# EXISTENCE THEOREMS OF NONLINEAR ASYMPTOTIC BVP FOR A HOMEOMORPHISM

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**Abstract.** In this work, we are concerned with the existence of solutions for the following $\varphi$ -Laplacian boundary value problem on the half-line

$$(\varphi(x'))' = f(t,x,x'), \quad x(0) = 0, \quad x'(\infty) = 0,$$

where $f : \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous. The results are proved using the properties of the Leray-Schauder topological degree.

**Keywords:** half-line, nonlinear, asymptotic boundary value problem, $\varphi$-Laplacian, Leray-Schauder degree.

**Mathematics Subject Classification:** 34B15, 34B40.

## 1. INTRODUCTION

Our aim is to study the existence of solutions for the following system of $k$ BVPs

$$(\varphi(x'))' = f(t,x,x'), \quad x(0) = 0, \quad x'(\infty) = 0, \tag{1.1}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous.

The function $\varphi : \mathbb{R}^k \to \mathbb{R}^k$ defines various boundary value problems associated with Laplacian-type operators. First, we shall assume that

$$\varphi(s) = \begin{cases} \frac{\beta(|s|)}{|s|} s, & s \neq 0, \\ 0, & s = 0, \end{cases} \tag{1.2}$$

where $\beta : [0,\infty) \to [0,\infty)$ is continuous, increasing, $\beta(0) = 0$ and $\beta(\infty) = \infty$. Observe that $\varphi$ is a generalization of a $p-$Laplacian operator of the form: $\psi_p(s) = |s|^{p-2}s$, for $s \neq 0$, $\psi_p(0) = 0$, $s \in \mathbb{R}^k$, $p > 1$. The function $\varphi$ defined by (1.2) has two properties:

 

strict monotonicity and coercivity and thus is a homeomorphism from $\mathbb{R}^k$ onto $\mathbb{R}^k$ (comp. [10, 11]).

In the second case,

$$\varphi(s) = (\varphi_1(s_1), \ldots, \varphi_k(s_k)) \tag{1.3}$$

is such that $\varphi_i(s_i)$ is a one dimensional increasing homeomorphism with $\varphi_i(0) = 0$, $i = 1, \ldots, k$. In this case $\varphi$ contains the following version of a $p-$Laplacian operator: $\varphi(s) = (\varphi_{p_1}(s_1), \ldots, \varphi_{p_k}(s_k))$, $i = 1, \ldots, k$, $s \in \mathbb{R}^k$, $p_i > 1$ and $\varphi_{p_i} : \mathbb{R} \to \mathbb{R}$ is the one dimensional $p_i$-Laplacian.

The BVP (1.1) with $\varphi(s) = s$ has been extensively studied in the literature. For instance, in [12] the authors established the existence of unbounded solutions. Results for problems where the nonlinearity may change sign one can find for example in [8, 14]. In [13], the asymptotic boundary condition $x'(\infty) = 0$ is replaced by $x \in H^2(\mathbb{R}_+)$. In [1–4, 6] authors also obtained some existence results for such problems. By applying a diagonalization procedure, in [5] authors established the existence of bounded solutions.

Recent papers have also investigated the case of the so-called $p$-Laplacian operator $\varphi(s) = |s|^{p-2}s$, $p > 1$ (see for instance [9]).

In [7], authors considered a homeomorphism $\varphi$ and proved the existence of at least one positive solution by application of the method of upper and lower solutions.

Known results for the BVP (1.1) refer to the scalar case. The problem (1.1) with $\varphi$ given by (1.2) has not been studied so far. In the case when $\varphi$ is given by (1.3) our assumptions are of a completely different kind. The most important here is the choice of the space (different than in the cited papers), which enabled us to get the existence under only two conditions: a linear growth condition and a sign condition for the nonlinear term $f$.

