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This paper is dedicated to Professor Edward Kącki
on the occasion of his 90th birthday.

PARETO OPTIMAL CONTROL PROBLEM AND ITS GALERKIN APPROXIMATION FOR A NONLINEAR ONE-DIMENSIONAL EXTENSIBLE BEAM EQUATION

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Abstract. Our goal is to study the Pareto optimal control system for a nonlinear one-dimensional extensible beam equation and its Galerkin approximation. First we consider a mathematical model of the beam equation which was obtained by S. Woinowsky-Krieger in 1950. Next we consider the Pareto optimal control problem based on this equation. Further, we describe the approximation of this system. We use the Galerkin method to approximate the solution of this control problem with respect to a spatial variable. Based on the standard finite dimensional approximation we prove that as the discretization parameters tend to zero then the weak accumulation point of the solutions of the discrete optimal control problems exist and each of these points is the solution of the original Pareto optimal control problem.

Keywords: nonlinear beam equation, Pareto optimal control, Galerkin approximation.

Mathematics Subject Classification: 49J20, 49M25, 58E17.

1. INTRODUCTION

In this paper, we consider a nonlinear extensible beam model with time and length finite. The one-dimensional nonlinear beam equation,

$$\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial^4 y}{\partial x^4} - \left[\beta + \gamma \int\limits_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx\right] \frac{\partial^2 y}{\partial x^2} = f(t,x) \tag{1.1}$$

was proposed by S. Woinowsky-Krieger [20] for the transverse deflection y at time t and position x along the extensible beam. The time $t \in [0,T]$ for $T < \infty$ and the point of position on beam $x \in [0,l]$ where $l < \infty$ is the length of the beam. The parameters α, β, and γ are positive physical constants dependent of the Young's modulus, the cross-sectional second moment of the area, the density and the cross-sectional area. The nonlinear term represents the change in the tension of the beam due to its extensibility. The term f represents an external load. The initial conditions are

$$y(0,x) = y_0(x) \quad \text{and} \quad \frac{\partial y(0,x)}{\partial t} = y_1(x). \tag{1.2}$$

We consider, from the mechanical point of view, the boundary conditions corresponding to clamped ends

$$y(t,0) = y(t,l) = \frac{\partial y(t,0)}{\partial x} = \frac{\partial y(t,l)}{\partial x} = 0 \tag{1.3}$$

and the boundary conditions corresponding to hinged ends

$$y(t,0) = y(t,l) = \frac{\partial^2 y(t,0)}{\partial x^2} = \frac{\partial^2 y(t,l)}{\partial x^2} = 0. \tag{1.4}$$

For a system governed by the nonlinear beam equation (1.1) with initial conditions (1.2) and boundary conditions (1.3) or (1.4) some results concerning the existence and uniqueness of solutions were published by many researchers. We will give several for example A.S. Ackleh *et al.* [1], J.M. Ball [4], M.L. Oliveira and O.A. Lima [16], D.C. Pereira [17], and their references. Some questions of optimal control problems for the beam equation were studied by M. Barboteu *et al.* [5], M. Galewski [9], I. Hlaváček and J. Lovišek [10], J. Hwang [11], I. Sadek *et al.* [18] and by many others.

The Galerkin approximation methods can be applied to the boundary problems as well as to control systems. The Galerkin method to solve the boundary beams system was investigated in articles [1,4,16,17]. Semidiscrete Galerkin approximation of control problems for linear and nonlinear elliptic, parabolic and second-order evolution equations was studied for example in [2,14,18,19]. In paper [3] the authors studied the effects of two damping parameters for a flexible beam. They used the Galerkin method in such a way that the partial differential equation transforms into ordinary differential equations in the time domain. In our papers [7,8,12] we applied the Galerkin method to different types of distributed parameter systems transforming them to lumped parameter systems. Pareto optimal control problems for distributed parameter systems have been studied for example in [13].

This paper is organized as follows: In Section 2 we analyze the properties of an operator from the control space into the spaces of solutions of equation (1.1). Next, in Section 3 we study the quadratic Pareto optimal control problem. In Section 4, we present the Galerkin approximation of our optimal problem. In Section 5 we prove the convergence of the solutions of discrete optimal problems to the solution for the original one.

2. PRELIMINARIES

Firstly, we establish the existence of a weak solution of equation (1.1) subject to the initial conditions (1.2) and the boundary conditions (1.3) or (1.4). Secondly, we define an operator F acting from the control space U into the space of solutions and we establish that F is Lipschitz continuous and a weakly continuous mapping.

