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FREE PROBABILITY ON HECKE ALGEBRAS AND CERTAIN GROUP C^*-ALGEBRAS INDUCED BY HECKE ALGEBRAS

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Abstract. In this paper, by establishing free-probabilistic models on the Hecke algebras $\mathcal{H}(GL_2(\mathbb{Q}_p))$ induced by p-adic number fields $\mathbb{Q}_p$, we construct free probability spaces for all primes p. Hilbert-space representations are induced by such free-probabilistic structures. We study C^*-algebras induced by certain partial isometries realized under the representations.

Keywords: free probability, free moments, free cumulants, Hecke algebras, normal Hecke subalgebras, representations, groups, group C^*-algebras.

Mathematics Subject Classification: 05E15, 11R47, 46L54, 47L15, 47L55.

1. INTRODUCTION

In this paper we study free-probabilistic models for *Hecke algebras* and study *representations* under the models, and investigate *groups* generated by certain *operators* under the representations. In [7], the author and Gillespie considered certain embedded free-probabilistic subalgebras of Hecke algebras induced by *p-adic number fields* for *primes p*. And, in [2], the author extended the free-probabilistic representations of [7] to those fully on the given Hecke algebras, and investigated elements of Hecke algebras as operators realized under the representations. Especially, the spectral theory of such *Hilbert-space operators* was considered in [2]. As a continuation, here, we keep studying free probability on the Hecke algebras in the extended sense of [2], and concentrate on studying certain *group* C^*-(*sub*-)*algebras* determined by the representations (under quotient).

1.1. BACKGROUND

We have considered how *primes* (or *prime numbers*) act on operator algebras. The relations between primes and operator algebra theory have been studied from various different approaches. For instance, in [1], we studied how primes act "on" certain von Neumann algebras generated by p -adic and Adelic measure spaces. Also, the primes as operators in certain von Neumann algebras, have been studied in [3] and [5].

Independently in [6] and [4] we have studied primes as linear functionals acting on *arithmetic functions*, i.e., each prime p induces a free-probabilistic structure $(\mathcal{A}, g_p)$ on the algebra $\mathcal{A}$ of all arithmetic functions. In such a case, one can understand arithmetic functions as *Krein-space operators* (for fixed primes) via certain representations (see [8]).

These studies are motivated by number-theoretic results (e.g., [9, 10] and [14]) under free probability techniques (e.g., [11, 12] and [13]).

1.2. MOTIVATION

In modern number theory and its applications, *p-adic analysis* provides important tools not only for studying mathematical *analysis*, *analytic number theory* and *non-Archimedian analysis* (e.g., [1, 3, 7, 9] and [10]), but also for studying geometry at small distances in *mathematical quantum physics* (e.g., [14]). So, it is interested in both various mathematical fields and related scientific fields.

In [2] we studied free probability on Hecke algebras (see Sections 3 and 4 below). From the free-probabilistic models on Hecke algebras, we established certain representations of Hecke algebras, and considered corresponding C^*-algebras of Hecke algebras obtained from the representations, i.e., we understand every Hecke-algebra element as a Hilbert-space operator. Especially in [2], spectral properties (self-adjointness, normality, isometry-property, unitarity, etc.) of such operators were characterized.

In this paper we are typically interested in *projections* and *partial isometries* induced by generating elements of $\mathcal{H}(G_p)$. By understanding them pure operator-theoretically we construct *group C^*-algebras* generated by certain "nice" partial isometries having their common initial-and-final projections. The operator-algebraic properties of such C^*-algebras will be studied as embedded C^*-subalgebras of the C^*-algebra induced by Hecke algebras.

Our study will provide bridges among number theory, operator algebra, operator theory and free probability.

1.3. OVERVIEW

In Section 2 we introduce definitions and fundamental properties for our work. In Sections 3 and 4 we briefly review our free probability models on Hecke algebras. Some free-moment and free-cumulant computations are provided for our main results. In Section 5 we establish Hilbert-space representations of Hecke algebras and construct corresponding C^*-algebras, as operator-algebraic structures containing full free-probabilistic information of Hecke algebras.

In Section 6 we study partial isometries and projections induced by generating elements of Hecke algebras under our representations in detail. Projections and partial isometries in our Hecke C^*-algebras have been considered in [2], but we here provide much more detailed properties and characterizations of them (Theorem 6.1 and Theorem 6.2) independently. Moreover, we fix finitely many partial isometries, having identical initial-and-final projections, and then construct groups generated by such partial isometries, as multiplicative subgroups of Hecke C^*-algebras. We study isomorphism theorems of such groups (see Theorem 6.3). Naturally, corresponding group C^*-algebras will be constructed as embedded C^*-subalgebras of the Hecke C^*-algebras. We consider structure theorems of such group C^*-algebras in Theorem 6.4 and Corollary 6.5.

In Section 7 free probability on these group C^*-algebras will be studied. We study free-distributional data of operators in the algebras by computing free-moments (Theorem 7.1 and Corollary 7.2), and consider freeness conditions (Theorem 7.6) on the group C^*-algebras by observing free-cumulants (Theorem 7.4) of generating operators.

2. DEFINITIONS AND BACKGROUND

In this section we review concepts and backgrounds of our proceeding works.

2.1. THE HECKE ALGEBRA OVER $GL_2(\mathbb{Q}_p)$

Throughout this section let p be a fixed *prime*, and let $\mathbb{Q}_p$ be the *p-adic number field* for p. This set $\mathbb{Q}_p$ is by definition the completion of the *rational numbers* $\mathbb{Q}$ with respect to the *p-adic norm*

$$|q|_p = \left|p^k \frac{a}{b}\right| = \left(\frac{1}{p}\right)^k$$

for $q = p^k \frac{a}{b} \in \mathbb{Q}$ and $k \in \mathbb{Z}$.

Define now the (multiplicative) group $GL_2(\mathbb{Q}_p)$ of all invertible (2×2)-matrices over the p-adic number field $\mathbb{Q}_p$,

$$GL_2(\mathbb{Q}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q}_p) \,\middle|\, \begin{array}{c} a,\ b,\ c,\ d \in \mathbb{Q}_p, \\ ad - bc \neq 0 \end{array} \right\},$$

where $M_2(\mathbb{Q}_p)$ means the set of all (2×2)-matrices over $\mathbb{Q}_p$.

In the rest of this paper we denote $GL_2(\mathbb{Q}_p)$ simply by G_p, if there is no confusion.

The group G_p is locally profinite coming from the topology on $\mathbb{Q}_p$, i.e., it has a neighborhood base of the identity u_p of G_p, consisting of the compact-open subgroups

$$K_k = u_p + (p^k)GL_2(\mathbb{Z}_p) \quad \text{for all} \quad k \in \mathbb{N},$$

where $GL_2(\mathbb{Z}_p)$ means the subset of $GL_2(\mathbb{Q}_p)$ whose elements have their entries in $\mathbb{Z}_p$, and where

$$u_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{is the identity matrix of } M_2(\mathbb{Q}_p).$$

Then the subgroup

$$K_0 = GL_2(\mathbb{Z}_p)$$

forms the maximal compact-open subgroup of G_p.

Now let (V, π) be a *representation* of G_p, that is V is a vector space, and π is a group action,

$$\pi : G_p \to GL(V)$$

acting on V, where $GL(V)$ is the set of all invertible linear transformations on V.

Definition 2.1. We say a representation (V, π) is a *smooth representation*, if given any vector $v \in V$, there is a compact-open subgroup K of G_p, such that

$$\pi(y)v = v \quad \text{for all} \quad y \in K.$$

Denote by V^K the set of vectors in V that are fixed by K under the action of π. Then the definition of smoothness implies that

$$V = \bigcup_{K \subseteq G_p:\ \text{compact-open}} V^K.$$

Given two smooth representations (V_1, π_1) and (V_2, π_2) of G_p, we denote by

$$Hom_{G_p}(\pi_1, \pi_2),$$

the set of $\mathbb{C}$-linear maps

$$f : V_1 \to V_2$$

such that

$$f \circ \pi_1(g) = \pi_2(g) \circ f$$

for all $g \in G_p$.

Definition 2.2. Define the *Hecke algebra* $\mathcal{H}(G_p)$ of G_p by

$$\mathcal{H}(G_p) = \{f : G_p \to \mathbb{C} \mid f \text{ has compact-open support, and it is } \rho\text{-smooth}\}. \quad (2.1)$$

The ρ-smoothness means that $\mathcal{H}(G_p)$ is a smooth representation of G_p under right translation. In other words, for any element $f \in \mathcal{H}(G_p)$, there is a compact-open subgroup K of G_p such that

$$\rho(y)f(g) = f(gy) = f(g) \quad (2.2)$$

for all $g \in G_p$. We sometimes say also that f is *locally constant*.

We make $\mathcal{H}(G_p)$ into an associative algebra by taking $f_1, f_2 \in \mathcal{H}(G_p)$ and defining *convolution* (as a vector multiplication)

$$(f_1 * f_2)(g) = \int_{G_p} f_1(x) f_2(x^{-1}g) d\mu_p(x), \quad (2.3)$$

where μ_p denotes a *left Haar measure* on the locally compact-open group G_p.

2.2. FREE PROBABILITY

Throughout this paper we use *Speicher*'s *combinatorial free probability* techniques in the sense of [12] (also, see cited papers therein). The original analytic *free probability* theory is established by Voiculescu, and since the mid 1980's, it has developed as one of the main branches of *operator algebra theory*. By replacing independence of classical probability theory to (noncommutative) *freeness*, we can have the noncommutative (and hence, possibly commutative) operator-algebraic and operator-theoretic probability and corresponding statistics (for instance, free stochastic calculus, etc). Such a noncommutative(-or-commutative)-algebraic extended probability theory, called free probability, has various applications not only in mathematics (operator theory, in particular, spectral theory, and operator algebra, see e.g. [11]), but also in related scientific fields (e.g., free entropy theory, quantum probability, and quantum statistics, etc).

In combinatorial free probability the free-probabilistic information of given operators in an algebra is determined by *free moments* or *free cumulants* (see e.g., [12]). In fact free moments and free cumulants are equivalent under the Möbius inversion; but free moments are used for studying free-distributional data of operators, while free cumulants are used for studying freeness among operators in the algebra.

We refer readers to [12] and [13] for more about free probability theory. Especially, we will use the same concepts and results of [12] in this paper (without introducing them precisely).

2.3. GROUP ALGEBRAS

Let G be a countable discrete group. Then one can construct the algebra $\mathcal{A}_G$ by

$$\mathcal{A}_G = \mathbb{C}[G] = \left\{ \sum_{g \in G} t_g g : t_g \in \mathbb{C} \text{ for all } g \in G \right\},$$

where $\sum$ means a finite sum, i.e., $\mathcal{A}_G$ is the algebra generated by G. We call $\mathcal{A}_G$, the *group algebra generated by* G.

Each group algebra $\mathcal{A}_G$ is understood as a $*$-algebra over $\mathbb{C}$, by defining *the adjoint* $(*)$ on it by

$$\left(\sum_{g \in G} t_g g \right)^* \overset{def}{=} \sum_{g \in G} \overline{t_g} g^{-1},$$

where g^{-1} in the right-hand side mean group-inverse of g.
All groups G of this paper are assumed to be countable discrete groups.

Every group algebra $\mathcal{A}_G$ acts on the Hilbert space $H_G = l^2(G)$ via a *group-action* u, under the *left regular unitary representation* denoted by (H_G, u), where $l^2(G)$ means the l^2-*space* with its *orthonormal basis* (or its *Hilbert basis*)

$$\{\xi_g : g \in G \backslash \{e_G\}\},$$

where e_G is the group-identity of G, satisfying

$$\langle \xi_{g_1}, \xi_{g_2} \rangle_2 = \delta_{g_1, g_2},$$

where $\langle \cdot, \cdot \rangle_2$ means the *inner product on* H_G and δ means the *Kronecker delta*.

In particular, the group-action u acts as follows: for each $g \in G$, the image $u(g)$, denoted by u_g, becomes a *unitary operator* in the sense that: $u_g^* = u_g^{-1}$, where u_g^* means the (*Hilbert-space-operator-*)*adjoint of* u_g, and u_g^{-1} means the (*operator-*)*inverse of* u_g on H_G. In particular, the unitary operators $\{u_g\}_{g \in G}$ satisfy

$$u_{g_1}(\xi_{g_2}) \overset{def}{=} \xi_{g_1}\xi_{g_2} = \xi_{g_1 g_2}$$

for all $g_1,\ g_2 \in G,$ and $\xi_{g_2} \in H_G$, and

$$u_{g_1} u_{g_2} = u_{g_1 g_2} \quad \text{for all} \quad g_1, g_2 \in G,$$

and

$$u_g^* = u_g^{-1} = u_{g^{-1}} \quad \text{for all} \quad g \in G,$$

where u_g^{-1} mean the operator-inverses of u_g for all $g \in G$.

By construction it is easy to check that a group algebra $\mathcal{A}_G$ is a ($*$-)subalgebra of the operator algebra $B(H_G)$, consisting of (bounded linear) operators on H_G (pure algebraically, without considering topology).

So under operator-norm topology of $B(H_G)$, we can have the *group* C^**-algebra* $\overline{\mathcal{A}_G}$; also, under weak-operator topology, one can have the *group von Neumann algebra* (or the *group* W^**-algebra*) $\overline{\mathcal{A}_G}^w$, etc.

Let $\mathcal{A}_G$ be the group algebra. Define a linear functional

$$tr_G : \mathcal{A}_G \to \mathbb{C}$$

by

$$tr_G \left(\sum_{g \in G} t_g g \right) \overset{def}{=} t_{e_G}.$$

Then it is a well-defined linear functional. Moreover, it satisfies

$$tr_G(x_1 x_2) = tr_G(x_2 x_1) \quad \text{for all} \quad x_1, x_2 \in \mathcal{A}_G,$$

even though $x_1 x_2 \neq x_2 x_1$ in $\mathcal{A}_G$, i.e., tr_G is a *trace on* $\mathcal{A}_G$. We usually call tr_G the *canonical trace on* $\mathcal{A}_G$ (e.g., [11]).