## 2. PRELIMINARIES

Throughout the paper $|\cdot|$ will denote the Euclidean norm on $\mathbb{R}^k$ (or alternatively on $\mathbb{R}$), while the scalar product in $\mathbb{R}^k$ corresponding to the Euclidean norm will be denoted by $(\cdot\,|\,\cdot)$. Let $\mathbb{R}_+ := [0, \infty)$. Denote by $C^1(\mathbb{R}_+, \mathbb{R}^k)$ the Banach space of all continuous functions $x : \mathbb{R}_+ \to \mathbb{R}^k$ which have continuous first derivatives $x'$.

In order to apply known topological methods, we need an appropriate Banach space. Let

$$\mathbb{X} = \left\{ x \in C^1(\mathbb{R}_+, \mathbb{R}^k) \;\middle|\; x(0) = 0,\ \lim_{t\to\infty} x'(t) = 0 \right\}$$

with the norm

$$\|x\|_{\mathbb{X}} = \sup_{t\in\mathbb{R}_+} |x'(t)|.$$

**Remark 2.1.** Notice that the above norm is actually of the form

$$\|x\|_{\mathbb{X}} = \max\left\{ |x(0)|, \sup_{t\in\mathbb{R}_+} |x'(t)| \right\}$$

but, since $x(0) = 0$, it is simplified.

The convergence of the sequence $(x_n)$ in the space $\mathbb{X}$ means: $(x_n|_K)$ is uniformly convergent for any compact set $K \subset [0,\infty)$ and $(x_n')$ is uniformly convergent.

Observe that $\mathbb{X}$ is a space of functions such that: if $x \in \mathbb{X}$ and $\|x\|_{\mathbb{X}} = M$, then

$$|x(t)| \leq Mt, \tag{2.1}$$

for any $t \in [0,\infty)$. Indeed, we have

$$|x(t)| \leq t \sup_{t\in\mathbb{R}_+} |x'(t)| + |x(0)| \leq Mt.$$

The following theorem gives a compactness criterion in $\mathbb{X}$:

**Theorem 2.2** ([14])**.** *For a set $A \subset \mathbb{X}$ to be relatively compact, it is necessary and sufficient that:*

(1) *there exists $M > 0$ that for any $x \in A$ and $t \in [0,\infty)$ we have $|x'(t)| \leq M$;*
(2) *for each $d > 0$, the family $A_d := \left\{ x'\big|_{[0,d]} : x \in A \right\}$ is equicontinuous;*
(3) *for any $\varepsilon > 0$ there exists $S > 0$ such that for all $t \geq S$ and $x \in A$ we have $|x'(t)| \leq \varepsilon$.*

Now, let us consider the asymptotic BVP (1.1).

By a solution to the problem (1.1) we mean a function $x \in \mathbb{X}$ with $\varphi(x') \in C^1(\mathbb{R}_+, \mathbb{R}^k)$, which satisfies the equation of (1.1) on $(0,\infty)$.

The following assumptions will be needed throughout the paper:

(i) $f : \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous;
(ii) $|f(t,x,y)| \leq a(t)|x| + b(t)|y| + c(t)$, where $a, b, c$ are nonnegative functions and $\int_0^\infty sa(s)\,ds < \infty$, $\int_0^\infty b(s)\,ds < \infty$, $\int_0^\infty c(s)\,ds < \infty$.

The BVP (1.1) is nonresonant, i.e. for $f = 0$, there is no nontrivial solutions. Hence, the problem is invertible.

Integrating both sides of equation $(\varphi(x'))' = f(t,x,x')$ from 0 to $t$, we get

$$\varphi(x'(t)) = \varphi(x'(0)) + \int_0^t f(s,x(s),x'(s))ds.$$

Since $x'(\infty) = 0$, we get

$$\varphi(x'(0)) = -\int_0^\infty f(s,x(s),x'(s))ds.$$

Hence, we obtain

$$x'(t) = \varphi^{-1}\left( -\int_t^\infty f(s,x(s),x'(s))ds \right). \tag{2.2}$$

Now, integrating (2.2) from 0 to $t$, we have

$$x(t) = \int_0^t \varphi^{-1}\left(-\int_s^\infty f(u, x(u), x'(u))du\right) ds.$$