In what follows we use the standard notation for the Lebesgue L^p and Sobolev H^k spaces. For brevity of notation, from now on primes denote differentiation with respect to time t, i.e. $y' = \frac{\partial y}{\partial t}$ and $y'' = \frac{\partial^2 y}{\partial t^2}$, while derivatives with respect to distance x along the beam are written by subscripts x, i.e. $y_x = \frac{\partial y}{\partial x}$ and $y_{xx} = \frac{\partial^2 y}{\partial x^2}$.

We set $S = (0,T)$, $Q = S \times (0,l)$ while $V = H_0^2(0,l)$ for clamped ends or $V = H^2(0,l) \cap H_0^1(0,l)$ (the closed subspace of $H^2(0,l)$) for hinged ends. These spaces are equipped with standard norms. The embedding $V \subset H$ is continuos, dense and compact. Identifying H with its dual we have the evolution triple $V \subset H \subset V^*$ (see [6, p. 391]). The duality pairing $\langle \varphi, \psi \rangle$ of V^* and V is identical with the inner product (φ, ψ) on H if $\varphi \in H$.

Let (see [15, p. 108])

$$L^2(S;W) = \left\{ \omega \colon S \to W \mid \int\limits_S \|\omega(t)\|_W^2 dt < \infty \right\}$$

and

$$L^\infty(S;W) = \left\{ \omega \colon S \to W \mid \operatorname{ess}\sup_{t \in S} \|\omega(t)\|_W < \infty \right\},$$

with the standard norms, where W is any Banach space.

We introduce still following a Sobolev-like space (see [6, pp. 394–395]).

$$\mathcal{W}(S) = \left\{ \omega \in L^2(S;V) \ , \ \omega' \in L^2(S;H) \quad \text{and} \quad \omega'' \in L^2(S;V^*) \right\}$$

with the norm

$$\|\omega\|_{\mathcal{W}} = \|\omega\|_{L^2(S;V)} + \|\omega'\|_{L^2(S;H)} + \|\omega''\|_{L^2(S;V^*)}.$$

We define now a weak (variational) formulation of the equation (1.1) with initial condition (1.2) and boundary conditions (1.3) or (1.4) (see [4, 11]).

Definition 2.1. A function y is said to be a weak solution of the equation (1.1) with the initial condition (1.2) and the boundary conditions (1.3) or (1.4) iff $y \in \mathcal{W}$ and y satisfies the equations:

$$\begin{aligned} &\langle y''(t), \psi \rangle + \alpha(y_{xx}(t), \psi_{xx}) - \left(\beta + \gamma \int\limits_0^l |y_x(t)|^2 dx \right) (y_{xx}(t), \psi) = (f(t), \psi), \\ &\text{for all } \psi \in V \text{ and a.e. } t \in S, \\ &y(0) = y_0 \ \text{ and } \ y'(0) = y_1 \quad \text{for } y_0 \in V, \ y_1 \in H, \end{aligned} \tag{2.1}$$

where $(\varphi, \psi) = \int_0^l \varphi(x)\psi(x)dx$ (the inner product on H).

Remark 2.2.

a) The weak formulation implies that the derivatives of the solution of the equation (1.1) are satisfied in the sense of distributions.
b) The solution y of (2.1) does not have to satisfy the boundary conditions $y_{xx}(0) = y_{xx}(l) = 0$ in any classical sense (see [4]).

We state the following existence theorem (see [4, 11]).

Theorem 2.3. *Let $f \in L^2(Q)$, $y_0 \in V$ and $y_1 \in H$. Then, there exists a unique weak solution y of the initial-boundary problems* (1.1)–(1.4) *with the following regularity $y \in \mathcal{W}(S) \cap L^\infty(S;V)$ and $y' \in L^\infty(S;H)$.*

Let us put in (2.1) $f = g + Bu$, where $g \in L^2(Q)$, $u \in U$ (the control space) and $B \in \mathcal{L}(U; L^2(Q))$. Now the equation (2.1) has a form

$$\begin{aligned} &\langle y''(t), \psi\rangle + \alpha(y_{xx}(t), \psi_{xx}) - \left(\beta + \gamma \int_0^l |y_x(t)|^2 dx\right)(y_{xx}(t), \psi) \\ &= \big(g(t) + (Bu)(t), \psi\big) \quad \text{for all } \psi \in V \text{ and a.e. } t \in S, \\ &y(0) = y_0 \ \text{ and } \ y'(0) = y_1 \quad \text{for } y_0 \in V,\ y_1 \in H. \end{aligned} \tag{2.2}$$