Thus, the pair $(\mathcal{A}_G, tr_G)$ forms a free probability space in the sense of Section 2.2. This free probability space $(\mathcal{A}_G,\ tr_G)$ is called the (*canonical*) *group*(*-algebra*)*free probability space* (under topologies, the *group* C^**-free probability space*, or the *group* W^**-probability space*, etc).

3. NORMAL HECKE PROBABILITY SPACES

In this section we review free-probabilistic structures obtained in [7], and main results of [7] will be introduced for our future work.

3.1. NORMAL HECKE SUBALGEBRAS $\mathcal{H}_{Y_p}$ OF $\mathcal{H}(G_p)$

Notice, first that, by the very definition (2.1), the Hecke algebra $\mathcal{H}(G_p)$ can be re-defined by

$$\mathcal{H}(G_p) = \mathbb{C}_*\left[\left\{ f = \sum_{j=1}^{N} t_j\ \chi_{x_j K} \middle| \begin{array}{l} N \in \mathbb{N},\ \text{and } t_j \in \mathbb{C},\ K \text{ is a compact-open} \\ \text{subgroup of } G_p, \text{ depending on } f \\ \text{for all } x_j \in G_p,\ j = 1, \ldots, N \end{array} \right\}\right], \tag{3.1}$$

where $\mathbb{C}_*[X]$ mean algebras generated by X under the usual functional addition and convolution in the sense of Section 2.1, and χ_Y mean characteristic functions of μ_p-measurable subsets Y of G_p, where μ_p is in the sense of (2.2). The set

$$X_p = \left\{ f = \sum_{j=1}^{N} t_j\ \chi_{x_j K} \middle| \begin{array}{l} N \in \mathbb{N},\ \text{and } t_j \in \mathbb{C}, K \text{ is a compact-open} \\ \text{subgroup of } G_p,\ \text{depending on } f \\ \text{for all } x_j \in G_p, j = 1, \ldots, N \end{array} \right\} \tag{3.2}$$

generating the Hecke algebra $\mathcal{H}(G_p)$, is said to be the *generating set of* $\mathcal{H}(G_p)$, and we call elements of X_p of (3.2) *generating elements of* $\mathcal{H}(G_p)$, i.e.,

$$\mathcal{H}(G_p) = \mathbb{C}_*[X_p]. \tag{3.3}$$

By (3.1) and (3.3), one may write

$$\mathcal{H}(G_p) = \left\{ \sum_{j=1}^{N} t_j\ \chi_{x_j K_j} \middle| \begin{array}{l} N \in \mathbb{N},\ \text{and } t_j \in \mathbb{C},\ \text{and} \\ K_j \text{ are compact-open subgroups of } G_p, \\ \text{for all } x_j \in G_p,\ j = 1, \ldots, N \end{array} \right\}, \tag{3.4}$$

set-theoretically.

By construction $\mathcal{H}(G_p)$ is a well-defined vector space over $\mathbb{C}$. As in Section 2.1, the convolution $(*)$ on $\mathcal{H}(G_p)$, as a vector multiplication, is defined by

$$(f_1 * f_2)(g) = \int_{G_p} f_1(x) f_2(x^{-1}g) d\mu_p(g)$$

for all $f_1, f_2 \in \mathcal{H}(G_p)$, for all $g \in G_p$.

Proposition 3.1 ([7])**.** *Let $\chi_{x_1K_1}, \chi_{x_2K_2}$ be generating elements of $\mathcal{H}(G_p)$, for $x_j \in G_p$, and compact-open subgroups K_j of G_p for $j = 1, 2$. Then*

$$\left(\chi_{x_1K_1} * \chi_{x_2K_2}\right)(g) = \mu_p\left(x_1K_1 \cap gK_2x_2^{-1}\right) \tag{3.5}$$

for all $g \in G_p$.

Thus by (3.5), we obtain the following general result; if $f_j = \sum_{k=1}^{n_j} t_{j,k}\chi_{x_{j,k}K_j}$ are generating elements of $\mathcal{H}(G_p)$ in X_p, for $j = 1, 2$, then

$$(f_1 * f_2)(g) = \sum_{k=1}^{n_1}\sum_{l=1}^{n_2} (t_{1,k}t_{2,l})\,\mu_p\left(x_{1,k}K_1 \cap gK_2x_{2,l}^{-1}\right)$$

for all $g \in G_p$.

Without loss of generality, for any $x \in G_p$, one can understand

$$\chi_{xK}(g) = \frac{\mu_p(xK \cap gK)}{\mu_p(xK)} = \frac{\mu_p(xK \cap gK)}{\mu_p(K)} \tag{3.6}$$

by (2.2).

We now consider specific generating elements χ_{xK} in X_p, where K are "normal" compact-open subgroups of G_p. Recall that a subgroup K is *normal* in an arbitrary group Γ, if $gK = Kg$ for all $g \in \Gamma$. As usual, we denote this normal subgroup-inclusion by $K \lhd \Gamma$.

Define a subset Y_p of the generating set X_p of $\mathcal{H}(G_p)$ by

$$Y_p \overset{def}{=} \left\{\sum_{j=1}^{N} t_j\chi_{x_jK} \in X_p \,|K \lhd G_p\right\}. \tag{3.7}$$

Then we have a subalgebra

$$\mathcal{H}_{Y_p} \overset{def}{=} \mathbb{C}_*[Y_p] \text{ of } \mathcal{H}(G_p). \tag{3.8}$$

Proposition 3.2 ([7]). *Let $\chi_{x_jK_j} \in \mathcal{H}_{Y_p}$, where $x_j \in G_p$, and $K_j \lhd G_p$ compact-open, for $j = 1, 2$. Then*

$$\chi_{x_1K_1} * \chi_{x_2K_2} = \mu_p(K_1 \cap K_2)\chi_{x_1x_2K_1K_2}, \tag{3.9}$$

where K_1K_2 is the product group of K_1 and K_2 in G_p.

Definition 3.3. Let Y_p be the subset (3.7) of the generating set X_p, and let $\mathcal{H}_{Y_p} = \mathbb{C}_*[Y_p]$ be the subalgebra (3.8) of the Hecke algebra $\mathcal{H}(G_p)$. Then we call Y_p and $\mathcal{H}_{Y_p}$, the normal sub-generating set of X_p, and the normal Hecke subalgebra of $\mathcal{H}(G_p)$, respectively.

For convenience, denote $\prod_{j=1}^{N} x_j$ and $\underset{j=1}{\overset{N}{\times}} K_j$ simply by $x_{1,\ldots,N}$ and $K_{1,\ldots,N}$, respectively, for all $N \in \mathbb{N}$, where $x_1, \ldots, x_N \in G_p$ and $K_1, \ldots, K_N$ are (normal) compact-open subgroups of G_p. Also, denote

$$K_{1,\ldots,(N-1)} \cap K_N \text{ by } K^o_{1,\ldots,N}$$

for all $N \in \mathbb{N}\backslash\{1\}$.

We obtain the following general computations.

Proposition 3.4. *Let $\chi_{x_jK_j}$ be generating elements of the normal Hecke subalgebra $\mathcal{H}_{Y_p}$ for $j \in \mathbb{N}$. Then*

$$\underset{j=1}{\overset{N}{*}} \chi_{x_jK_j} = \left(\mu_p(K^o_{1,2})\mu_p(K^o_{1,2,3}) \dots \mu_p(K^o_{1,\dots,N})\right) \chi_{x_{1,\dots,N}K_{1,\dots,N}} \tag{3.10}$$

for all $N \in \mathbb{N}$.

Proof. The proof of (3.12) is done by (3.9), inductively (e.g., [2] and [7]). □

From now on, let us denote the convolution $f* \dots *f$ of n-copies of f simply by $f^{(n)}$ for all $n \in \mathbb{N}$ and $f \in \mathcal{H}(G_p)$.

3.2. FREE-PROBABILISTIC MODELS ON $\mathcal{H}_{Y_p}$

Let $\mathcal{H}(G_p)$ be the Hecke algebra generated by the generalized linear group $G_p = GL_2(\mathbb{Q}_p)$ over the p-adic number field $\mathbb{Q}_p$, for a fixed prime p. From Section 3.1, we start to understand this algebra $\mathcal{H}(G_p)$ as an algebra $\mathbb{C}_*[X_p]$ generated by X_p of (3.1), consisting of $\mathbb{C}$-valued functions f formed by

$$f = \sum_{j=1}^{N} t_j \chi_{x_jK} \quad \text{for} \quad t_j \in \mathbb{C}, x_j \in G_p, \tag{3.11}$$

where K is a compact-open subgroup of G_p, for $N \in \mathbb{N}$. So, to consider free-distributional data, we concentrate on generating elements χ_{xK}'s and e_{xK}'s, for $x \in G_p$, and compact-open subgroups K. Moreover, in this section, we restrict further our interests to the normal Hecke subalgebra $\mathcal{H}_{Y_p}$ of $\mathcal{H}(G_p)$, for a fixed prime p.

Let u_p be the group-identity of G_p, i.e.,

$$u_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G_p = GL_2(\mathbb{Q}_p).$$

For the fixed u_p define now a linear functional φ_p on $\mathcal{H}_{Y_p}$ by

$$\varphi_p(f) \overset{def}{=} f(u_p) \quad \text{for all} \quad f \in \mathcal{H}_{Y_p}. \tag{3.12}$$

The construction of the linear functional φ_p on $\mathcal{H}_{Y_p}$ (originally introduced in [7]) is motivated by the *canonical traces* on *group von Neumann algebras* (e.g., [11]), and the*point-evaluation linear functionals* on arithmetic functions in the sense of [4–6] and [8]. Clearly, the morphism φ_p is a well-defined linear functional on $\mathcal{H}_{Y_p}$, and hence, the pair $(\mathcal{H}_{Y_p}, \varphi_p)$ forms a free probability space in the sense of Section 2.2.

Definition 3.5. We call the linear functional φ_p of (3.12) on the normal Hecke subalgebra $\mathcal{H}_{Y_p}$, the canonical linear functional. And the corresponding free probability space $(\mathcal{H}_{Y_p}, \varphi_p)$ is said to be the normal Hecke probability space.

Then we obtain the following fundamental free-moment computations.

Proposition 3.6 ([7]). *Let χ_{xK}, $\chi_{x_jK_j}$, e_{xK}, $e_{x_jK_j}$ be generating free random variables in the normal Hecke probability space $(\mathcal{H}_{Y_p}, \varphi_p)$ for all $j \in \mathbb{N}$. Then*

$$\varphi_p\left(\underset{j=1}{\overset{N}{\ast}}\chi_{x_jK_j}\right) = \frac{\mu_p(K^o_{1,2})\dots\mu_p(K^o_{1,\dots,N})\mu_p(x_{1,\dots,N}K_{1,\dots,N}\cap K_{1,\dots,N})}{\mu_p(K_{1,\dots,N})} \tag{3.13}$$

for all $N \in \mathbb{N}$.

Indeed,

$$\varphi_p\left(\underset{j=1}{\overset{N}{\ast}}\chi_{x_jK_j}\right) = \varphi_p\left(\mu_p(K^o_{1,2})\mu_p(K^o_{1,2,3})\dots\mu_p(K^o_{1,\dots,N})\chi_{x_{1,\dots,N}K_{1,\dots,N}}\right)$$

by (3.10)

$$= \mu_p(K^o_{1,2})\mu_p(K^o_{1,2,3})\dots\mu_p(K^o_{1,\dots,N})\varphi_p\left(\chi_{x_{1,\dots,N}K_{1,\dots,N}}\right)$$
$$= \mu_p(K^o_{1,2})\mu_p(K^o_{1,2,3})\dots\mu_p(K^o_{1,\dots,N})\chi_{x_{1,\dots,N}K_{1,\dots,N}}(u_p)$$

by (3.12)

$$= \mu_p(K^o_{1,2})\mu_p(K^o_{1,2,3})\dots\mu_p(K^o_{1,\dots,N})\frac{\mu_p(x_{1,\dots,N}K_{1,\dots,N}\cap K_{1,\dots,N})}{\mu_p(K_{1,\dots,N})}$$

by (3.6)

$$= \frac{\mu_p(K^o_{1,2})\mu_p(K^o_{1,2,3})\dots\mu_p(K^o_{1,\dots,N})\mu_p(x_{1,\dots,N}K_{1,\dots,N}\cap K_{1,\dots,N})}{\mu_p(K_{1,\dots,N})}$$

for all $N \in \mathbb{N}$.

Let $\chi_{x_1K_1}, \dots, \chi_{x_NK_N} \in (\mathcal{H}_{Y_p}, \varphi_p)$ for $N \in \mathbb{N}$. Then

$$k^p_N\left(\chi_{x_1K_1}, \dots, \chi_{x_NK_N}\right) = \sum_{\pi\in NC(N)}\left(\prod_{V\in\pi}\varphi_p\left(\underset{j\in V}{\ast}\chi_{x_{i_j}K_{i_j}}\right)\mu\left(0_{|V|},\ 1_{|V|}\right)\right)$$

by the Möbius inversion of Section 2.2

$$= \sum_{\pi\in NC(N)}\left(\prod_{V=(i_1,\dots,i_{|V|})\in\pi}(\mu_p(V))\,\mu\left(0_{|V|},\ 1_{|V|}\right)\right), \tag{3.14}$$

by (3.13), where

$$\mu_p(V) = \frac{\mu_p(K^o_{i_1,i_2})\dots\mu_p(K^o_{i_1,\dots,i_{|V|}})\mu_p(x_{i_1,\dots,i_{|V|}}K_{i_1,\dots,i_{|V|}}\cap K_{i_1,\dots,i_{|V|}})}{\mu_p(K_{i_1,\dots,i_{|V|}})},$$

are the block-depending free moments for all $V \in \pi$ and $\pi \in NC(N)$, where $k_n^p(\ldots)$ means free cumulant determined by φ_p as in Section 2.2.

By (3.14) one can get the following freeness condition (3.15) on the normal Hecke subalgebra $\mathcal{H}_{Y_p}$. And this freeness condition shows that classical independence guarantees our freeness.