Now, is easy to see that the following lemma holds:

**Lemma 2.3.** *Let* (i) *hold. A function $x \in \mathbb{X}$ is a solution to the problem* (1.1) *if and only if $x$ satisfies the following integral equation*

$$x(t) = \int_0^t \varphi^{-1}\left(-\int_s^\infty f(u, x(u), x'(u))du\right) ds.$$

Let

$$\int_0^\infty sa\,(s)\,ds = M_1, \quad \int_0^\infty b\,(s)\,ds = M_2, \quad \int_0^\infty c\,(s)\,ds = M_3. \tag{2.3}$$

Hence, under assumption (ii), (2.1) and (2.3), we get

$$\begin{aligned} &\left|-\int_t^\infty f\left(s, x\left(s\right), x'\left(s\right)\right) ds\right| \leq \\ &\leq \int_0^\infty a\left(s\right)\left|x\left(s\right)\right| ds + \int_0^\infty b\left(s\right)\left|x'\left(s\right)\right| ds + \int_0^\infty c\left(s\right) ds \leq \\ &\leq M \int_0^\infty sa\left(s\right) ds + M \int_0^\infty b\left(s\right) ds + \int_0^\infty c\left(s\right) ds \leq \\ &\leq M\left(M_1 + M_2\right) + M_3 < \infty. \end{aligned} \tag{2.4}$$

Now, let $T : [0,1] \times \mathbb{X} \to \mathbb{X}$ be such that

$$T(\lambda, x)\,(t) := \int_0^t \varphi^{-1}\left(-\lambda \int_s^\infty f(u, x(u), x'(u))du\right) ds. \tag{2.5}$$

Then

$$T(\lambda, x)'\,(t) = \varphi^{-1}\left(-\lambda \int_t^\infty f(s, x(s), x'(s))ds\right). \tag{2.6}$$

The functions $T, (T)'$ are continuous. Moreover, $(T(\lambda, x))(0) = 0$ and $(T(\lambda, x))'\,(\infty) = 0$. Finally, by (2.4), it follows that the operator $T$ is well-defined.

Let $x_n \subset \mathbb{X}$, $(x_n) \to x$ and $\lambda_n \to \lambda$. Observe that

$$\int_0^\infty \left|\lambda_n f\left(s, x_n\left(s\right), x_n'\left(s\right)\right) - \lambda f\left(s, x\left(s\right), x'\left(s\right)\right)\right| ds < \infty,$$

which is clear from (2.4). From (2.6), the fact that $\varphi$ is a homeomorphism and the Lebesgue Dominated Convergence Theorem the operator $T$ is continuous.

Now, we shall prove that $T$ is completely continuous.

**Lemma 2.4.** *Under assumptions* (i) *and* (ii) *the operator* $T$ *is completely continuous.*

*Proof.* For the proof is sufficient to show that the image of

$$B := \{(\lambda, x) \in [0,1] \times \mathbb{X} \mid \|x\|_{\mathbb{X}} \leq M\}$$

under $T$ is relatively compact.

First, observe that condition (1) of Theorem 2.2 holds true. Indeed, from (2.4) we know that there exists $L > 0$ such that for any $x \in B$, $t \in [0, \infty)$ and $\lambda \in [0,1]$ we have $\left|(T(\lambda, x))'(t)\right| \leq L$.

Now, we will prove condition (2). By assumption (ii) we get that for every $\epsilon > 0$ there exists $\delta_1 > 0$ such that, if $|t - t_0| < \delta_1$ then $\int_{\min\{t_0,t\}}^{\max\{t_0,t\}} sa\left(s\right) ds < \frac{\epsilon}{3M}$, $\delta_2 > 0$ such that, if $|t - t_0| < \delta_2$ then $\int_{\min\{t_0,t\}}^{\max\{t_0,t\}} b\left(s\right) ds < \frac{\epsilon}{3M}$, and $\delta_3 > 0$ such that, if $|t - t_0| < \delta_3$ then $\int_{\min\{t_0,t\}}^{\max\{t_0,t\}} c\left(s\right) ds < \frac{\epsilon}{3}$. Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Hence, we obtain