We define a nonlinear operator F from the separable Hilbert space U into a space $X = \prod_{i=1}^{4} L^2(S;H)$ by

$$F(u) = (y, y', y_x, y_{xx}),$$

where y is the unique weak solution of (2.2). The norm in the space X is given by the form

$$\begin{aligned} \|F(u)\|_X^2 &= \int_0^T \left[\|y(t)\|_H^2 + \|y'(t)\|_H^2 + \|y_x(t)\|_H^2 + \|y_{xx}(t)\|_H^2\right] dt \\ &= \int_0^T \left[\|y(t)\|_V^2 + \|y'(t)\|_H^2\right] dt. \end{aligned}$$

Lemma 2.4. *If the assumptions of Theorem 2.3 are satisfied with $f = g + Bu$, where $g \in L^2(Q)$, U is a separable Hilbert space and the operator B is linear and bounded, then the operator F is locally Lipschitz continuous and a weakly continuous map.*

Proof. The proof is split into three steps. Firstly, we prove a priori estimates. We start by deriving some a priori bounds for the solution y and its derivatives. In first part of (2.1) we set $\psi = y'(t)$ and using the integration by parts formula (see [6, p. 397]), we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\|y'(t)\|_H^2 + \alpha\|y_{xx}(t)\|_H^2\right] - \left(\beta + \gamma\|y_x(t)\|_H^2\right)\big(y_{xx}(t), y'(t)\big) = \big(g(t) + (Bu)(t), y'(t)\big). \tag{2.3}$$

By using the formulas:

$$\frac{d}{dt}\|y_x(t)\|_H^2 = -2\big(y_{xx}(t), y'(t)\big)$$

and

$$\frac{d}{dt}\|y_x(t)\|_H^4 = -4\|y_x(t)\|_H^2\big(y_{xx}(t), y'(t)\big)$$

(see [4]) and integrating (2.3) over an arbitrary interval $[0,t] \subset [0,T]$ we obtain

$$\begin{aligned}
&\|y'(t)\|_H^2 + \alpha\|y_{xx}(t)\|_H^2 + \beta\|y_x(t)\|_H^2 + \frac{1}{2}\gamma\|y_x(t)\|_H^4 \\
&= \|y_1\|_H + \alpha\|y_{0xx}\|_H^2 + \beta\|y_{0x}\|_H^2 + \frac{1}{2}\gamma\|y_{0x}\|_H^4 + 2\int_0^t \big(g(s) + (Bu)(s), y'(s)\big)ds.
\end{aligned}$$

From this by Schwartz's and Gronwall's inequalities (see [6, pp. 127–128]) and by $2ab \le \frac{1}{\varepsilon}a^2 + \varepsilon b^2$ for $\varepsilon > 0$ (see [15, p. 112]) we have

$$\|y'(t)\|_H^2 + \|y_{xx}(t)\|_H^2 + \|y_x(t)\|_H^2 \le C_1(1 + \|u\|_U^2) \tag{2.4}$$

for any constant $C_1 > 0$ (C_1 dependent on $\|y_0\|_V$, $\|y_1\|_H$ and $\|g\|_{L^2(Q)}$).

From (2.4) and the Poincaré inequality (see [21, p. 59]) we obtain

$$\|y'(t)\|_H^2 + \|y(t)\|_V^2 \le C_2(1 + \|u\|_U^2) \tag{2.5}$$

for $C_2 > 0$ and a.e. $t \in [0,T]$. Integrating (2.5) over the interval $[0,T]$ we state the following inequality

$$\|F(u)\|_X^2 \le C_3(1 + \|u\|_U^2) \le C_3(1 + \|u\|_U)^2 \tag{2.6}$$

for any constant $C_3 > 0$.

Now, we shall prove that the operator F is a locally Lipschitz's map. Let $u_1, u_2 \in U$. For $i = 1, 2$ we have from (2.1) with $f = g + Bu$

$$\begin{aligned}
&\langle y_i''(t), \psi\rangle + \alpha\big(y_{ixx}(t), \psi_{xx}\big) - \big(\beta + \gamma\|y_{ix}(t)\|_H^2\big)\,(y_{ixx}(t), \psi) \\
&= (g(t) + (Bu_i)(t), \psi) \quad \text{for all } \psi \in V \text{ and a.e. } t \in S, \\
&y_i(0) = y_0 \;\text{ and }\; y_i'(0) = y_1 \quad \text{for } i = 1, 2.
\end{aligned} \tag{2.7}$$

From Theorem 2.3 we know that the problem (2.7) for $i = 1, 2$ has exactly one solution $y_1, y_2 \in \mathcal{W}(S) \cap L^\infty(S; V)$ and $y_1', y_2' \in L^\infty(S; H)$.