Proposition 3.7 ([7]). *Let $f_j = \chi_{K_j}$ be free random variables in the normal Hecke free probability space $(\mathcal{H}_{Y_p}, \varphi_p)$ for $j = 1, 2$. Then*

$$f_1 \text{ and } f_2 \text{ are free in } (\mathcal{H}_{Y_p}, \varphi_p) \Leftrightarrow \mu_p(K_{1,2}^o) = \mu_p(K_1)\mu_p(K_2). \tag{3.15}$$

4. FREE PROBABILITY ON $\mathcal{H}(G_p)$

In this section we extend the free probability on the normal Hecke subalgebra $\mathcal{H}_{Y_p}$ of Section 3.2 to free probability fully on the Hecke algebra $\mathcal{H}(G_p)$. For more information about such extensions, see [2].

Let G be an arbitrary group and let K be a subgroup of G. The *normal core* $\mathrm{Core}_G(K)$ *of K in G* is defined by the subgroup of G,

$$\mathrm{Core}_G(K) \overset{def}{=} \underset{g \in G}{\cap} \left(g^{-1}Kg\right). \tag{4.1}$$

Then the normal core $\mathrm{Core}_G(K)$ is the maximal normal subgroup of G contained in K, i.e.,

$$\mathrm{Core}_G(K) \lhd G \text{ and } \mathrm{Core}_G(K) \leqslant K. \tag{4.2}$$

For convenience, we denote the normal core $\mathrm{Core}_G(K)$ of (4.1) satisfying (4.2) simply by K_G.

Define now a linear transformation E_p on the Hecke algebra $\mathcal{H}(G_p)$ by a morphism satisfying (4.3) and (4.4) below:

$$E_p\left(\chi_{xK}\right) = \begin{cases} \chi_{xK_{G_p}} & \text{if } xK = Kx, \\ 0_{\mathcal{H}(G_p)} & \text{otherwise} \end{cases} \tag{4.3}$$

and

$$E_p\left(\chi_{x_1K_1} * \chi_{x_2K_2}\right) = \begin{cases} \mu_p(K_{1,2}^o)\chi_{x_{1,2}K_{1,2:G_p}} & \text{if } x_iK_j = K_jx_j \text{ for all } i,\ j \in \{1,2\}, \\ 0_{\mathcal{H}(G_p)} & \text{otherwise,} \end{cases} \tag{4.4}$$

where K_{G_p} and $K_{1,2:G_p}$ mean the normal cores of K and $K_{1,2}$ in G_p, respectively, and where $0_{\mathcal{H}(G_p)}$ is the zero element of $\mathcal{H}(G_p)$.

By (4.3) and (4.4), if K_j are compact-open subgroups of G_p, and $x_i \in G_p$, and if

$$x_iK_j = K_jx_i \quad \text{for all} \quad i, j = 1, \ldots, N, \tag{4.5}$$

for $N \in \mathbb{N}$, then

$$\begin{aligned} & E_p\left(\chi_{x_1K_1} * \ldots * \chi_{x_NK_N}\right) \\ & = E_p\left(\mu_p(K_{1,2}^o)\mu_p(K_{1,2,3}^o)\ldots\mu_p(K_{1,\ldots,N}^o)\,\chi_{x_{1,\ldots,N}K_{1,\ldots,N}}\right) && (4.6) \\ & = \mu_p(K_{1,2}^o)\mu_p(K_{1,2,3}^o)\ldots\mu_p(K_{1,\ldots,N}^o)\chi_{x_{1,\ldots,N}K_{1,\ldots,N:G_p}} && (4.7) \end{aligned}$$

inductively by (4.4). Remark that if the condition (4.5) holds, then the formula

$$\underset{j=1}{\overset{N}{\circledast}} \chi_{x_jK_j} = \mu_p(K_{1,2}^o)\mu_p(K_{1,2,3}^o)\ldots\mu_p(K_{1,\ldots,N}^o)\chi_{x_{1,\ldots,N}K_{1,\ldots,N}} \tag{4.8}$$

holds in $\mathcal{H}(G_p)$, without normality of $K_1, \ldots, K_N$ in G_p (see [2]), and hence, the formula (4.6) holds, and hence the equality (4.7) holds, by (4.3) and (4.6).

Proposition 4.1. *Let $f_j = \chi_{x_jK_j}$ be generating elements of the Hecke algebra $\mathcal{H}(G_p)$, for $j = 1, \ldots, N$, for $N \in \mathbb{N}$, and let E_p be the linear transformation* (4.4) *on $\mathcal{H}(G_p)$. If*

$$x_iK_j = K_jx_i \quad \textit{for all} \quad i, j = 1, \ldots, N,$$

then

$$E_p\left(\underset{j=1}{\overset{N}{\circledast}} f_j\right) = \left(\prod_{j=2}^{N} \mu_p(K_{1,\ldots,j}^o)\right)\chi_{x_{1,\ldots,N}K_{1,\ldots,N:G_p}}. \tag{4.9}$$

Otherwise, they are identical to the zero element $0_{\mathcal{H}(G_p)}$ of the Hecke algebra $\mathcal{H}(G_p)$.

Proof. The proof of (4.9) is done by (4.5) and (4.8). See [2] for more details. □

By construction it is not difficult to check that the linear transformation E_p maps $\mathcal{H}(G_p)$ onto the normal Hecke subalgebra $\mathcal{H}_{Y_p}$. Moreover, this morphism E_p is idempotent in the sense that

$$E_p^2(f) = E_p\left(E_p(f)\right) = E_p(f)$$

for all $f \in \mathcal{H}(G_p)$, because normal cores are normal subgroups of G_p.

Definition 4.2. We will call the morphism E_p of (4.2), the normal-coring on $\mathcal{H}(G_p)$.

Define now a linear functional ψ_p on the Hecke algebra $\mathcal{H}(G_p)$ by

$$\psi_p \overset{def}{=} \varphi_p \circ E_p \text{ on } \mathcal{H}(G_p). \tag{4.10}$$

By the linearity of both the canonical linear functional φ_p on $\mathcal{H}_{Y_p}$ and the normal-coring E_p on $\mathcal{H}(G_p)$, the morphism ψ_p is a linear functional on $\mathcal{H}(G_p)$. We call the linear functional ψ_p of (4.10), the *normal-cored* (*canonical*) *linear functional on* $\mathcal{H}(G_p)$. So, the pair $(\mathcal{H}(G_p), \psi_p)$ forms a free probability space.

Definition 4.3. The free probability space $(\mathcal{H}(G_p), \psi_p)$ of the Hecke algebra $\mathcal{H}(G_p)$ and the normal-cored linear functional ψ_p of (4.10) is said to be the normal-cored Hecke probability space.

Generally we obtain the following joint free-moment computations.

Theorem 4.4. *Let* $(\mathcal{H}(G_p), \psi_p)$ *be the normal-cored Hecke probability space, and let* $f_j = \chi_{x_j K_j}$ *be generating free random variables in* $(\mathcal{H}(G_p), \psi_p)$ *for* $j \in \mathbb{N}$. *If the condition* (4.5) *holds for* $N \in \mathbb{N}$, *then we obtain*

$$\psi_p\left(\underset{j=1}{\overset{N}{*}} f_j\right) = \frac{\left(\mu_p(K_{1,2}^o)\mu_p(K_{1,2,3}^o)\dots\mu_p(K_{1,\dots,N}^o)\right)\mu_p\left(x_{1,\dots,N}K_{1,\dots,N:G} \cap K_{1,\dots,N:G_p}\right)}{\mu_p\left(K_{1,\dots,N:G_p}\right)} \tag{4.11}$$

for all $N \in \mathbb{N}$, *where* $K_{1,\dots,N:G_p}$ *is in the sense of* (4.2). *If there exists at least one pair* $(i,j) \in \{1,\dots,N\}^2$, *for* $N \in \mathbb{N}$, *such that* $x_i K_j \neq K_j x_i$ *in* G_p, *then the formulas* (4.11) *vanish in* $\mathcal{H}(G_p)$.

Proof. Suppose first that

$$x_i K_j = K_j x_i \quad \text{for all} \quad i,j = 1,\dots,N,$$

for $N \in \mathbb{N}$, i.e., assume that the condition (4.5) holds. Then we have

$$\psi_p\left(\underset{j=1}{\overset{N}{*}} f_j\right) = \psi_p\left(\mu_p(K_{1,2}^o)\mu_p(K_{1,2,3}^o)\dots\mu_p(K_{1,\dots,N}^o)\,\chi_{x_{1,\dots,N}K_{1,\dots,N}}\right)$$

by (4.6)

$$\begin{aligned}
&= \mu_p(K_{1,2}^o)\mu_p(K_{1,2,3}^o)\dots\mu_p(K_{1,\dots,N}^o)\psi_p\left(\chi_{x_{1,\dots,N}K_{1,\dots,N}}\right)\\
&= \mu_p(K_{1,2}^o)\dots\mu_p(K_{1,\dots,N}^o)\;\varphi_p\left(E_p(\chi_{x_{1,\dots,N}K_{1,\dots,N}})\right)\\
&= \mu_p(K_{1,2}^o)\dots\mu_p(K_{1,\dots,N}^o)\;\varphi_p\left(\chi_{x_{1,\dots,N}K_{1,\dots,N:G_p}}\right)\\
&= \mu_p(K_{1,2}^o)\dots\mu_p(K_{1,\dots,N}^o)\;\left(\frac{\mu_p\left(x_{1,\dots,N}K_{1,\dots,N:G}\cap K_{1,\dots,N:G_p}\right)}{\mu_p\left(K_{1,\dots,N:G_p}\right)}\right)
\end{aligned}$$

by (3.9)

$$= \frac{\mu_p(K_{1,2}^o)\dots\mu_p(K_{1,\dots,N}^o)\mu_p\left(x_{1,\dots,N}K_{1,\dots,N:G}\cap K_{1,\dots,N:G_p}\right)}{\mu_p\left(K_{1,\dots,N:G_p}\right)}.$$

So, the formula (4.11) holds.

Of course if there exists at least one pair (i,j), such that $x_i K_j \neq K_j x_i$, then the formulas (4.11) and (4.12) simply vanish, by (4.3) and (4.4). □

So we obtain that

$$\begin{aligned}
\psi_p\left(\underset{j=1}{\overset{N}{*}} \chi_{K_j}\right) &= \frac{\mu_p(K_{1,2}^o)\mu_p(K_{1,2,3}^o)\dots\mu_p(K_{1,\dots,N}^o)\mu_p(K_{1,\dots,N:G_p})}{\mu_p(K_{1,\dots,N:G_p})}\\
&= \mu_p(K_{1,2,}^o)\mu_p(K_{1,2,3,}^o)\dots\mu_p(K_{1,\dots,N}^o),
\end{aligned} \tag{4.12}$$

by (4.11).

Now let K_1 and K_2 be compact-open subgroups of G_p, and let χ_{K_j} be corresponding free random variables in the normal-cored Hecke probability space $(\mathcal{H}(G_p), \psi_p)$. Suppose $k_N(\dots)$ is the free cumulant for the normalized linear functional ψ_p. Then, for any $(i_1, \ldots, i_N) \in \{1,2\}^N$, for all $N \in \mathbb{N}$, we obtain the following free cumulant computation:

$$k_N\left(\chi_{K_{i_1}}, \ldots, \chi_{K_{i_N}}\right) = \sum_{\pi \in NC(N)} \left(\prod_{V \in \pi} \mu_p(V)\, \mu\left(0_{|V|},\ 1_{|V|}\right)\right) \tag{4.13}$$

with

$$\mu_p(V) = \mu_p(K^o_{i_{j_1}, i_{j_2}})\mu_p\left(K^o_{i_{j_1}, i_{j_2}, i_{j_3}}\right) \ldots \mu_p\left(K^o_{i_{j_1}, \ldots, i_{j_k}}\right),$$

by (4.12), whenever $V = (j_1, \ldots, j_k) \in \pi$ for all $\pi \in NC(N)$ and for all $N \in \mathbb{N}$, where $\mu_p(V)$ are the V-block-depending free moments.

By the above joint free-cumulant formula (4.13), we obtain the following freeness condition on the normalized Hecke probability space $(\mathcal{H}(G_p), \psi_p)$.

Theorem 4.5 ([2])**.** *Let $f_j = \chi_{K_j}$ and $h_j = e_{K_j}$ be free random variables in the normal-cored Hecke probability space $(\mathcal{H}(G_p), \psi_p)$ for $j = 1, 2$. Then*

$$f_1 \text{ and } f_2 \text{ are free in } (\mathcal{H}(G_p), \psi_p) \Leftrightarrow \mu_p(K^o_{1,2}) = \mu_p(K_1)\mu_p(K_2). \tag{4.14}$$

5. REPRESENTATIONS ON NORMAL-CORED HECKE PROBABILITY SPACES

In this section we introduce representations of the normal-cored Hecke probability spaces $(\mathcal{H}(G_p), \psi_p)$, for primes p. Let p be a fixed prime, and let $(\mathcal{H}(G_p), \psi_p)$ be the corresponding normal-cored Hecke probability space.

Define a *sesqui-linear form* on the Hecke algebra $\mathcal{H}(G_p)$,

$$[\cdot, \cdot]_p : \mathcal{H}(G_p) \times \mathcal{H}(G_p) \to \mathbb{C}$$

by

$$[f_1, f_2]_p \overset{def}{=} \psi_p(f_1 * f_2^*) \quad \text{for all} \quad f_1, f_2 \in \mathcal{H}(G_p), \tag{5.1}$$

where

$$f^*(x) \overset{def}{=} \overline{f(x)} \text{ in } \mathbb{C} \quad \text{for all} \quad x \in G_p,$$

where $\overline{z}$ means the conjugate of z for all $z \in \mathbb{C}$. We call the above unary operation

$$f \in \mathcal{H}(G_p) \longmapsto f^* \in \mathcal{H}(G_p), \tag{5.2}$$

the *adjoint*. And the element f^* of (5.2) is said to be the *adjoint of* f. Since the adjoint (5.2) is well-defined on $\mathcal{H}(G_p)$, one may understand our Hecke algebra $\mathcal{H}(G_p)$ as a $*$-*algebra* over $\mathbb{C}$.