$$\begin{aligned}
&\left| -\int_t^\infty \lambda f\left(s, x\left(s\right), x'\left(s\right)\right) ds + \int_{t_0}^\infty \lambda f\left(s, x\left(s\right), x'\left(s\right)\right) ds \right| \\
&= \left| \int_{t_0}^t \lambda f\left(s, x\left(s\right), x'\left(s\right)\right) ds \right| \leq \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} \left|f\left(s, x\left(s\right), x'\left(s\right)\right)\right| ds \\
&\leq M \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} sa\left(s\right) ds + M \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} b\left(s\right) ds + \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} c\left(s\right) ds \\
&< M\frac{\epsilon}{3M} + M\frac{\epsilon}{3M} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

As $\varphi$ is a homeomorphism, one can see that $(Tx)'$ is equicontinuous on $[0, d]$.

It remains to prove condition (3). By assumption (ii) for every $\epsilon > 0$ there exists $t_1$, $t_2$, $t_3$ large enough and such that

$$\int_{t_1}^\infty sa\left(s\right) ds < \frac{\epsilon}{3M}, \quad \int_{t_2}^\infty b\left(s\right) ds < \frac{\epsilon}{3M} \quad \int_{t_3}^\infty c\left(s\right) ds < \frac{\epsilon}{3}.$$

Let $S = \max\{t_1, t_2, t_3\}$. For $t \geq S$ we get

$$\left| -\int_t^\infty \lambda f(s, x(s), x'(s))\, ds \right| \leq M \int_S^\infty s\, a(s)\, ds + M \int_S^\infty b(s)\, ds + \int_S^\infty c(s)\, ds$$
$$< M\frac{\epsilon}{3M} + M\frac{\epsilon}{3M} + \frac{\epsilon}{3} = \epsilon.$$

Since $\varphi$ is a homeomorphism, we get condition (3) of Theorem 2.2, what completes the proof. □

## 3. EXISTENCE THEOREMS

**Theorem 3.1.** *Let assumptions* (i)–(ii) *hold. Moreover, assume that*

(iii) *there exists* $M > 0$ *such that* $(y \mid f(t, x, y)) > 0$ *for* $t \geq 0$ , $x, y \in \mathbb{R}^k$ *and* $|y| \geq M$.

*Then the problem* (1.1) *with* $\varphi$ *given by* (1.2) *has at least one solution.*

*Proof.* Consider the following family of BVPs:

$$(\varphi(x'))' = \lambda f(t, x, x'), \qquad x(0) = 0, \qquad \lim_{t\to\infty} x'(t) = 0, \tag{3.1}$$

depending on a parameter $\lambda \in [0, 1]$. Then the problem (3.1) is equivalent to an integral equation

$$x(t) := \int_0^t \varphi^{-1}\left(-\lambda \int_s^\infty f(u, x(u), x'(u))du\right) ds.$$

By Lemma 2.4, we get that operator

$$T(\lambda, x)(t) := \int_0^t \varphi^{-1}\left(-\lambda \int_s^\infty f(u, x(u), x'(u))du\right) ds$$

is completely continuous. Let us consider homotopy $H : [0, 1] \times \mathbb{X} \to \mathbb{X}$ given by

$$H(\lambda, x) = x - T(\lambda, x)$$

in the ball $\Omega = B(0, M)$, where $M$ is the positive constant from assumption (iii).

If $H(\lambda, x) = 0$ for $\lambda = 0$ and $x \in \partial\Omega$, then the BVP (3.1) has only a trivial solution, which does not lie on the boundary of $\Omega$, a contradiction.