From that we have (subtracting the two equations)

$$\begin{aligned}
&\langle y_1''(t) - y_2''(t), \psi\rangle + \alpha\big(y_{1xx}(t) - y_{2xx}(t), \psi_{xx}\big) = \\
&= \Big(\big(\beta + \gamma\|y_{1x}(t)\|_H^2\big)y_{1xx}(t) - \big(\beta + \gamma\|y_{2x}(t)\|_H^2\big)y_{2xx}(t), \psi\Big) \\
&= ((B(u_1 - u_2))(t), \psi) \quad \text{for all } \psi \in V \text{ and for a.e. } t \in S.
\end{aligned} \tag{2.8}$$

Let us put $\psi = y_1'(t) - y_2'(t)$. (2.8) gives

$$\begin{aligned}
&\langle y_1''(t) - y_2''(t), y_1'(t) - y_2'(t)\rangle + \alpha(y_{1xx}(t) - y_{2xx}(t), y_{1xx}'(t) - y_{2xx}'(t)) = \\
&= \beta\,(y_{1xx}(t) - y_{2xx}(t), y_1'(t) - y_2'(t)) \\
&\quad + \big(\gamma\|y_{1x}(t)\|_H^2\ y_{1xx}(t) - \gamma\|y_{2x}(t)\|_H^2\ y_{2xx}(t), y_1'(t) - y_2'(t)\big) + \\
&\quad + ((B(u_1 - u_2))(t), y_1'(t) - y_2'(t)) \quad \text{for all } \psi \in V \text{ and for a.e. } t \in S.
\end{aligned} \tag{2.9}$$

Now we take up the nonlinear part of (2.9).

$$\begin{aligned}
&(\gamma\|y_{1x}(t)\|_H^2\ y_{1xx}(t) - \gamma\|y_{2x}(t)\|_H^2\ y_{2xx}(t), y_1'(t) - y_2'(t)) \\
&= (\gamma\|y_{1x}(t)\|_H^2(y_{1xx}(t) - y_{2xx}(t)) \\
&\quad + \gamma\left(\|y_{1x}(t)\|_H^2 - \|y_{2x}(t)\|_H^2\right)y_{2xx}(t), y_1'(t) - y_2'(t)\big) \\
&= \gamma\|y_{1x}(t)\|_H^2(y_{1xx}(t) - y_{2xx}(t), y_1'(t) - y_2'(t)) + \\
&\quad + \gamma(\|y_{1x}(t)\|_H - \|y_{2x}(t)\|_H)(\|y_{1x}(t)\|_H \\
&\quad + \|y_{2x}(t)\|_H)(y_{2xx}(t), y_1'(t) - y_2'(t)) \\
&\leq \gamma\|y_{1x}(t)\|_H^2\|y_{1xx}(t) - y_{2xx}(t)\|_H\|y_1'(t) - y_2'(t)\|_H + \\
&\quad + \gamma(\|y_{1x}(t)\|_H + \|y_{2x}(t)\|_H)\|y_{2xx}(t)\|(\|y_{1x}(t)\|_H \\
&\quad - \|y_{2x}(t)\|_H)\|y_1'(t) - y_2'(t)\| \\
&\leq \gamma\|y_{1x}(t)\|_H^2(\|y_{1xx}(t) - y_{2xx}(t)\|_H^2 + \|y_1'(t) - y_2'(t)\|_H^2) \\
&\quad + \gamma(\|y_{1x}(t)\|_H + \|y_{2x}(t)\|_H)\,\|y_{2xx}(t)\|\,(\|y_{1x}(t) - y_{2x}(t)\|_H^2 \\
&\quad + \|y_1'(t) - y_2'(t)\|_H^2) \\
&\leq C_4(\|y_1'(t) - y_2'(t)\|_H^2 + \|y_{1xx}(t) - y_{2xx}(t)\|_H^2),
\end{aligned} \tag{2.10}$$

where $C_4 > 0$ is a constant depending only on the data and the imbedding constant from Sobolev spaces H^2 into H^1.

Finally, combining (2.10) with (2.9) we arrive at the following equality

$$\begin{aligned}
&\tfrac{d}{dt}\left[\|y_1'(t) - y_2'(t)\|_H^2 + \alpha\|y_{1xx}(t) - y_{2xx}(t)\|_H^2\right] \leq \\
&\leq C_5(\|y_1'(t) - y_2'(t)\|_H^2 + \|y_{1xx}(t) - y_{2xx}(t)\|_H^2) + \|(B(u_1 - u_2))(t)\|_H^2
\end{aligned} \tag{2.11}$$

for a.e. $t \in S$ and a constant $C_5 > 0$.