The form $[\cdot, \cdot]_p$ of (5.1) is indeed sesqui-linear, since

$$[t_1 f_1 + t_2 f_2, f_3]_p = t_1[f_1, f_3] + t_2[f_2, f_3]$$

and

$$[f_1, t_2 f_2 + t_3 f_3]_p = \overline{t_2}[f_1, f_2]_p + \overline{t_3}[f_1, f_3]_p$$

for all $f_1, f_2, f_3 \in \mathcal{H}(G_p)$ and $t_1, t_2, t_3 \in \mathbb{C}$.

Consider now that, for any fixed generating element χ_{xK} of $\mathcal{H}(G_p)$, for $x \in G_p$, and a compact-open subgroup K of G_p, we have

$$[t\chi_{xK}, t\chi_{xK}]_p = \psi_p \left(t\chi_{xK} * \overline{t}\chi_{xK}\right) = |t|^2\, \psi_p\left(\chi_{xK} * \chi_{xK}\right)$$

by the sesqui-linearity of $[\cdot,\cdot]_p$, where $|t|$ means the modulus $\sqrt{t\,\overline{t}}$ of t,

$$= \begin{cases} |t|^2\, \psi_p\left(\mu_p(K)\ \chi_{x^2K}\right) & \text{if } xK = Kx \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \left(\mu_p(K)\,|t|^2\right)\left(\frac{\mu_p(x^2K_{G_p} \cap K_{G_p})}{\mu_p(K_{G_p})}\right) & \text{if } xK = Kx \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} |t|^2\left(\frac{\mu_p(K)\ \mu_p(x^2K_{G_p} \cap K_{G_p})}{\mu_p(K_{G_p})}\right) & \text{if } xK = Kx \\ 0 & \text{otherwise} \end{cases}$$

by (4.11), i.e.,

$$[t\chi_{xK}, t\chi_{xK}]_p = |t|^2\left(\frac{\mu_p(K)\mu_p(x^2K_{G_p} \cap K_{G_p})}{\mu_p(K_{G_p})}\right), \text{ or } 0, \tag{5.3}$$

where K_{G_p} is the normal core of K in G_p. So, by (5.3), we obtain that

$$[t\chi_{xK}, t\chi_{xK}]_p \geqslant 0 \tag{5.4}$$

for all $x \in G_p$, for all compact-open subgroups K of G_p, for all $t \in \mathbb{C}$.

By (5.4) one can get in general that

$$[f, f]_p \geqslant 0 \quad \text{for all} \quad f \in \mathcal{H}(G_p). \tag{5.5}$$

Proposition 5.1 ([2]). *The sesqui-linear form $[\cdot,\cdot]_p$ on the Hecke algebra $\mathcal{H}(G_p)$ forms a pseudo-inner product on $\mathcal{H}(G_p)$.*

Suppose K is a nonempty proper "normal" compact-open subgroup of G_p and let xK be the left coset of K by $x \in G_p$. As "non-empty subsets" of G_p, it is possible that

$$xK \cap K = \varnothing, \text{ and hence, } \mu_p(xK \cap K) = 0.$$

In such a case we have

$$\begin{aligned}[][\chi_{xK}, \chi_{xK}]_p &= \psi_p\left(\mu_p(K)\chi_{xK}\right) = \varphi_p\left(\mu_p(K)\ \chi_{xK}\right) \\ &= \frac{\mu_p(K)\mu_p(xK \cap K)}{\mu_p(K)} = \mu_p(xK \cap K) = 0,\end{aligned}$$

i.e., there exist nonzero elements f of $\mathcal{H}(G_p)$ such that

$$[f, f]_p = 0.$$

Indeed, if $xK \neq Kx$ in G_p, then, by the very definition of E_p,

$$E_p \left(\chi_{xK} * \chi_{xK} \right) = 0_{\mathcal{H}(G_p)},$$

and hence,

$$\psi_p(\chi_{xK} * \chi_{xK}^*) = \varphi_p \left(0_{\mathcal{H}(G_p)} \right) = 0,$$

even though $\chi_{xK} \neq 0_{\mathcal{H}(G_p)}$, i.e.,

$$\exists f \neq 0_{\mathcal{H}(G_p)} \;:\; [f, f]_p = 0. \tag{5.6}$$

So the pseudo-inner product space $(\mathcal{H}(G_p), [\cdot, \cdot]_p)$ is not an inner product space, by (5.6).

When we understand our Hecke algebra $\mathcal{H}(G_p)$ as a pseudo-inner product space, we denote it by $\mathcal{H}_p$.

On the pseudo-inner product space $\mathcal{H}_p$ define a relation $\mathcal{R}_p$ by

$$f_1 \mathcal{R}_p f_2 \overset{def}{\Longleftrightarrow} [f_1, f_1]_p = [f_2, f_2]_p. \tag{5.7}$$

By the very definition (5.7) of $\mathcal{R}_p$, it is an equivalence relation on $\mathcal{H}_p$.

Definition 5.2. Let $\mathcal{H}_p$ be the pseudo-inner product space (5.6), and let $\mathcal{R}_p$ be the equivalence relation (5.7) on $\mathcal{H}_p$. Define the quotient space $\mathfrak{H}_p$ by

$$\mathfrak{H}_p = \mathcal{H}_p / \mathcal{R}_p, \tag{5.8}$$

equipped with the inherited pseudo-inner product, also denoted by $[\cdot, \cdot]_p$ on it. Then

$$\mathfrak{H}_p = (\mathfrak{H}_p, [\cdot, \cdot]_p) = (\mathcal{H}_p / \mathcal{R}_p, [\cdot, \cdot]_p)$$

is called the (normal-cored) Hecke inner product space.

From now on, if there is no confusion we denote equivalence classes

$$[f]_{\mathcal{R}_p} = \{h \in \mathcal{H}_p : h \mathcal{R}_p f\}$$

simply by f in the Hecke inner product space $\mathfrak{H}_p$.□

Indeed, our Hecke inner product space $\mathfrak{H}_p$ is an inner product space, by $\mathcal{R}_p$ of (5.7), i.e., it satisfies

$$[f, f]_p = 0 \Longleftrightarrow f = 0_{\mathfrak{H}_p} = 0_{\mathcal{H}_p} / \mathcal{R}_p, \tag{5.9}$$

where $0_{\mathcal{H}_p}$ is the zero element of $\mathcal{H}_p$.

For the given inner product space $\mathfrak{H}_p$, one can define the corresponding norm $\|\cdot\|_p$ on $\mathfrak{H}_p$ by

$$\|f\|_p \overset{def}{=} \sqrt{[f,\ f]_p} \quad \text{for all} \quad f \in \mathfrak{H}_p, \tag{5.10}$$

and the corresponding metric d_p on $\mathfrak{H}_p$ by

$$d_p \left(f_1,\ f_2 \right) = \|f_1 - f_2\|_p \quad \text{for all} \quad f_1, f_2 \in \mathfrak{H}_p. \tag{5.11}$$

Definition 5.3. Construct the d_p-metric topology closure of $\mathfrak{H}_p$, also denoted by $\mathfrak{H}_p$, where d_p is in the sense of (5.11) induced by the norm $\|\cdot\|_p$ of (5.10). It is called the (normal-cored) Hecke Hilbert space.

Then by the very construction of the Hecke Hilbert space $\mathfrak{H}_p$ from the normal-cored Hecke probability space $(\mathcal{H}(G_p), \psi_p)$, the algebra $\mathcal{H}(G_p)$ acts on $\mathfrak{H}_p$ via an algebra-action α^p;

$$\alpha^p(f)(h) = f * h \quad \text{for all} \quad h \in \mathfrak{H}_p, \tag{5.12}$$

for all $f \in \mathcal{H}(G_p)$. More precisely, the above relation (5.12) means

$$\alpha^p(f)(h) = \alpha^p(f)\left([h]_{\mathcal{R}_p}\right) = [f * h]_{\mathcal{R}_p} \tag{5.13}$$

in $\mathfrak{H}_p$ for $f \in \mathcal{H}(G_p)$. For convenience, we denote $\alpha^p(f)$ by α^p_f for all $f \in \mathcal{H}(G_p)$.

The above morphism α^p of (5.12) and (5.13) is indeed a well-defined algebra-action of $\mathcal{H}(G_p)$ acting on $\mathfrak{H}_p$, since

$$\begin{aligned}\alpha^p_{f_1 * f_2}(h) &= f_1 * f_2 * h = f_1 * (f_2 * h)\\ &= f_1 * \left(\alpha^p_{f_2}(h)\right) = \alpha^p_{f_1}\left(\alpha^p_{f_2}(h)\right) = \left(\alpha^p_{f_1}\alpha^p_{f_2}\right)(h)\end{aligned}$$

for all $h \in \mathfrak{H}_p$ and $f_1, f_2 \in \mathcal{H}(G_p)$, i.e.,

$$\alpha^p_{f_1 * f_2} = \alpha^p_{f_1}\alpha^p_{f_2} \text{ on } \mathfrak{H}_p \tag{5.14}$$

for all $f_1, f_2 \in \mathcal{H}(G_p)$. Also, α^p satisfies that

$$\begin{aligned}\left[\alpha^p_f(h_1),\ h_2\right]_p &= [f * h_1,\ h_2]_p\\ &= \psi_p\left((f * h_1) * h_2^*\right)\\ &= \psi_p\left(h_1 * f * h_2^*\right)\\ &= \psi_p\left(h_1 * (h_2^* * f)\right)\psi_p\left(h_1 * (f^* * h_2)^*\right)\\ &= [h_1, f^* * h_2]_p = \left[h_1,\ \alpha^p_{f^*}(h_2)\right]_p\end{aligned}$$

for all $h_1, h_2 \in \mathfrak{H}_p$ and $f \in \mathcal{H}(G_p)$, i.e.,

$$\left(\alpha^p_f\right)^* = \alpha^p_{f^*} \text{ on } \mathfrak{H}_p \quad \text{for all} \quad f \in \mathcal{H}(G_p). \tag{5.15}$$

Therefore, the morphism α^p of (5.12) is a $*$-algebra-action of $\mathcal{H}(G_p)$ acting on $\mathfrak{H}_p$, by (5.14) and (5.15).

Theorem 5.4. *The pair $(\mathfrak{H}_p, \alpha^p)$ of the Hecke Hilbert space $\mathfrak{H}_p$ and the morphism α^p of (5.12) forms a Hilbert-space representation of the Hecke algebra $\mathcal{H}(G_p)$ acting on $\mathfrak{H}_p$.*

Proof. The proof is done by (5.13), (5.14) and (5.15). (See [2] for more details.) □

We call the algebra-action α^p of (5.12) the (*normal-cored*) *Hecke*(*-algebra*) *action of* $\mathcal{H}(G_p)$ *acting on* $\mathfrak{H}_p$.

Definition 5.5. The Hilbert-space representation $(\mathfrak{H}_p, \alpha^p)$ of the Hecke algebra $\mathcal{H}(G_p)$ is called the (normal-cored) Hecke representation (of the normal-cored Hecke probability space $(\mathcal{H}(G_p), \psi_p)$).

6. CERTAIN PROJECTIONS AND PARTIAL ISOMETRIES ON $\mathfrak{H}_p$

In this section under the Hecke representation $(\mathfrak{H}_p, \alpha^p)$ of the Hecke probability space $(\mathcal{H}(G_p), \psi_p)$, certain generating elements of $\mathcal{H}(G_p)$ will be considered as Hilbert-space operators on $\mathfrak{H}_p$ (under quotient). In particular, we are interested in partial isometries induced by generating elements and their initial and final projections.

Already in [2] we studied some operator-theoretic information; self-adjointness, normality, unitarity, isometry-property and hyponormality; of such operators. In particular, we realized that, by the very constructions of the Hecke algebra $\mathcal{H}(G_p)$ and our representation $(\mathfrak{H}_p, \alpha^p)$, there are no isometries (and hence, no unitaries) formed by $\alpha^p_{t\chi_{xK}}$, for $t \in \mathbb{C}, x \in G_p$, and compact-open subgroups K of G_p. However, operators $\alpha^p_{t\chi_{xK}}$ are always normal on $\mathfrak{H}_p$.

Since there are neither isometries nor unitaries we are interested in the operators $\alpha^p_{t\chi_{xK}}$ which are projections, and partial isometries having their identical initial-and-final projections on $\mathfrak{H}_p$.

Recall that an operator T on a Hilbert space H is said to be a *partial isometry*, if T^*T is a projection on H. It is well-known that: T is a partial isometry, if and only if $TT^*T = T$ on H, if and only if T^* is a partial isometry on H, if and only if $T^*TT^* = T^*$ on H, if and only if TT^* is a projection on H. i.e., a partial isometry T is a unitary from $T^*T(H)$ onto $TT^*(H)$.

If T is a partial isometry on H, then the projection T^*T is called the *initial projection of* T, and the projection TT^* is called the *final projection of* T on H. Also, the (*closed*) *subspaces* $T^*T(H)$ and $TT^*(H)$ of H are called the *initial subspace* and the *final subspace* of T in H, respectively.

If T is a partial isometry on H, then it is a unitary from its initial subspace onto its final subspace, in the sense that:

$$T^*T = 1_{T^*T(H)} \quad \text{and} \quad TT^* = 1_{TT^*(H)},$$

where 1_K means the identity operators on Hilbert (sub-)spaces K (in H). Thus, if T has identical initial and final subspaces K in H, then

$$T^*T = 1_K = TT^*,$$

and hence, one can understand T as unitary in the operator subalgebra $B(K)$ of $B(H)$.

Notice that in Section 5 (and [2]), we observed that:

$$\left(\alpha^p_{f_1}\right)\left(\alpha^p_{f_2}\right) = \alpha^p_{f_1 * f_2} \quad \text{for all} \quad f_1, f_2 \in \mathcal{H}(G_p), \tag{6.1}$$

$$\left(\alpha_f^p\right)^* = \alpha_{f^*}^p \quad \text{for all} \quad f \in \mathcal{H}(G_p). \tag{6.2}$$

Theorem 6.1. *Let $f = \chi_{xK}$ be a generating element of $\mathcal{H}(G_p)$ for $x \in G_p$, and a compact-open subgroup K of G_p. Assume $xK = Kx$ in G_p, and let α_f^p be the corresponding operator on the Hecke Hilbert space $\mathfrak{H}_p$.*

$$\alpha_f^p \text{ is a projection on } \mathfrak{H}_p \Longleftrightarrow \mu_p(K) = 1, \text{ and } x \in K. \tag{6.3}$$

Proof. Recall that an operator T on an arbitrary Hilbert space H is a projection, if (i) T is self-adjoint in the sense that $T^* = T$ on H, where T^* is the adjoint of T, and (ii) T is idempotent in the sense that $T^2 = T$ on H.

