(\int_0^1 r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} \left[ \lambda \sum_{i=1}^m \int_0^1 \exp\left(-\int_0^s r(\eta) d\eta\right) f_i(s, x(s)) ds + \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)) \right] \\ &\leq c_4, \end{aligned}$$

that is,  $A : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$ . In a similar way, using  $(H_3)$ , one can prove condition (ii) of Theorem 2.1, that  $A : \overline{K}_{c_1} \rightarrow K_{c_1}$ .

In order to verify condition (i) of Theorem 2.1, we choose  $x_k(t) = \lambda c_2$  for  $t \in [0, 1]$ . Since  $\psi(x_k(t)) = \min_{t \in [0, 1]} x_k(t) = \lambda c_2 > c_2$ ;  $c_2 \leq \psi(x)$ ,  $\|x\| = \lambda c_2$ , then the set  $\{x \in K; c_2 \leq \psi(x), \|x\| \leq \lambda c_2\} \neq \emptyset$ . Let  $x \in K(\psi, c_2, c_3)$ . Then  $c_2 \leq \psi(x) \leq x \leq \|x\| = x(1) = \lambda c_2$  for  $t \in [0, 1]$ , that is,  $c_2 \leq x(t) \leq \lambda c_2$  for  $t \in [0, 1]$  holds. Hence

$$\begin{aligned} \psi(Ax) &= Ax(0) \\ &= \sum_{i=1}^m \int_0^1 G(0, s) f_i(s, x(s)) ds + \frac{\sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} \\ &= \sum_{i=1}^m \int_0^1 \frac{\exp\left(\int_s^0 r(\eta) d\eta\right) \exp\left(\int_0^1 r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} f_i(s, x(s)) ds + \sum_{j=1}^n \frac{\Lambda_j(\tau_j, x(\tau_j))}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} \\ &= \frac{1}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} \left[ \sum_{i=1}^m \int_0^1 \exp\left(\int_s^1 r(\eta) d\eta\right) f_i(s, x(s)) ds + \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)) \right] \end{aligned}$$

holds. Then, using  $(H_2)$ , the above inequality yields that

$$\psi(Ax) > c_2 \text{ for } x \in \psi(K, c_2, c_3), t \in [0, 1],$$

that is, condition (i) of Theorem 2.1 is satisfied.

Finally, suppose that  $x \in K(\psi, c_2, c_4)$  with  $\|Ax\| > \lambda c_2$ . Then

$$\psi(Ax) = Ax(0) \geq \frac{1}{\lambda} Ax(1) = \frac{\|Ax\|}{\lambda} = \frac{\lambda c_2}{\lambda} = c_2$$

implies that condition (iii) of Theorem 2.1 is satisfied. Consequently, the boundary value problem (1.1) and (1.2) has at least three positive solutions. This completes the proof of the theorem.

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### REFERENCES

- [1] D.R. Anderson, *Existence of three solutions for a first-order problem with nonlinear nonlocal boundary conditions*, J. Math. Anal. Appl. **408** (2013), 318–323.
- [2] D. Bai, Y. Xu, *Periodic solutions of first order functional differential equations with periodic deviations*, Comp. Math. Appl. **53** (2007), 1361–1366.
- [3] J.G. Dix, S. Padhi, *Existence of multiple positive periodic solutions for delay differential equation whose order is a multiple of 4*, Appl. Math. Comput. **216** (2010), 2709–2717.
- [4] J.R. Graef, S. Padhi, S. Pati, *Periodic solutions of some models with strong Allee effects*, Nonlinear Anal. Real World Appl. **13** (2012), 569–581.
- [5] J.R. Graef, S. Padhi, S. Pati, *Existence and nonexistence of multiple positive periodic solutions of first order differential equations with unbounded Green's kernel*, Panamer. Math. J. **23** (2013) 1, 45–55.
- [6] J.R. Graef, S. Padhi, S. Pati, *Multiple positive periodic solutions of first order ordinary differential equations with unbounded Green's Kernel*, Commun. Appl. Anal. **17** (2013), 319–330.
- [7] J.R. Graef, S. Padhi, S. Pati, P.K. Kar, *Positive solutions of differential equations with unbounded Green's Kernel*, Appl. Anal. Discrete Math. **6** (2012), 159–173.
- [8] R.W. Leggett, L.R. Williams, *Multiple positive fixed points of nonlinear operators on ordered Banach Spaces*, Indiana Univ. Math. J. **28** (1979), 673–688.
- [9] S. Padhi, C. Qian, S. Srivastava, *Multiple periodic solutions for a first order nonlinear functional differential equation with applications to population dynamics*, Commun. Appl. Anal. **12** (2008) 3, 341–352.

- [10] S. Padhi, S. Srivastava, *Existence of three periodic solutions for a nonlinear first order functional differential equation*, J. Franklin Inst. **346** (2009), 818–829.
- [11] S. Padhi, S. Srivastava, J.G. Dix, *Existence of three nonnegative periodic solutions for functional differential equations and applications to hematopoiesis*, Panamer. Math. J. **19** (2009) 1, 27–36.
- [12] S. Padhi, P.D.N. Srinivasu, G.K. Kumar, *Periodic solutions for an equation governing dynamics of a renewable resource subjected to Allee effects*, Nonlinear Anal. Real World Appl. **11** (2010), 2610–2618.
- [13] S. Padhi, S. Srivastava, S. Pati, *Three periodic solutions for a nonlinear first order functional differential equation*, Appl. Math. Comput. **216** (2010), 2450–2456.
- [14] S. Padhi, S. Srivastava, S. Pati, *Positive periodic solutions for first order functional differential equations*, Commun. Appl. Anal. **14** (2010), 447–462.
- [15] S. Pati, J.R. Graef, S. Padhi, P.K. Kar, *Periodic solutions of a single species renewable resources under periodic habitat fluctuations with harvesting and Allee effect*, Comm. Appl. Nonl. Anal. **20** (2013), 1–16.

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