By integration (2.11) over an arbitrary interval $[0, t] \subset [0, T]$ we obtain

$$\begin{aligned}
&\|y_1'(t) - y_2'(t)\|_H^2 + \alpha\|y_{1xx}(t) - y_{2xx}(t)\|_H^2 \leq \\
&\leq C_5 \textstyle\int_0^t(\|y_1'(s) - y_2'(s)\|_H^2 + \|y_{1xx}(s) - y_{2xx}(s)\|_H^2)ds + L\|u_1 - u_2\|_U^2
\end{aligned} \tag{2.12}$$

for a.e. $t \in S$ and a constant $C_5 > 0$, where $L > 0$ (because $B \in \mathcal{L}(U; L^2(Q))$.

From (2.12) we can prove that the operator F is a locally Lipschtz map by analogy with the proof of inequality (2.6).

Thirdly, we shall prove that the operator F is a weakly continuous mapping. Let (u_n) denote a sequence such that

$$u_n \to \overline{u} \quad \text{weakly in } U. \tag{2.13}$$

Let $y_n = y(u_n)$ satisfy equation (2.1) with $f = g + Bu_n$, i.e.

$$\begin{aligned}&\langle y_n''(t), \psi\rangle + \alpha\big(y_{nxx}(t), \psi_{xx}\big) - \big(\beta + \gamma\|y_{nx}(t)\|_H^2\big)\big(y_{nxx}(t), \psi\big)\\ &= \big(g(t) + (Bu_n)(t), \psi\big) \quad \text{for all } \psi \in V \text{ and a.e. } t \in S,\\ &y_n(0) = y_0 \text{ and } y_n'(0) = y_1.\end{aligned} \tag{2.14}$$

From Theorem 2.3 we know that problem (2.14) has exactly one weak solution y_n for $n \in N$. From the assumptions of the Lemma and from the first part of the proof we obtain

$$\|y_n'(t)\|_H^2 + \|y_n(t)\|_V^2 \le C_6\big(1 + \|u_n\|_U^2\big) \tag{2.15}$$

for certain $C_6 > 0$ and a.e. $t \in [0, T]$. From the last inequality it follows that there exists a subsequence, that we also denote (y_n), converging weakly to an element $\overline{y}$ in $L^2(S, V)$ and strongly to $\overline{y}$ in $L^2(S; H)$ since the embedding $V \subset H$ is compact. From (2.15) we obtain also that the subsequence of derivatives (y_n') converges weakly to $\overline{y'}$ in $L^2(S, H)$ and the nonlinear term $\|y_{nx}\|^2 y_{nxx}$ converges weakly to $\|\overline{y}_x\|^2 \overline{y}_{xx}$ in $L^2(S; H)$ (see proof of Theorem 1 in [4]). These convergences show that the function $\overline{y}$ verifies the first part of (2.1).

It remains to be shown that the initial conditions (1.2) are satisfied by $\overline{y}$ (the second part of (2.1)).

As $y_n \to \overline{y}$ and $y_n' \to \overline{y}'$ weakly in $L^2(S; H)$ and $(y_n''(t), \varphi) \to (\overline{y}_n''(t), \varphi)$ for all $\varphi \in D(0, T)$ are satisfied then $y_n(0) \to \overline{y}(0) = y_0$ and $y_n'(0) \to \overline{y}'(0) = y_1$ (see proof Theorem 1 in [4]). We conclude that $\overline{y}$ is the solution of (2.1) for $u = \overline{u}$. From the fact that there is only one solution of problem (2.1), we deduce that not only a subsequence, but the original sequence (y_n) converges weakly to $\overline{y}$. Also the sequences (y_n'), (y_{nx}), (y_{nxx}) converge weakly in $L^2(S; H)$ to $\overline{y}'$, $\overline{y}_x$, $\overline{y}_{xx}$, respectively. This completes the proof of Lemma 2.4. □

Remark 2.5. Lemma 2.4 gives continuous dependence of the state of the system from the control variable.

3. QUADRATIC PARETO OPTIMAL CONTROL PROBLEM

Many engineering and economic applications such as optimal design problems, environmental control problems, production problems can lead to an optimal control formulation where several objective functions that need to be optimized simultaneously. The statement of the Pareto optimal problems for distributed parameter systems are defined for example in [13, pp. 28–41].