Observe now that

$$\left(\alpha_f^p\right)^* = \alpha_{f^*}^p = \alpha_{(\chi_{xK})^*}^p = \alpha_{\chi_{xK}}^p = \alpha_f^p,$$

by (6.2). Thus, the operator α_f^p is self-adjoint on $\mathfrak{H}_p$. So, the given operator α_f^p satisfies the self-adjointness condition (i) automatically.

Now observe that

$$\left(\alpha_f^p\right)^2 = \alpha_{f*f}^p = \alpha_{\mu_p(K)\chi_{x^2K}}^p \text{ on } \mathfrak{H}_p, \tag{6.4}$$

by (6.1), and by the assumption: $xK = Kx$ in G_p.

So to satisfy the idempotence condition (ii), the operator α_f^p must satisfy

$$\alpha_{\mu_p(K)\ \chi_{x^2K}}^p = \alpha_{\chi_{xK}}^p \text{ on } \mathfrak{H}_p, \tag{6.5}$$

by (6.4).

($\Leftarrow$) If $\mu_p(K) = 1$, and $x \in K$, then $xK = K$, and hence, $x^2K = K$, moreover,

$$\alpha_{\mu_p(K)\chi_{x^2K}}^p = \alpha_{\chi_K}^p = \alpha_{\chi_{xK}}^p.$$

Therefore, the relation (6.5) holds, and hence α_f^p is a projection on $\mathfrak{H}_p$.

($\Rightarrow$) Suppose the relation (6.5) holds, and assume that either $\mu_p(K) \neq 1$, or $x \notin K$ in G_p.

Let $x \notin K$ in G_p. Then, in general, $xK \neq x^2K$, and hence, $\chi_{x^2K} \neq \chi_{xK}$. So, the relation (6.5) does not hold true, and it contradicts our assumption.

Assume now that $\mu_p(K) \neq 1$. Then, clearly,

$$\mu_p(K)\chi_{x^2K} \neq \chi_{xK},$$

in general, thus the relation (6.5) does not hold either. It again contradicts our assumption.

Therefore, we obtain the characterization

$$\alpha_f^p \text{ is an idempotent } \Longleftrightarrow \mu_p(K) = 1, \text{ and } x \in K. \tag{6.6}$$

By the self-adjointness of α_f^p, and by (6.5) and (6.6), one can conclude that: α_f^p is a projection on $\mathfrak{H}_p$, if and only if

$$\mu_p(K) = 1, \text{ and } x \in K.$$

□

The above characterization (6.3) shows that the generating elements $f = \chi_{xK}$ of the normal-cored Hecke probability space $(\mathcal{H}(G_p), \psi_p)$ assign projections α_f^p on the Hecke Hilbert space $\mathfrak{H}_p$, whenever

$$f = \chi_K \quad \text{with} \quad \mu_p(K) = 1. \tag{6.7}$$

Let $f_j = \chi_{K_j}$ be non-zero generating elements of $(\mathcal{H}(G_p), \psi_p)$, where $\mu_p(K_j) = 1$, equivalently, $\alpha_{f_j}^p$ are projections on $\mathfrak{H}_p$, by (6.3) and (6.7), for $j = 1, 2$. Also, let $f = \chi_{xK} \in (\mathcal{H}(G_p), \psi_p)$, and α_f^p, the corresponding operator on $\mathfrak{H}_p$, where

$$xK = Kx \quad \text{in} \quad G_p.$$

Consider the following functional equation:

$$f^* * f = f_1 \quad \text{and} \quad f * f^* = f_2 \quad \text{on} \quad \mathcal{H}(G_p). \tag{6.8}$$

Observe that

$$f^* * f = \mu_p(K)\chi_{x^2K} = f * f^* \quad \text{in} \quad \mathcal{H}(G_p). \tag{6.9}$$

Consider the equality (6.10) below:

$$\mu_p(K)\chi_{x^2K} = \chi_K. \tag{6.10}$$

To satisfy (6.10), one must have that:

$$\mu_p(K) = 1, \text{ and } x^2K = K. \tag{6.11}$$

By (6.8), (6.9) and (6.10), we obtain the following theorem.

Theorem 6.2. *Let $x_0 \in G_p$, and K_0, K, compact-open subgroups of G_p, where $x_0K_0 = K_0x_0$ in G_p. If*

$$x_0K_0 = x_0^{-1}K \text{ in } G_p, \text{ with } \mu_p(K_0) = 1 = \mu_p(K), \tag{6.12}$$

then $\alpha^p_{\chi_{x_0K_0}}$ is a partial isometry with its initial and final projections $\alpha^p_{\chi_K}$ on $\mathfrak{H}_p$.

Proof. By (6.3) and (6.7), if $\mu_p(K) = 1$, then $\alpha^p_{\chi_K}$ is a projection on $\mathfrak{H}_p$. Assume now that

$$x_0^2K_0 = K \text{ in } G_p, \text{ where } \mu_p(K_0) = 1,$$

for some $x_0 \in G_p$. Then we have

$$\chi^*_{x_0K_0} * \chi_{x_0K_0} = \chi_{x_0K_0} * \chi_{x_0K_0} = \mu_p(K_0)\chi_{x_0^2K_0} = \chi_{x_0^2K_0} = \chi_K$$

on $\mathfrak{H}_p$, by (6.9), (6.10) and (6.11). Similarly, one obtains that

$$\chi_{x_0K_0} * \chi^*_{x_0K_0} = \chi_{x_0^2K_0} = \chi_K \text{ on } \mathfrak{H}_p.$$

Thus, the operator $\alpha^p_{\chi_{x_0K_0}}$ satisfies

$$\left(\alpha^p_{\chi_{x_0K_0}}\right)^* \left(\alpha^p_{\chi_{x_0K_0}}\right) = \alpha^p_{\chi_K} = \left(\alpha^p_{\chi_{x_0K_0}}\right)\left(\alpha^p_{\chi_{x_0K_0}}\right)^* \tag{6.13}$$

on $\mathfrak{H}_p$, by the assumption that $x_0K_0 = K_0x_0$ in G_p.

The relation (6.13) shows that the operator $\alpha^p_{\chi_{x_0K_0}}$ is a partial isometry with its initial and final projections identified with the projection $\alpha^p_{\chi_K}$, on $\mathfrak{H}_p$. □

The above necessary condition (6.12) shows that, whenever we fix a projection $\alpha^p_{\chi_K}$ on $\mathfrak{H}_p$ (with $\mu_p(K) = 1$), one may take a partial isometry $\alpha^p_{\chi_{x_0 K_0}}$ on $\mathfrak{H}_p$, whenever

$$x_0^2 K_0 = K,$$

having its both initial and final projections $\alpha^p_{\chi_K}$. By the property of μ_p, one automatically obtains that

$$\mu_p(x_0^2 K_0) = \mu_p(K_0) = \mu_p(K) = 1.$$

Notice that the choice of K_0, for a fixed K, is not unique, i.e., one may have multi-partial isometries having both initial and final projections $\alpha^p_{\chi_K}$ on $mathfrakH_p$. Assume now that, for a fixed compact-open subgroup K of G_p with $\mu_p(K) = 1$, there are "distinct" compact-open subgroups K_j of G_p such that

$$x_j K_j = x_j^{-1} K \text{ and } \mu_p(K_j) = 1, \tag{6.14}$$

for some $x_j \in G_p$, for $j = 1, \ldots, N$, for $N \in \mathbb{N}$.

Then by (6.12), the operators $\alpha^p_{\chi_{x_j K_j}}$ are self-adjoint partial isometries having their initial and final projections $\alpha^p_{\chi_K}$ on $\mathfrak{H}_p$, for $j = 1, \ldots, N$. And, by (6.14), one can understand the partial isometries $\alpha^p_{\chi_{x_j K_j}}$ as certain perturbed operators $\alpha^p_{\chi_{x_j^{-1} K}}$ induced by $x_j^{-1}K$, satisfying (6.14) for all $j = 1, \ldots, N$, i.e.,

$$\alpha^p_{\chi_{x_j K_j}} = \alpha^p_{\chi_{x_j^{-1} K}} \text{ on } \mathfrak{H}_p \quad \text{for all} \quad j = 1, \ldots, N.$$

The above equality holds by the quotient relation $\mathcal{R}_p$ on the normal-cored Hecke Hilbert space $\mathfrak{H}_p$.

Let us denote these partial isometries $\alpha^p_{\chi_{x_j K_j}} = \alpha^p_{\chi_{x_j^{-1} K}}$ simply by T_j^K for $j = 1, \ldots, N$.

Theorem 6.3. *Let T_j^K be distinct partial isometries $\alpha^p_{\chi_{x_j K_j}} = \alpha^p_{\chi_{x_j^{-1} K}}$ satisfying* (6.14)*, whose initial and final projections $\alpha^p_{\chi_K}$, for $j = 1, \ldots, N$, for $N \in \mathbb{N}$, where*

$$K_j \lhd G_p \text{ for } j = 1, \ldots, N$$

(*and hence, $K \lhd G_p$, too, by* (6.14)). *Then the subgroup generated by $\{T_j^K\}_{j=1}^N$ (under the operator-multiplication on the operator algebra $B(\mathfrak{H}_p)$) is group-isomorphic to a quotient group $\mathfrak{T}_N$,*

$$\mathfrak{T}_N = \mathcal{F}\left(\{a_j\}_{j=1}^N\right) / \{a_j^2 = e_N\}_{j=1}^N$$

where $\mathcal{F}\left(\{a_j\}_{j=1}^N\right)$ is the free group generated by $\{a_j\}_{j=1}^N$, and $\{a_j^2 = e_N\}_{j=1}^N$ is the relator set of $\mathfrak{T}_N$, where e_N is the group-identity of $\mathfrak{T}_N$.

Proof. Let $T_j^K = \alpha^p_{\chi_{x_j K_j}}$ be given as above, and let

$$\alpha^p_{\chi_K}(\mathfrak{H}_p) \overset{denote}{=} \mathfrak{H}_p^K$$

be the subspace of $\mathfrak{H}_p$. Since $\alpha^p_{\chi_K}$ is a well-defined projection on $\mathfrak{H}_p$, its image $\mathfrak{H}_p^K$ is indeed a well-determined (closed) subspace of $\mathfrak{H}_p$. Moreover, it is both the initial and final subspaces of T_j^K, by (6.12) and (6.14), for all $j = 1, \ldots, N$, in $\mathfrak{H}_p$.

So without loss of generality, one can understand T_j^K are operators in the operator (sub-)algebra $B(\mathfrak{H}_p^K)$ of $B(\mathfrak{H}_p)$ for $j = 1, \ldots, N$. By understanding $\{T_j^K\}_{j=1}^N$ as a subset of $B(\mathfrak{H}_p^K)$, one can define the (multiplicative) subgroup $\mathfrak{T}_N^K$ (under operator multiplication on $B(\mathfrak{H}_p^K)$), by the group generated finitely by $\{T_j^K\}_{j=1}^N$, i.e.,

$$\mathfrak{T}_N^K \overset{def}{=} \left\langle \{T_j^K\}_{j=1}^N \right\rangle \subseteq B(\mathfrak{H}_p^K) \subseteq B(\mathfrak{H}_p), \tag{6.15}$$

where $\langle X \rangle$ mean here the groups generated by sets X.

Now let $\mathfrak{T}_N$ be the group,

$$\mathfrak{T}_N = \mathcal{F}\left(\{a_j\}_{j=1}^N\right) / \{a_j^2 = e_N\}_{j=1}^N, \tag{6.16}$$

where $\mathcal{F}(X)$ mean the (noncommutative) free groups generated by sets X.

Define now a morphism

$$\Omega : \mathfrak{T}_N^K \to \mathfrak{T}_N$$

by the binary–operation-preserving map such that

$$\Omega\left(T_j^K\right) = a_j \quad \text{for} \quad j = 1, \ldots, N \tag{6.17}$$

(with possible re-arrangements), where $\mathfrak{T}_N^K$ is in the sense of (6.15), and $\mathfrak{T}_N$ is in the sense of (6.16).

Since both $\mathfrak{T}_N^K$ and $\mathfrak{T}_N$ have N-generators, the generator-and-operation-preserving morphism Ω of (6.17) is bijective. It also satisfies that

$$\Omega\left(\left(T_j^K\right)^2\right) = a_j^2 = e_N \quad \text{for all} \quad j = 1, \ldots, N. \tag{6.18}$$

Indeed, by definition, one has

$$\left(T_j^K\right)^2 = \left(\alpha^p_{\chi_{x_j K_j}}\right)^2 = \alpha^p_{\chi_{x_j K_j} * \chi_{x_j K_j}} = \alpha^p_{\mu_p(K_j)\chi_{x_j^2 K_j}} = \alpha^p_{\chi_K} = 1_{\mathfrak{H}_p^K},$$

where $1_{\mathfrak{H}_p^K}$ means the identity operator on the subspace $\mathfrak{H}_p^K$ (in $B(\mathfrak{H}_p^K)$) of $\mathfrak{H}_p$. Thus, the formula (6.18) holds.

Remark that even though $K_1, \ldots, K_N$ are normal in G_p, one has

$$T_i^K T_j^K = \alpha^p_{\chi_{x_1 K_1} * \chi_{x_2 K_2}} = \alpha^p_{\mu_p(K^o_{1,2})\chi_{x_{1,2} K_{1,2}}} \neq \alpha^p_{\mu_p(K^o_{2,1})\chi_{x_{2,1} K_{2,1}}} = T_j^K T_i^K,$$

in general, in $\mathfrak{T}_N^K$, because $x_{1,2} \neq x_{2,1}$ in G_p, while $K_{1,2} = K_{2,1}$ in G_p.