Assume that $H(\lambda, x) = 0$ for $\lambda \in (0, 1]$ and $x \in \partial\Omega$. Let us consider a function

$$\psi(t) := (\beta(|x'(t)|))^2 = \left(\frac{\beta(|x'(t)|)}{|x'(t)|}x'(t) \,\middle|\, \frac{\beta(|x'(t)|)}{|x'(t)|}x'(t)\right)$$

and observe that $\lim_{t\to\infty}\psi(t) = 0$. Hence $\psi$ has a maximum equal to $\beta^2(M)$ for certain $t_0 \in \mathbb{R}_+$. If $t_0 = 0$, then from assumption (iii) and the fact that $|x'(0)| = M$, we have

$$0 \geq \psi'(0) = 2\lambda \frac{\beta(|x'(0)|)}{|x'(0)|} \left(x'(0) \mid f(0, x(0), x'(0))\right) > 0,$$

a contradiction. If $t_0 > 0$, then by (iii) we also reach a contradiction

$$0 = \psi'(t_0) > 0.$$

Hence homotopy $H$ does not vanish on the boundary of $\Omega$ for $\lambda > 0$. Finally $H(\lambda, x) \neq 0$ for $\lambda \in [0,1]$ and $x \in \partial\Omega$.

Therefore, by the properties of the Leray-Schauder topological degree, we have

$$\deg(I - T(1,\cdot), \Omega) = \deg(H(1,\cdot), \Omega) = \deg(H(0,\cdot), \Omega) = \deg(I, \Omega) = 1 \neq 0.$$

Hence $T(1,\cdot)$ has a fixed point in $\Omega$, what means that the problem (1.1) has at least one solution. □

**Theorem 3.2.** *Let assumptions* (i)–(ii) *hold. Moreover, let $f$ satisfy the following condition*

(iii) *there exists $M_i > 0$ such that $y_i \cdot f_i(t, x, y) > 0$ for $t \geq 0$ , $x, y \in \mathbb{R}^k$ and $|y_i| \geq M_i$, $i = 1, \ldots, k$.*

*Then the problem* (1.1) *with $\varphi$ given by* (1.3) *has at least one solution.*

*Proof.* A part of the proof of this theorem is similar to the proof of Theorem 3.1. Therefore, we consider here only that part of the proof, which differs from the previous one. Set

$$\Omega = \left\{ x \in \mathbb{X} \;\middle|\; \sup_{t\in\mathbb{R}_+} |x_i'(t)| < M_i,\ i = 1, \ldots, k \right\},$$

where $M_i$ are as in (iii).

Assume that $H(\lambda, x) = 0$ for $\lambda \in (0,1]$ and $x \in \partial\Omega$. This means that for some index $i \in \{1, \ldots, k\}$ we have $\sup_{t\in\mathbb{R}_+} |x_i'(t)| = M_i$.

Let us consider a function $x_i'(t)$. If for some $t_0$ the function $x_i'(t)$ has a maximum equal to $\pm M_i$, then $\varphi_i(x_i'(t_0))$ has a maximum too. Hence, we get $(\varphi_i(x_i'(t_0)))' = 0$. On the other hand, by (iii), we have

$$0 = x_i'(t_0)(\varphi_i(x_i'(t_0)))' = \lambda\, x_i'(t_0) f_i(t_0, x(t_0), x'(t_0)) > 0,$$

a contradiction.

Now, let $x_i'(0) = M_i$ and let $x_i'$ be decreasing on a neighborhood of zero. Then, by (iii) we also reach a contradiction. Indeed, we have $(\varphi_i(x_i'(t_0)))' < 0$ and

$$0 > x_i'(0)(\varphi_i(x_i'(0)))' = \lambda\, x_i'(0) f_i(0, x(0), x'(0)) > 0.$$

The proof of the case when $x_i'(0) = -M_i$ follows in the same way.

Finally, the homotopy $H$ does not vanish on the boundary of $\Omega$ for $\lambda\,(0,1]$. □

**Example 3.3.** Let us consider the problem (1.1), where

$$f(t,(x_1,x_2),(y_1,y_2)) = \alpha(t)g(x_1,x_2)\,(y_1+y_2, y_2-y_1+1)\,.$$

Moreover, assume that

— $\alpha$ is positive, continuous and integrable on $[0,\infty)$;
— function $g:\mathbb{R}^2\to\mathbb{R}$ is continuous;
— there exist $l,L>0$ such that the function $g$ satisfies $l\le g(x_1,x_2)\le L$.