We consider the following objective functions:

$$J_1(u) = \int_0^T \int_0^l |y(t,x) - y_d|^2 dxdt = \|y - y_d\|^2_{L^2(Q)},$$

$$J_2(u) = \int_0^T \int_0^l |y'(t,x) - y_d^1|^2 dxdt = \|y' - y_d^1\|^2_{L^2(Q)},$$

$$J_3(u) = \int_0^T \int_0^l |y_x(t,x) - y_d^2|^2 dxdt = \|y_x - y_d^2\|^2_{L^2(Q)},$$

$$J_4(u) = \int_0^T \int_0^l |y_{xx}(t,x) - y_d^3|^2 dxdt = \|y_{xx} - y_d^3\|^2_{L^2(Q)},$$

$$J_5(u) = \|u\|^2_U.$$

The function $y = y(u)$ is the weak solution of the state equations

$$\begin{aligned} &\langle y''(t), \psi\rangle + \alpha\big(y_{xx}(t), \psi_{xx}\big) - \big(\beta + \gamma\|y_x(t)\|^2\big)\big(y_{xx}(t), \psi\big) \\ &= \big(g(t) + (Bu)(t), \psi\big) \text{ for all } \psi \in V \text{ and a.e. } t \in S, \\ &y(0) = y_0 \text{ and } y'(0) = y_1. \end{aligned} \tag{3.1}$$

The $u \in U$ is a control and $y_d,\ y_d^1,\ y_d^2,\ y_d^3,\ y_0,\ y_1$ are desired functions with respective spaces.
The objective funtionals J_i for $i \in N_5 = \{1,2,3,4,5\}$ represent the different types of energy of the beam.

Problem (P). We study the following Pareto optimal control problem:
minimize the vector objective function

$$J(u) = \big(J_1(u), J_2(u), J_3(u), J_4(u), J_5(u)\big) \tag{3.2}$$

for $u \in U$, where $y = y(u)$ is a unique solution of equation (3.1).

Definition 3.1. A control $u^\circ \in U$ is called a Pareto optimal control for Problem (P) iff there is no $u^1 \in U$ and $u^1 \neq u^\circ$ such that $J_i(u^1) \leq J_i(u^\circ)$ for $i \in N_5$ with strict inequality for at least one $i \in N_5$ (see [13, p. 14] et next).

In general, the multiobjective optimization problems are usually solved by scalarization. The method of scalar optimization depends on certain auxiliary parameters λ_i for $i \in N_5$ (see [13, pp. 14–15]).

With Problem (P) the following scalar one is associated:

Problem (S). We study the following optimal control problem:
minimize the scalar function

$$J_\lambda(u) = \sum_{i=1}^{5} \lambda_i J_i(u) \tag{3.3}$$

for $u \in U$, $\lambda_i > 0$ for $i \in N_5$ and $\sum_{i=1}^{5} \lambda_i = 1$, where $y = y(u)$ is a unique solution of the state equations (3.1).

The objective functional J_λ represents the total energy of the beam.

Remark 3.2. In general, we can consider the Pareto and scalar control problems with the admissible set of controls $U_{ad} \subset U$, where U_{ad} is convex, closed with non-empty interior. The theorems and proofs can be easily extended for this case.

Lemma 3.3. *If u° is the solution of Problem (S) with fixed $\lambda_i > 0$ for $i \in N_5$ and $\sum_{i=1}^{5} \lambda_i = 1$, then u° is the Pareto optimal solution to Problem (P).*

The proof is immediate.

Theorem 3.4. *Let $g \in L^2(Q)$, $y_0 \in V$, $y_1 \in H$, and the operator $B \in \mathcal{L}(U; L^2(Q))$, where U is a separable Hilbert space and y_d, y_d^1, y_d^2, $y_d^3 \in L^2(Q)$, then the scalar control Problem (S) has at least one optimal solution $u^\circ \in U$ such that $J_\lambda(u^\circ) = \inf_{u \in U} J_\lambda(u)$.*

Proof. Let (u_n) be a minimizing sequence for functional (3.3), i.e. $u_n \in U$ for $n \in N$ and $\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in U} J_\lambda(u)$.

Since the functional J_λ is coercive, then the sequence (u_n) is bounded in U (see [15, p. 14]). Therefore, there exists a subsequence, which we also denote by (u_n) such that $u_n \to v$, weakly in U. Let $y_n = y(u_n)$ be the solution of (3.1) for $u = u_n$. Lemma 2.4 implies that the sequences of weak solutions y_n corresponding to controllers u_n satisfy the convergences:

$$\begin{aligned} y_n &\longrightarrow y = y(v) \quad \text{strongly in } L^2(S;H),\\ y_n' &\longrightarrow y' = y'(v) \quad \text{weakly in } L^2(S;H),\\ y_{nx} &\longrightarrow y_x = y_x(v) \quad \text{weakly in } L^2(S;H) \end{aligned}$$

and

$$y_{nxx} \longrightarrow y_{xx} = y_{xx}(v) \quad \text{weakly in } L^2(S;H).$$

Since the norm is weakly lower semicontinuous, we get

$$\inf_{u\in U} J_\lambda(u_n) = \lim_{n\to\infty} J_\lambda(u_n) = \lim_{n\to\infty} \inf J_\lambda(u_n) \geq J_\lambda(v).$$

From this $J_\lambda(v) = \inf_{u\in U} J_\lambda(u) = J_\lambda(u^\circ)$. $\square$

4. GALERKIN APPROXIMATION OF THE PARETO OPTIMAL CONTROL PROBLEM

Here we recall some known results concerning the finite dimensional Galerkin approximation (see [21, p. 271]). They are basic for the convergence analysis of our optimal problem.