Therefore, the bijective generator-and-operation-preserving morphism Ω also preserves the relations between $\mathfrak{T}_N^K$ and $\mathfrak{T}_N$, and hence, it is a well-determined group-isomorphism from $\mathfrak{T}_N^K$ onto $\mathfrak{T}_N$, i.e., two groups $\mathfrak{T}_N^K$ and $\mathfrak{T}_N$ are group-isomorphic. □

Notice that in the above theorem, the normality condition for $K_1, \ldots, K_N$ is crucial.

By the above theorem we obtain the following sub-structure theorem in $\alpha^p(\mathcal{H}(G_p))$ in $B(\mathfrak{H}_p)$.

Theorem 6.4. *Under the same hypothesis with the above theorem, the C^*-subalgebra generated by $\{T_j^K\}_{j=1}^N$ in $B(\mathfrak{H}_p)$ is $*$-isomorphic to the group C^*-algebra $C^*_{l^2(\mathfrak{T}_N)}(\mathfrak{T}_N)$ in the sense of Section 2.3, i.e.,*

$$C^*_{\mathfrak{H}_p^K}\left(\mathfrak{T}_N^K\right) \overset{*\text{-}iso}{=} C^*_{l^2(\mathfrak{T}_N)}(\mathfrak{T}_N), \tag{6.19}$$

where $C^*_H(X)$ mean the C^*-subalgebras of $B(H)$ generated by subsets X of $B(H)$ over Hilbert spaces H.

Proof. By the above theorem the (sub)group $\mathfrak{T}_N^K$ of (6.14) generated by $\{T_j^K\}_{j=1}^N$ (in $B(\mathfrak{H}_p^K) \subseteq B(\mathfrak{H}_p)$) is group-isomorphic to the group $\mathfrak{T}_N$ of (6.16), by the group-isomorphism Ω of (6.17), i.e.,

$$\mathfrak{T}_N^K \overset{\text{Group}}{=} \mathcal{F}\left(\{a_j\}_{j=1}^N\right)/\{a_j^2 = a_j\}_{j=1}^N = \mathfrak{T}_N.$$

Therefore, the group C^*-algebra

$$C^*\left(\mathfrak{T}_N^K\right) \overset{denote}{=} C^*_{\mathfrak{H}_p^K}\left(\mathfrak{T}_N^K\right) = \overline{\mathbb{C}\left[\mathfrak{T}_N^K\right]} \text{ of } B(\mathfrak{H}_p^K)$$

is $*$-isomorphic to the group C^*-algebra

$$C^*(\mathfrak{T}_N) \overset{denote}{=} C^*_{l^2(\mathfrak{T}_N)}(\mathfrak{T}_N) = \overline{\mathbb{C}\left[u(\mathfrak{T}_N)\right]} \text{ of } B\left(l^2(\mathfrak{T}_N)\right),$$

where u means the left-regular unitary representation in the sense of Section 2.3. Indeed, one can extend the group-isomorphism Ω of (6.17) under linearization, i.e., we have a morphism

$$\Omega_o : C^*\left(\mathfrak{T}_N^K\right) \to C^*(\mathfrak{T}_N),$$

such that

$$\Omega_o\left(\sum_{j=1}^n t_j\, T_j^K\right) \overset{def}{=} \sum_{j=1}^N t_j \Omega\left(T_j^K\right) = \sum_{j=1}^n t_j u(a_j),$$

for $t_j \in \mathbb{C}, j = 1, \ldots, n$ and $n \in \mathbb{N} \cup \{\infty\}$ (under C^*-topology).

It is not difficult to check Ω_o is a $*$-isomorphism. □

The characterization (6.19) shows that $\alpha^p(\mathcal{H}(G_p))$ contains group C^*-algebras ($*$-isomorphic to) $C^*(\mathfrak{T}_N)$, for $N \in \mathbb{N}$, where $\mathfrak{T}_N$ are in the sense of (6.16), whenever there are compact-open normal subgroups K with $\mu_p(K) = 1$, and distinct compact-open subgroups K_j with $\mu_p(K_j) = 1$, satisfying

$$x_j K_j = x_j^{-1} K \quad \text{for} \quad j = 1, \ldots, N.$$

As in above theorems we assume K is a normal compact-open subgroup of G_p with $\mu_p(K) = 1$, and

$$x_j K_j = x_j^{-1} K \quad \text{with} \quad \mu_p(K_j) = 1$$

for all $j = 1, \ldots, N$.

As a special case we consider the following conditions (6.20) and (6.21) below; suppose that the non-identity group elements x_j of G_p are self-invertible in the sense that:

$$x_j = x_j^{-1} \Longleftrightarrow x_j^2 = u_p = x_j^{-2}, \text{ the group-identity of } G_p \tag{6.20}$$

for all $j = 1, \ldots, N$.

And for the compact-open normal subgroup K, take

$$K_j = x_j K \quad \text{for all} \quad j = 1, \ldots, N. \tag{6.21}$$

Then automatically we have that

$$\mu_p(K_j) = 1 \quad \text{for all} \quad j = 1, \ldots, N.$$

Remark 6.5. Indeed, such group elements x_j exist in G_p. For instance, if we let

$$x = \begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix} \in G_p,$$

for $a, b \in \mathbb{Q}_p$, then $x^2 = u_p$ in G_p. So, one may take finitely many distinct elements $x_1, \ldots, x_N$ in G_p, for some $N \in \mathbb{N}$.

Moreover, for a fixed normal subgroup K of G_p, we can take such $x_1, \ldots, x_N$ in G_p, which are not contained in K. For instance, if K is the normal core U_{G_p} of $U = GL_2(\mathbb{Z}_p)$, then we can take

$$x_1 = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} 3 & 8 \\ -1 & -3 \end{pmatrix} \text{ in } G_p,$$

satisfying $x_1, x_2 \notin U_{G_p}$ and hence, $x_1 U_{G_p}$ and $x_2 U_{G_p}$ are as in (6.21).

Remark that

$$x_1 x_2 = \begin{pmatrix} 3 & 7 \\ -1 & -2 \end{pmatrix} \neq \begin{pmatrix} -2 & -7 \\ 1 & 3 \end{pmatrix} = x_2 x_1,$$

in G_p. So, the group generated by $\{x_1 U_{G_1}, x_2 U_{G_2}\}$ is group-isomorphic to the noncommutative group

$$\mathcal{F}(\{a_1, a_2\}) / \{a_j^{-1} = a_j\}_{j=1}^{2}.$$

The corresponding operators $T_j^K = \alpha_{\chi_{K_j}}^p$ are partial isometries on $\mathfrak{H}_p$, whose initial and final projections are the projection $\alpha_{\chi_K}^p$ on $\mathfrak{H}_p$. Therefore, one can obtain the group,

$$\mathfrak{T}_N^K = \left\langle \{T_j^K = \alpha_{\chi_{x_j K}}^p\}_{j=1}^{N} \right\rangle, \tag{6.22}$$

generated by $\{T_j^K\}_{j=1}^N$, as a multiplicative subgroup of the operator algebra $B(\mathfrak{H}_p^K)$, where $\mathfrak{H}_p^K = \alpha_{\chi_K}^p(\mathfrak{H}_p)$ is the subspace of $\mathfrak{H}_p$. Note that

$$T_j^K T_j^K = \alpha_{\chi_{K_i}}^p \alpha_{\chi_{K_j}}^p = \alpha^p \alpha_{\mu_p(K \cap K)\chi_{x_1 x_2 KK}}^p = \alpha_{\chi_{x_1 x_2 K}}^p. \tag{6.23}$$

Assumption and Notation 6.6 (in short, AN 6.6 from below)**.** In the rest of this paper if we write a group $\mathfrak{T}_N^K$, then it means a group (6.22), which is a special case of the general construction (6.15), satisfying (6.23), i.e.,

$$K_j = x_j K$$

of (6.21), where x_j satisfy (6.20), for $j = 1, \ldots, N$. But if we need to handle general cases as in (6.15) and (6.19), we will state clearly in the text.

By the group-isomorphic relation in the general format of (6.15) with (6.16), a group $\mathfrak{T}_N^K$ of AN 6.6 is group-isomorphic to the group $\mathfrak{T}_N$ of (6.16), too.

Recall that the group $\mathfrak{T}_N$ of (6.16) is defined to be the quotient group

$$\mathcal{F}\left(\{a_j\}_{j=1}^N\right)/\{a_j^{-1} = a_j\}_{j=1}^N.$$

In fact, the group $\mathfrak{T}_N$ is naturally group-isomorphic to the finitely presented group $\mathfrak{F}_N$,

$$\mathfrak{F}_N = \left\langle \{w_j\}_{j=1}^N, \left\{ \begin{array}{c} w_j^2 = e_N, \text{ and} \\ w_i w_j = w_j w_i \end{array} \right\}_{i,j=1}^N \right\rangle, \tag{6.24}$$

i.e.,

$$\mathfrak{T}_N \overset{\text{Group}}{=} \mathfrak{F}_N.$$

By the above discussions, we obtain the following refined results under AN 6.6.

Corollary 6.7. *Let $\mathfrak{T}_N^K$ be a group in the sense of* (6.22) *under AN 6.6. Then it is group-isomorphic to the finitely generated group $\mathfrak{F}_N$ of* (6.24)*. Moreover, the group C^*-algebra $C^*_{\mathfrak{H}_p^K}\left(\mathfrak{T}_N^K\right)$ is $*$-isomorphic to the group C^*-algebra $C^*_{l^2(\mathfrak{F}_N)}(\mathfrak{F}_N)$, i.e.,*

$$\mathfrak{T}_N^K \overset{\textit{Group}}{=} \mathfrak{F}_N \overset{\textit{def}}{=} \left\langle \{a_j\}_{j=1}^N, \left\{ \begin{array}{c} a_j = a_j^{-1} \textit{ and} \\ a_i a_j = a_j a_i \end{array} \right\}_{i,\, j=1}^N \right\rangle, \tag{6.25}$$

and

$$C^*_{\mathfrak{H}_p^K}\left(\mathfrak{T}_N^K\right) \overset{*\text{-}iso}{=} C^*_{l^2(\mathfrak{F}_N)}(\mathfrak{F}_N).$$

Proof. By the discussion in the above paragraphs, the group $\mathfrak{T}_N$ of (6.16) is group-isomorphic to $\mathfrak{F}_N$ of (6.24), by (6.20), (6.21) and (6.23) (under AN 6.6).

So one can define a morphism $\Psi : \mathfrak{T}_N \to \mathfrak{F}_N$ by a generator-preserving bijection between the two finite sets,

$$\Psi(a_j) = w_j \quad \text{for all} \quad j = 1, \ldots, N,$$

such that

$$\Psi(a_i a_j) = \Psi(a_i)\Psi(a_j) = w_i w_j$$

(under possible re-arrangements) for all $i, j = 1, \ldots, N$.

Therefore, one has that

$$\mathfrak{T}_N^K \overset{\text{Group}}{=} \mathfrak{T}_N \overset{\text{Group}}{=} \mathfrak{F}_N.$$

By the above group-isomorphic relations we obtain

$$C^*_{\mathfrak{H}_p^K}\left(\mathfrak{T}_N^K\right) \overset{*\text{-iso}}{=} C^*_{l^2(\mathfrak{T}_N)}\left(\mathfrak{T}_N\right) \overset{*\text{-iso}}{=} C^*_{l^2(\mathfrak{F}_N)}\left(\mathfrak{F}_N\right). \qquad \square$$

7. FREE STRUCTURES ON $C^*\left(\mathfrak{T}_N^K\right)$

In this section we study freeness conditions on our group C^*-algebras and their structure theorems.

Now let K be a fixed normal compact-open subgroup of G_p, with $\mu_p(K) = 1$, and hence, the corresponding operator $T^K = \alpha^p_{\chi_K}$ is a projection on the Hecke Hilbert space $\mathfrak{H}_p$, acting as the identity operator on the subspace $\mathfrak{H}_p^K = T^K(\mathfrak{H}_p)$ in $\mathfrak{H}_p$. Assume further that there exist distinct self-invertible group elements $x_j \in G_p$ in the sense that: $x_j^{-1} = x_j$, and distinct subsets K_j of G_p with $\mu_p(K_j) = 1$, such that

$$K_j = x_j^{-1}K = x_j K \quad \text{for all} \quad j = 1, \ldots, N,$$

as in AN 6.6. Then, by (6.12), the corresponding operators $T_j^K = \alpha^p_{\chi_{K_j}}$ are the partial isometries on $\mathfrak{H}_p$ with their initial and final projections identified with $T^K = \alpha^p_{\chi_K}$, for $j = 1, \ldots, N$.

We have seen in (6.19) and (6.25) the C^*-algebra $C^*\left(\mathfrak{T}_N^K\right)$ is $*$-isomorphic to the group C^*-algebra $C^*(\mathfrak{T}_N)$ generated by the finitely generated group,

$$\mathfrak{T}_N \overset{\text{Group}}{=} \left\langle \{a_j\}_{j=1}^N, \left\{ \begin{array}{c} a_j^2 = e_N \text{ and} \\ a_i a_j = a_j a_i \end{array} \right\}_{i,j=1}^N \right\rangle.$$

Let's denote $C^*\left(\mathfrak{T}_N^K\right)$ and $C^*(\mathfrak{T}_N)$ simply by $\mathfrak{C}^*_{K,N}$, and $\mathfrak{C}^*_N$, respectively. Because of the $*$-isomorphic relations between $\mathfrak{C}^*_{K,N}$ and $\mathfrak{C}^*_N$ we sometimes use $\mathfrak{C}^*_{K,N}$ and $\mathfrak{C}^*_N$, alternatively, as a same object. However, whenever we emphasize such C^*-algebras $\mathfrak{C}^*_N$ are constructed from our Hecke representational setting we will precisely use the term $\mathfrak{C}^*_{K,N}$.