Obviously, (i) holds. Moreover, one can see that

$$|f(t,(x_1,x_2),(y_1,y_2))| \le \sqrt{2}L\alpha(t)\,|y| + L\alpha(t).$$

Hence, assumption (ii) is satisfied. It remains to show that assumption (iii) of Theorem 3.1 holds. Indeed, for any $M>1$ and $|y|\ge M$ we get

$$((y_1,y_2)\mid f(t,(x_1,x_2),(y_1,y_2))) \ge l\alpha(t)\left(y_1^2+y_2^2+y_2\right).$$

Observe that $y_1^2+y_2^2+y_2>0$ for $y_2\in(-\infty,-1]\cup[0,\infty)$. If $y_2\in(-1,0)$, we get

$$((y_1,y_2)\mid f(t,(x_1,x_2),(y_1,y_2))) > l\alpha(t)(1+y_2) > 0.$$

Finally, $((y_1,y_2)\mid f(t,(x_1,x_2),(y_1,y_2)))>0$.

Hence, by Theorem 3.1, there exists at least one nontrivial solution of the BVP (1.1).

## REFERENCES


[1] J. Andres, G. Gabor, L. Górniewicz, *Boundary value problems on infinite intervals*, Trans. Amer. Math. Soc. **351** (1999), 4861–4903.

[2] C.Z. Bai, J.X. Fang, *On positive solutions of boundary value problems for second-order functional differential equations on infinite intervals*, J. Math. Anal. Appl. **282** (2003), 711–731.

[3] J.V. Baxley, *Existence and uniqueness of nonlinear boundary value problems on infinite intervals*, J. Math. Anal. Appl. **147** (1990), 127–133.

[4] J.W. Bebernes, L.K. Jackson, *Infinite interval boundary value problems for $y''=f(x,y)$*, Duke Math. J. **34** (1967), 39–47.

[5] L.E. Bobisud, *Existence of positive solutions to some nonlinear singular boundary value problems on finite and infinite intervals*, J. Math. Anal. Appl. **173** (1993), 69–83.

[6] A. Constantin, *On an infinite interval boundary value problem*, Ann. Mat. Pura Appl. **176** (1999) 4, 379–394.

[7] S. Djebali, O. Saifi, *Positive solutions for singular $\varphi$-Laplacian BVPs on the positive half-line*, Electron. J. Qual. Theory Differ. Equ. **56** (2009), 24 pp.

[8] S. Djebali, O. Saifi, *Positive solutions for singular BVPs with sign changing and derivative depending nonlinearity on the positive half line*, Acta Appl. Math. **110** (2010) 2, 639–665.

[9] S. Liang, J. Zhang, *The existence of countably many positive solutions for one-dimensional p-Laplacian with infinitely many singularities on the half line*, Appl. Math. Comput. **201** (2008), 210–220.

[10] A. Lipowski, B. Przeradzki, K. Szymańska-Dębowska, *Periodic solutions to differential equations with a generalized p-Laplacian*, Discrete Contin. Dyn. Syst. Ser. B **19** (2014) 8, 2593–2601.

[11] R. Manásevich, J. Mawhin, *Periodic solutions for nonlinear systems with p-Laplacian-like operators*, J. Diff. Equ. **145** (1998), 367–393.

[12] D. O'Regan, B. Yan, R.P. Agarwal, *Solutions in weighted spaces of singular boundary value problems on the half-line*, J. Comput. Appl. Math. **205** (2007), 751–763.

[13] P.J. Rabier, C.A. Stuart, *A Sobolev space approach to boundary value problems on the half-line*, Comm. in Contemp. Math. **7** (2005) 1, 1–36.

[14] K. Szymańska, *On an asymptotic boundary value problem for second order differential equation*, J. Appl. Anal. **12** (2006) 1, 109–119.

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