We consider a family $\{V_n\}_{n\in G}$ of finite dimensional subspaces of V which satisfies the following conditions:

$$\forall h_1, h_2 \in G \quad (h_1 > h_2 \Longrightarrow V_{h_1} \subset V_{h_2}) \quad \text{and} \quad \overline{\bigcup_{h\in G} V_h} = V, \tag{4.1}$$

where the set $G \subset (0,1]$ of parameters h has an accumulation point at 0. The approximation of space H is the same family $\{V_h\}_{h\in G}$ with an induced norm with H. The approximation of the spaces $L^2(S;V)$ and $L^2(S;H)$ is understood here as a family of spaces $\{L^2(S;V_h)\}_{h\in G}$ from respective norms.

As an approximate solution of equations (3.1) we mean the family of functions $y_h \in L^2(S;V_h)$ which are the solutions of the following system

$$\begin{aligned}&\langle y_h''(t), \psi_h\rangle + \alpha\big(y_{hxx}(t), \psi_{hxx}\big) - \big(\beta + \gamma\|y_{hx}(t)\|_H^2\big)\big(y_{hxx}(t), \psi_h\big)\\ &= \big(g(t) + (Bu)(t), \psi_h\big) \quad \text{for all } \psi_h \in V_h \text{ and for a.e. } t \in S,\\ &y_h(0) = y_{0h} \text{ and } y_h'(0) = y_{1h},\end{aligned} \tag{4.2}$$

where y_{0h} and y_{1h} are the orthogonal projections y_0 and y_1 onto V_h with the respective norms. From Theorem 2.3 we conclude that for each $h \in G$ the equation (4.2) has a unique solution $y_h \in L^2(S;V_h)$.

As an approximation of control space U we take a family $\{U_k\}_{k\in K}$ of finite dimensional subspaces of U which satisfies the following conditions:

$$\forall k_1, k_2 \in K \quad (k_1 > k_2 \Longrightarrow U_{k_1} \subset U_{k_2}) \quad \text{and} \quad \overline{\bigcup_{k\in K} U_k} = U, \tag{4.3}$$

where the set $K \subset (0,1]$ of parameters k has an accumulation point at 0.

Our objective approximated functionals have the following forms

$$\begin{aligned}J_{1h}(u_k) &= \|y_h - y_{dh}\|^2_{L^2(Q)},\\ J_{2h}(u_k) &= \|y_h' - y^1_{dh}\|^2_{L^2(Q)},\\ J_{3h}(u_k) &= \|y_{hx} - y^2_{dh}\|^2_{L^2(Q)},\\ J_{4h}(u_k) &= \|y_{hxx} - y^3_{dh}\|^2_{L^2(Q)},\\ J_{5k}(u_k) &= \|u_k\|^2_U,\end{aligned}$$

where y_{dh}, y^1_{dh}, y^2_{dh}, y^3_{dh} are the orthogonal projections of the elements y_d, y^1_d, y^2_d, y^3_d, respectively, onto the space $L^2(S;V_h)$ with the norm from $L^2(S;H)$ and $u_k \in U_k$.

We have to study the following approximated scalar control Problem (S_{hk}): find a minimizer $u^\circ_{kh} \in U_k$ such that

$$\inf_{u_k \in U_k} J_{\lambda hk}(u_k) = J_{\lambda hk}(u^\circ_{kh}),$$

where $J_{\lambda hk}(u_k) = \sum_{i=1}^{4} \lambda_i J_{ih}(u_k) + \lambda_5 J_{5k}(u_k)$ and $y_{hk} = y_h(u_k)$ is a solution of the equation (4.2) for $u = u_k \in U_k$. In other words y_{hk} is the approximate state associated witch control $u_k \in U_k$. The scalar controls Problems (S_{hk}) are the lumped parameter systems.

The optimal solution of the approximate Problem (S_{hk}) can be characterized analogously to the considered continuos Problem (S) in Section 3.

Theorem 4.1. *Under the assumptions of Theorem 3.4, the approximated scalar control Problem (S_{hk}) has at least one solution $u^\circ_{kh} \in U_k$.*

The theorem can be proved in the same way as Theorem 3.4.