7.1. FREE-DISTRIBUTIONAL DATA ON $\mathfrak{C}^*_{K,N}$

Let $\mathfrak{T}_N^K$ be the group in the general sense of (6.14) and $\mathfrak{C}^*_{K,N}$, the corresponding group C^*-algebra generated by $\mathfrak{T}_N^K$ (without AN 6.6). On the C^*-subalgebra $\mathfrak{C}^*_{K,N}$

of $B(\mathfrak{H}_p^K) \subseteq B(\mathfrak{H}_p)$, define a linear functional, also denoted by ψ_p, by a morphism satisfying

$$\begin{aligned}\psi_p\left(T_j^K\right) = \psi_p\left(\alpha^p_{\chi_{x_jK_j}}\right) &\overset{def}{=} \psi_p\left(\chi_{x_jK_j}\right) = \varphi_p\left(\chi_{x_jK_{j:G_p}}\right) \\ &= \varphi_p\left(\chi_{x_jK_j}\right) = \chi_{x_jK_j}(u_p) = \frac{\mu_p\left(x_jK_j \cap K_j\right)}{\mu_p(K_j)},\end{aligned} \tag{7.1}$$

by the normality conditions for $K_1, \ldots, K_N$, where $K_{j:G_p}$ means the normal core $\mathrm{Core}_{G_p}(K_j)$ of K_j in G_p, as in Section 3 and where ψ_p in the second equality $\overset{def}{=}$ of (7.1) means the normal-cored linear functional $\varphi_p \circ E_p$ on the Hecke algebra $\mathcal{H}(G_p)$ in the sense of (4.10) and φ_p is the canonical linear functional on the normal Hecke algebra $\mathcal{H}_{Y_p}$ in the sense of (3.12).

The pair $\left(\mathfrak{C}^*_{K,N},\ \psi_p\right)$ becomes a well-determined a C^*-probability space in the sense of [12] and [13].

Definition 7.1. The C^*-probability space $\left(\mathfrak{C}^*_{K,N},\ \psi_p\right)$ is called the K(-concentrated--C^*)-Hecke probability space on $\mathfrak{H}_p^K$ (or, on $\mathfrak{H}_p$).

Remark that since

$$x_jK_j = x_j^{-1}K \quad \text{for all} \quad j = 1, \ldots, N,$$

one has that

$$K_j = x_j^{-2}K \quad \text{for all} \quad j = 1, \ldots, N, \tag{7.2}$$

and hence,

$$\psi_p\left(T_j^K\right) = \frac{\mu_p\left(x_jK_j \cap K_j\right)}{\mu_p(K_j)} = \frac{\mu_p\left(x_j^{-1}K \cap x_j^{-2}K\right)}{\mu_p(x_j^{-2}K)} = \mu_p\left(x_j^{-1}K \cap x_j^{-2}K\right) \tag{7.3}$$

by (7.2), for all $j = 1, \ldots, N$.

Notice here in (7.3) that

$$\begin{aligned} x \in gK \cap g^2K &\Leftrightarrow x = gk_1 \text{ and } x = g^2k_2, \text{ for some } k_1, k_2 \in K \\ &\Leftrightarrow g^{-1}x = k_1 \text{ and } g^{-1}x = gk_2 \\ &\Leftrightarrow g^{-1}x \in K \cap gK \\ &\Leftrightarrow x \in g\left(K \cap gK\right), \end{aligned}$$

and hence one has

$$gK \cap g^2K \subseteq g\left(K \cap gK\right) \quad \text{for} \quad g \in G_p.$$

Similarly,

$$\begin{aligned} x \in g\left(K \cap gK\right) &\Leftrightarrow x = gv \text{ with } v = k_1 = gk_2, \text{ for some } k_1, k_2 \in K \\ &\Leftrightarrow x = gk_1 \text{ and } x = g^2k_2 \\ &\Leftrightarrow x \in gK \cap g^2K, \end{aligned}$$

and hence we have

$$g\,(K \cap gK) \subseteq gK \cap g^2K \quad \text{for} \quad g \in G_p.$$

Therefore,

$$gK \cap g^2K = g\,(K \cap gK)\,,$$

for a compact-open subgroup K of G_p, and $g \in G_p$. So, the second equality of (7.3) indeed holds.

It shows that the formula (7.3) can be re-written by

$$\psi_p\left(T_j^K\right) = \mu_p\left(x_j^{-1}K \cap x_j^{-2}K\right) = \mu_p\left(x_j^{-1}\left(K \cap x_j^{-1}K\right)\right) = \mu_p\left(K \cap x_j^{-1}K\right),$$

i.e.,

$$\psi_p\left(T_j^K\right) = \mu_p\left(K \cap x_j^{-1}K\right) \tag{7.4}$$

for all $j = 1, \ldots, N$, since $\mu_p(K) = 1$. So, one can conclude that

$$\begin{aligned}\psi_p\left(T_j^K\right) = \mu_p\left(x_jK \cap K\right) &= \frac{\mu_p\left(x_jK \cap K\right)}{\mu_p(K)} \\ &= \frac{\mu_p\left(x_jK \cap u_pK\right)}{\mu_p(K)} = \psi_p\left(\chi_{x_jK}\right) = \varphi_p\left(\chi_{x_jK}\right),\end{aligned} \tag{7.5}$$

by the normality of K, where u_p is the group-identity of G, by the normality of K in G_p. By (7.5), it is not difficult to check that

$$\psi_p\left(T_K\right) = \psi_p\left(\alpha^p_{\chi_K}\right) = \psi_p\left(\chi_K\right) = \chi_K(u_p) = \frac{\mu_p(K \cap u_pK)}{\mu_p(K)} = 1.$$

It shows that the K-Hecke probability space $(\mathfrak{C}^*_{K,N}, \psi_p)$ is *unital* in the sense that

$$\psi_p(T^K) = \psi_p\left(1_{\mathfrak{C}^*_{K,N}}\right) = 1,$$

because T^K is the identity operator $1_{\mathfrak{C}^*_{K,N}}$ on $\mathfrak{H}_p^K$ in $\mathfrak{C}^*_{K,N}$.

Observe now that

$$\psi_p\left(\prod_{k=1}^{n} T_{i_k}^K\right) = \psi_p\left(\underset{k=1}{\overset{n}{*}}\chi_{x_{i_k}K_{i_k}}\right) = \psi_p\left(\underset{k=1}{\overset{n}{*}}\ \chi_{x_{i_k}^{-1}K}\right)$$

by (7.2)

$$= \psi_p\left(\mu_p(K)^{n-1}\chi_{x_{i_1}^{-1}x_{i_2}^{-1}\dots x_{i_n}^{-1}K}\right)$$

by the normality condition for K

$$\begin{aligned} &= \frac{\mu_p(K)^{n-1}\mu_p\left((x_{i_n}\dots x_{i_1})^{-1}K \cap K\right)}{\mu_p(K)} \\ &= \frac{\mu_p\left((x_{i_n}\dots x_{i_1})^{-1}K \cap K\right)}{\mu_p(K)} \\ &= \mu_p\left((x_{i_n}\dots x_{i_2}x_{i_1})^{-1}K \cap K\right) \end{aligned} \tag{7.6}$$

refining (7.4) and (7.5).

The above formulas (7.5) and (7.6) are also obtained under AN 6.6, too.

Theorem 7.2. *If $(i_1, \dots, i_n) \in \{1, \dots, N\}^n$, for $n \in \mathbb{N}$, then*

$$\psi_p\left(\prod_{k=1}^{n}\ T_{i_k}^K\right) = \mu_p\left(\left(\prod_{k=0}^{n-1} x_{i_{n-k}}\right)^{-1} K \cap K\right). \tag{7.7}$$

Proof. The proof of (7.7) is done by formula (7.6). □

So one obtains the following corollary immediately.

Corollary 7.3. *Under AN 6.6, if $(i_1, \dots, i_n) \in \{1, \dots, N\}^n$ for $n \in \mathbb{N}$, then*

$$\psi_p\left(\prod_{k=1}^{n}\ T_{i_k}^K\right) = .\mu_p\left(\left(\Pi_{k=0}^{n-1} x_{i_{n-k}}\right) K \cap K\right). \tag{7.8}$$

The above formula (7.7) (or (7.8)) characterizes the free-distributional data of our partial isometries $\{T_j^K\}_{j=1}^N$ (resp., under AN 6.6).

For $(i_1, \dots, i_n) \in \{1, \dots, N\}^n$, for $n \in \mathbb{N}$, consider now the free cumulants,

$$k_n\left(T_{i_1}^K,\ T_{i_2}^K, \dots, T_{i_n}^K\right) = \sum_{\pi \in NC(n)}\left(\Pi_{V\in\pi}\left(\psi_p\left(\Pi_{j\in V}^{o} T_{i_j}^K\right)\mu\left(0_{|V|},\ 1_{|V|}\right)\right)\right)$$

by the Möbius inversion of Section 2.2, where $k_n(\dots)$ means the free cumulant for ψ_p on $\mathfrak{C}_{K,N}^2$

$$= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \left(\frac{\mu_p \left(\left(\prod_{j \in V}^{o} x_{i_j}^{-1} \right) K_{G_p} \cap K_{G_p} \right)}{\mu_p(K_{G_p})} \mu\left(0_{|V|},\ 1_{|V|}\right) \right) \right). \tag{7.9}$$

By the free cumulant formula (7.9), we obtain the following equivalent free-distributional data with (7.7) for the partial isometries $\{T_j^K\}_{j=1}^N$ generating $\mathfrak{C}_{K,N}^*$ in the K-Hecke probability space $(\mathfrak{C}_{K,N}^*, \psi_p)$.

Proposition 7.4. *Under the same hypothesis with* (7.8) *one has*

$$k_n\left(T_{i_1}^K,\ T_{i_2}^K, \ldots, T_{i_n}^K\right) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \left(\mu_p \left(\left(\prod_{j \in V} x_{i_j}^{-1} \right) K \cap\ K \right) \mu\left(0_{|V|},\ 1_{|V|}\right) \right) \right) \tag{7.10}$$

for all $(i_1,\ \ldots,\ i_n) \in \{1, \ldots, N\}^n$ *and* $n \in \mathbb{N}$.

Proof. The proof of (7.10) is done by (7.9). □

The above computation (7.10) provides the following freeness necessary condition on our group C^*-probability space $(\mathfrak{C}_{K,N}^*, \psi_p)$.

Theorem 7.5. *Let $\mathfrak{C}_{K,N}^*$ be the group C^*-subalgebra of $B(\mathfrak{H}_p^K)$ generated by the group $\mathfrak{T}_N^K$. Assume that the generators $T_j^K = \alpha_{\chi_{x_j^{-1}K}}^p$ satisfy that*

$$\mu_p\left(x_{i_1}^{-1} K \cap K\right) = \mu_p\left(x_{i_2}^{-1} K \cap K\right) \tag{7.11}$$

for all $i_1, i_2 = 1, \ldots, N$ and

$$\mu_p\left(\left(x_{j_1}^{-1} x_{j_2}^{-1} \ldots x_{j_k}^{-1}\right) K \cap K\right) = \mu_p\left(x_{j_1}^{-1} K \cap K\right) \tag{7.12}$$

for all $(j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$, where the entries $j_1,\ \ldots,\ j_k$ are all mutually distinct in the k-tuples for all $k \in \mathbb{N}$. Then the family $\{T_j^K\}_{j=1}^N$ is a free family in $(\mathfrak{C}_{K,N}^*, \psi_p)$, in the sense that: all elements of the family are free in $(\mathfrak{C}_{K,N}^*, \psi_p)$ from each other.

Proof. Assume the generator set $\{T_j^K\}_{j=1}^N$ of the group $\mathfrak{T}_N^K$ satisfies the above two conditions (7.11) and (7.12). Then by (7.10) we obtain a quantity β_o such that

$$\beta_o = \mu_p(x_j^{-1} K \cap K) \text{ for any } j = 1, \ldots, N.$$

Thus for any "mixed" n-tuple, $(i_1,\ \ldots,\ i_n) \in \{1,\ \ldots,\ N\}^n$, one has

$$\begin{aligned} k_n\left(T_{i_1}^K,\ T_{i_2}^K, \ldots, T_{i_n}^K\right) &= \beta_o \left(\sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \mu\left(0_{|V|},\ 1_{|V|}\right) \right) \right) \\ &= \beta_o \left(\sum_{\pi \in NC(n)} \mu\left(\pi,\ 1_n\right) \right) = 0, \end{aligned}$$

by Section 2.2, for all $n \in \mathbb{N} \setminus \{1\}$. Therefore the generator set $\{T_j^K\}_{j=1}^N$ of $\mathfrak{T}_N^K$ is a free family. □

7.2. FREENESS ON $\mathfrak{C}^*_{K,N}$

In this section we concentrate on freeness on our graph C^*-subalgebra $\mathfrak{C}^*_{K,N}$ generated by $\mathfrak{T}^K_N$ in $B(\mathfrak{H}^K_p)$. Throughout this section we restrict our interests to the special case where $\mathfrak{T}^K_N$ are under AN 6.6, for convenience. Remark that even though we are in the general setting, the main results of this section would be similar.

Recall the $*$-isomorphic relation between $\mathfrak{C}^*_{K,N}$ and $\mathfrak{C}^*_N$, where $\mathfrak{C}^*_N$ is the group C^*-algebra generated by the group,

$$\begin{aligned}\mathfrak{T}_N &\overset{def}{=} \mathcal{F}\left(\{a_j\}_{j=1}^N\right) / \{a_j = a_j^{-1}\}_{j=1}^N \\ &\overset{\text{Group}}{=} \left\langle \{a_j\}_{j=1}^N, \left\{ \begin{array}{c} a_j^{-1} = a_j \text{ and} \\ a_i a_j = a_j a_i \end{array} \right\}_{i,\,j=1}^N \right\rangle .\end{aligned} \tag{7.13}$$

Like the above necessary freeness conditions (7.11) and (7.12), one can verify that in some cases, the generator set $\{T_j^K\}_{j=1}^N$ of the group $\mathfrak{T}^K_N$ forms a free family in our K-Hecke probability settings.