5. CONVERGENCE OF THE SOLUTIONS OF APPROXIMATED PROBLEMS TO THE ORIGINAL

In this section, we prove the main result of our paper, convergence of solutions of approximated Problems (S_{hk}) to the original Problem (S). We start with a Lemma, whose proof follows immediately from Lemma 2.4 and from our assumptions of Galerkin approximations.

Lemma 5.1. *Let (u_k) be any sequence of elements in U_k and (y_{hk}) be the sequence of solutions of system* (4.2) *for $u = u_k$. Let the assumptions of Lemma 2.4 and the properties of Galerkin approximations* (4.1) *and* (4.3) *be satisfied.*

(i) *If $u_k \xrightarrow[k\to 0]{} \overline{u}$ weakly in U, then*

$$y_{hk} \xrightarrow[h,k\to 0]{} \overline{y} \quad \text{weakly in } L^2(S;V),$$
$$y'_{hk} \xrightarrow[h,k\to 0]{} \overline{y}' \quad \text{weakly in } L^2(S;H),$$
$$y_{hkx} \xrightarrow[h,k\to 0]{} \overline{y}_x \quad \text{weakly in } L^2(S;H)$$

and

$$y_{hkxx} \xrightarrow[h,k\to 0]{} \overline{y}_{xx} \quad \text{weakly in } L^2(S;H).$$

(ii) *If $u_k \xrightarrow[k\to 0]{} \overline{u}$ strongly in U, then*

$$y_{hk} \xrightarrow[h,k\to 0]{} \overline{y} \quad \text{strongly in } L^2(S;V),$$

where the function $\overline{y}$ is the unique solution of system (3.1) *for $u = \overline{u}$.*

Now, we have to analyse the question of convergence of approximate solutions u°_{kh} of Problem (S_{hk}) to solution u° of Problem (S).

Theorem 5.2. *Let the assumptions of Theorem 2.3 and approximated conditions* (4.1) *and* (4.3) *be satisfied. Then there exist weak condensation points of a set of solutions of the scalar control Problems* (S_{hk}) *in* U *and each of these points is the solution of the scalar optimal Problem* (S).

Proof. The proof is split into two steps. First, we have to prove that the sequence (u°_{kh}) of solutions of the scalar control Problem (S_{hk}) is a minimizing sequence for functional (3.3). Indeed. For $u^\circ \in U$, the solution of the scalar control Problem (S), according to (4.3), there exists a sequence (u°_k) such that $u^\circ_k \in U_k$ for all $k \in K$, and $u^\circ_k \xrightarrow[k\to 0]{} u^\circ$ strongly in U. By Lemma 5.1 (second part), the solution of (4.2), corresponding to the control $u = u^\circ_k$, $y^\circ_{hk} = y_h(u^\circ_k) \xrightarrow[h,k\to 0]{} y^\circ$ strongly in $L^2(S;V)$, where $y^\circ = y(u^\circ)$ is the solution of (3.1) corresponding to the control $u = u^\circ \in U$. Then, as

$$\inf_{u\in U} J_\lambda(u) = J_\lambda(u^\circ) \le J_\lambda(u^\circ_{kh}) \le J_\lambda(u^\circ_k)$$

and functional J_λ is continuous, we have $\lim\limits_{k,h\to 0} J_\lambda(u^\circ_{kh}) = J_\lambda(u^\circ)$. Hence we obtain that (u°_{kh}) is a minimizing sequence for functional J_λ.

Second, we have to prove that the sequence (u°_{kh}) has a subsequence weakly convergent to one of the solutions of the scalar control Problem (S). Since functional (3.3) is coercive, then the sequence (u°_{kh}) is bounded in U. There is a subsequence, denoted again by (u°_{kh}), such that $u^\circ_{kh} \xrightarrow[k,h\to 0]{} \overline{u}$ weakly in U. Then Lemma 5.1 (first part) implies that the solutions of (3.1) corresponding to controls u°_{kh} are convergent weakly to $\overline{y} = y(\overline{u})$ and their derivatives, too. Since, the norm is functional weakly lower-semicontinuous,

$$\inf_{u\in U} J_\lambda(u) = \lim_{k,h\to 0} J_\lambda(u^\circ_{kh}) = \lim_{k,h\to 0} \inf J_\lambda(u^\circ_{kh}) \ge J_\lambda(\overline{u}).$$

This implies that $\overline{u}$ is one of solutions of the scalar control Problem (S). □

Remark 5.3. From Lemma 3.3 we obtain that the results of Theorem 5.2 are also the conclusion to our Pareto optimal Problem (P).

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