Corollary 7.6. *Under AN 6.6, assume that the conditions* (7.11) *and* (7.12) *hold. Then the subgroup $\mathfrak{T}^K_N$ of* (6.22) *in $B(\mathfrak{H}^K_p)$ is group-isomorphic to the quotient group*

$$G_N^2 = \underset{j=1}{\overset{N}{\star}} \left\langle a_j : a_j^{-1} = a_j \right\rangle, \tag{7.14}$$

where $(\star)$ in (7.13) *means the "free product of groups" for $i, j = 1, \dots, N$. Therefore, in this case, the C^*-algebra $\mathfrak{C}^*_{K,N}$ is $*$-isomorphic to the group C^*-algebra $C^*\left(G_N^2\right)$, i.e.,*

$$\mathfrak{C}^*_{K,N} \overset{*\text{-}iso}{=} C^*\left(G_N^2\right). \tag{7.15}$$

Proof. If the conditions (7.11) and (7.12) hold, then the generators $\{T_j^K\}_{j=1}^N$ of the subgroup $\mathfrak{T}^K_N$ of (7.13) are free from each other in $(\mathfrak{C}^*_{K,N}, \psi_p)$. Moreover, in such a case, the group $\mathfrak{T}^K_N$ is group-isomorphic to G_N^2 of (7.13), since $\mathfrak{T}^K_N$ forms a free family (under quotient). Thus, the group-isomorphic relation (7.14) holds.

Therefore, in this case, one has

$$C^*\left(\mathfrak{T}^K_N\right) = \mathfrak{C}^*_{K,N} \overset{*\text{-iso}}{=} C^*(G_N^2),$$

by (7.14). So, the $*$-isomorphic relation (7.16) holds. □

In the proof of (7.16) the freeness on $\mathfrak{T}^K_N$ (from (7.11) and (7.12)) in $(\mathfrak{C}^*_{K,N}, \psi_p)$ is critical i.e., If $\mathfrak{T}^N_K$ is generated by a free family $\{T_j\}_{j=1}^N$, then

$$\mathfrak{T}^K_N \overset{\text{Group}}{=} G_N^2, \text{ and } \mathfrak{C}^*_{K,N} \overset{*\text{-iso}}{=} C^*\left(G_N^2\right).$$

Theorem 7.7. *Under AN 6.6, if the set $\{T_j^K\}_{j=1}^N$ of partial isometries forms a free family in $(\mathfrak{C}^*_{K,N}, \psi_p)$, then the subgroup $\mathfrak{T}^K_N$ of* (6.22) *in $B(\mathfrak{H}^K_p)$ is group-isomorphic to the quotient group*

$$G_N^2 = \underset{j=1}{\overset{N}{\star}} \left\langle a_j : a_j^{-1} = a_j \right\rangle, \tag{7.16}$$

where $(\star)$ *means the "commutative" group-free product. And the corresponding group* C^**-algebra* $\mathfrak{C}^*_{K,N}$ *is* $*$*-isomorphic to the group* C^**-algebra* $C^*\left(G^2_N\right)$*,*

$$\mathfrak{C}^*_{K,N} \overset{*\text{-}iso}{=} C^*\left(G^2_N\right) \overset{*\text{-}iso}{=} \underset{j=1}{\overset{N}{\star_{\mathbb{C}}}} C^*\left(\left\langle a_j : a_j^{-1} = a_j\right\rangle\right).$$

Proof. The proof is done by similar arguments for the above corollary under arbitrary freeness on $\{T^K_j\}^N_{j=1}$ in $(\mathfrak{C}^*_{K,N}, \psi_p)$. Remark that in the above corollary, we give necessary freeness condition from (7.11) and (7.12), while here we simply assume the generators of $\mathfrak{T}^K_N$ are free from each other under AN 6.6. □

Assume now that $\{T^K_j\}^N_{j=1}$ under AN 6.6 forms a free family in $(\mathfrak{C}^*_{K,N}, \psi_p)$. Then for any n-tuple $(i_1, \ldots, i_n)$ of $\{1, \ldots, N\}^n$, for $n \in \mathbb{N}$, one has

$$\psi_p\left(T^K_{i_1} T^K_{i_2} \ldots T^K_{i_n}\right) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} k_V\right), \tag{7.17}$$

where

$$k_V = k_{|V|}\left(T^K_{i_{k_1}}, T^K_{i_{k_2}}, \ldots, T^K_{i_{k_{|V|}}}\right),$$

whenever $V = \left(i_{k_1}, \ldots, i_{k_{|V|}}\right)$ in π, for all $\pi \in NC(n)$, where $k_n(\ldots)$ means the free cumulant in terms of the linear functional ψ_p.

By the freeness (7.16) under (7.7), all mixed free cumulants of $\{T^K_j\}^N_{j=1}$ vanish. Let $(i_1, \ldots, i_n) \in \{1, \ldots, N\}^n$, for $n \in \mathbb{N}$, and assume $\pi_{(i_1,\ldots,i_n)}$ is a noncrossing partition in $NC(n)$ with its blocks $V_1, \ldots, V_{\left|\pi_{(i_1,\ldots,i_n)}\right|}$, where $|\pi|$ mean the numbers of blocks of noncrossing partitions π, such that: (i) each block has its form

$$V_j = (k_j, k_j, \ldots, k_j), \text{ for } k_j \in \{1, \ldots, N\} \tag{7.18}$$

for all $j = 1, \ldots, \left|\pi_{(i_1,\ldots,i_n)}\right|$, and (ii) such a block V_j is maximal, under noncrossing ordering, satisfying (7.18), i.e., each block V_j of $\pi_{(i_1,\ldots,i_n)}$ is the maximal block, consisting only of one number in $\{1, \ldots, N\}$ for all $j = 1, \ldots, \left|\pi_{(i_1,\ldots,i_n)}\right|$.

Example 7.8. For example, if $N = 3$, and $(1, 1, 2, 2, 2, 1, 3)$ is fixed as a 7-tuple, then the corresponding partition $\pi_{(1,1,2,2,2,1,3)}$ in $NC(7)$ has its blocks,

$$(1, 1), (2, 2, 2), (1) \text{ and } (3),$$

i.e.,

$$\pi_{(1,1,2,2,2,1,3)} = \{(1, 1),\ (2, 2, 2),\ (1),\ (3)\} \text{ in } NC(7).$$

Also under same hypothesis, if $(1, 1, 1, 2, 2, 1, 1, 1, 2)$ is fixed as a 9-tuple, then the corresponding partition $\pi_{(1,1,1,2,2,1,1,1,2)}$ is

$$\pi_{(1,1,1,2,2,1,1,1,2)} = \{(1, 1, 1),\ (2, 2),\ (1, 1, 1),\ (2)\}.$$

We call such noncrossing partitions $\pi_{(i_1,\dots,i_n)}$ the *free-depending partition of $\{T_j^K\}_{j=1}^N$ for $(i_1,\dots,i_n)$ in $NC(n)$.*

Therefore by [12] and by (7.17) and (7.18), one has that

$$\begin{aligned}\psi_p\left(T_{i_1}^K T_{i_2}^K \dots T_{i_n}^K\right) &= \sum_{\pi\in NC(n)} k_\pi\left(T_{i_1}^K,\dots,T_{i_n}^K\right)\\ &= \sum_{\pi\in NC(i_1,\dots,i_n)} k_\pi\left(T_{i_1}^K,\dots,T_{i_n}^K\right),\end{aligned}\tag{7.19}$$

because all mixed free cumulants of $\{T_j^K\}_{j=1}^N$ vanish under assumed freeness, where

$$NC(i_1,\dots,i_n) \overset{def}{=} \left\{\theta\in NC(n)\,\middle|\,\theta\leqslant\pi_{(i_1,\dots,i_n)}\right\},$$

where $\pi_{(i_1,\dots,i_n)}$ is the free-depending partition of $\{T_j^K\}_{j=1}^N$ for $(i_1,\dots,i_n)$ in $NC(n)$, and where the inclusion $\leqslant$ on $NC(n)$ is in the sense of [12].

Thus, one can obtain that if the partial isometries $\{T_j^K\}_{j=1}^N$ forms a free family then

$$\psi_p\left(T_{i_1}^K\, T_{i_2}^K\, \dots T_{i_n}^K\right) = \sum_{\pi\in NC(i_1,\dots,i_n)} k_\pi\left(T_{i_1}^K,\dots,T_{i_n}^K\right)$$

by (7.19)

$$= \sum_{V\in\pi_{(i_1,\dots,i_n)}}\left(\sum_{\theta\in NC(|V|)} k_\theta\left(T_{i_1}^K,\dots,T_{i_n}^K\right)\right) = \sum_{V\in\pi_{(i_1,\dots,i_n)}}\psi_{p:V},$$

where

$$\psi_{p:V} = \psi_p\left(T_{i_{k_1}}^K T_{i_{k_2}}^K \dots T_{i_{k_{|V|}}}^K\right),$$

whenever

$$V = (i_{k_1}, i_{k_2},\dots,i_{k_{|V|}}) \text{ in } \pi_{(i_1,\dots,i_n)}.$$

Proposition 7.9. *Let $\{T_j^K\}_{j=1}^N$ be a family of partial isometries on $\mathfrak{H}_p$ with their initial and final projections identified with T^K, satisfying AN 6.6. If this family forms a free family in $(\mathfrak{C}_{K,N}^*,\psi_p)$, then the joint-free-moment computations* (7.19) *becomes*

$$\psi_p\left(T_{i_1}^K T_{i_2}^K \dots T_{i_n}^K\right) = \sum_{V\in\pi_{(i_1,\dots,i_n)}}\psi_{p:V},\tag{7.20}$$

where

$$\psi_{p:V} = \psi_p\left(T_{i_{k_1}}^K \dots T_{i_{k_{|V|}}}^K\right) = \mu_p\left(\left(x_{i_{k_{|V|}}}^{|V|}\right)^{-1} K\cap K\right),$$

whenever $V = (i_{k_1}, i_{k_2}, \dots, i_{k_{|V|}})$ in the free-depending partition $\pi_{(i_1,\dots,i_n)}$ of $(i_1,\dots,i_n)$ in $NC(n)$ for all $(i_1,\dots,i_n)\in\{1,\dots,N\}^n$ and $n\in\mathbb{N}$.

Proof. The proof of (7.20) is done by (7.7) and (7.19), as we have discussed in the above paragraph. □

Example 7.10. Assume again that $N = 3$, and let $\{T_1^K, T_2^K, T_3^K\}$ be a family of partial isometries satisfying both AN 6.0, and the conditions (7.11) and (7.12). Then, one can compute the following free moments as follows:

$$\begin{aligned}&\psi_p\left((T_1^K)^2(T_2^K)^3(T_1^K)(T_3^K)\right)\\&=\psi_p\left((T_1^K)^2\right)+\psi_p\left((T_2^K)^3\right)+\psi_p\left(T_1^K\right)+\psi_p(T_3^K)\end{aligned}$$

by (7.20)

$$=\psi_p\left(\chi_{x_1K_1}^{(2)}\right)+\psi_p\left(\chi_{x_2K_2}^{(3)}\right)+\psi_p\left(\chi_{x_1K_1}\right)+\psi_p\left(\chi_{x_3K_3}\right)$$

by (7.1)

$$\begin{aligned}&=\mu_p\left((x_1^2)^{-1}K\cap K\right)+\mu_p\left((x_2^3)^{-1}K\cap K\right)\\&\quad+\mu_p\left(x_1^{-1}K\cap K\right)+\mu_p\left(x_3^{-1}K\cap K\right)\end{aligned}$$

by (7.7).

Similarly,

$$\begin{aligned}&\psi_p\left((T_1^K)^3(T_2^K)^2(T_1^K)^3(T_2^K)\right)\\&=\psi_p\left((T_1^K)^3\right)+\psi_p\left((T_2^K)^2\right)+\psi_p\left(\left(T_1^K\right)^3\right)+\psi_p(T_2^K)\\&=\mu_p\left(\left(x_1^3\right)^{-1}K\cap K\right)+\mu_p\left(\left(x_2^2\right)^{-1}K\cap K\right)\\&\quad+\mu_p\left(x_1^{-1}K\cap K\right)+\mu_p\left(x_2^{-1}K\cap K\right).\end{aligned}$$

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REFERENCES

[1] I. Cho, *Operators induced by prime numbers*, Methods Appl. Math. **19** (2013) 4, 313–340.

[2] I. Cho, *Representations and corresponding operators induced by Hecke algebras*, Complex. Anal. Oper. Theory, DOI: 10.1007/s11785-014-0418-7, (2014).

[3] I. Cho, *p-adic Banach space operators and adelic Banach space operators*, Opuscula Math. **34** (2014) 1, 29–65.

[4] I. Cho, *Free distributional data of arithmetic functions and corresponding generating functions*, Complex. Anal. Oper. Theory **8** (2014) 2, 537–570.

[5] I. Cho, *Dynamical systems of arithmetic functions determined by primes*, Banach J. Math. Anal. **9** (2015), 173–215.

[6] I. Cho, *Classification on arithmetic functions and corresponding free-moment L-functions*, Bulletin Korean Math. Soc. (2015), to appear.

[7] I. Cho, T. Gillespie, *Free probability on the Hecke algebra*, Complex Anal. Oper. Theory, DOI: 10.1007/s11785-014-0403-1, (2014).

[8] I. Cho, P.E.T. Jorgensen, *Krein-space representations of arithmetic functions dertermined by primes*, Alg. Rep. Theo. **17** (2014) 6, 1809–1841.

[9] T. Gillespie, *Superposition of zeroes of automorphic L-functions and functoriality*, PhD Thesis, Univ. of Iowa, (2010).

[10] T. Gillespie, *Prime number theorems for Rankin-Selberg L-functions over number fields*, Sci. China Math. **54** (2011) 1, 35–46.

[11] F. Radulescu, *Random matrices, amalgamated free products and subfactors of the C^*-algebra of a free group of nonsingular index*, Invent. Math. **115** (1994), 347–389.

[12] R. Speicher, *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory*, Amer. Math. Soc. Mem. **132** (1998) 627.

[13] D. Voiculescu, K. Dykemma, A. Nica, *Free Random Variables*, vol. 1, CRM Monograph Series, 1992.

[14] V.S. Vladimirov, I.V. Volovich, E.I. Zelenov, *p-Adic Analysis and Mathematical Physics*, vol. 1, Ser. Soviet & East European Math., World Scientific, 1994.

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