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ENTROPY OF FOLIATIONS WITH LEAFWISE FINSLER STRUCTURE

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Abstract. We extend the notion of the geometric entropy of foliation to foliated manifolds equipped with leafwise Finsler structure. We study the relation between the geometric entropy and the topological entropy of the holonomy pseudogroup. The case of a foliated manifold with leafwise Randers structure is considered. In this case the estimates for one dimensional foliation defined by a vector field in terms of the topological entropy of a flow are presented.

Keywords: geometric entropy, leafwise Finsler structure.

Mathematics Subject Classification: 58B20, 37A35, 53C12, 53C60.

1. INTRODUCTION

The notion of the topological entropy was introduced by Adler, Konheim and McAndrew in 1965 in [1]. Another approach was presented by Bowen [2] in the early 70's. Ghys, Langevin and Walczak, in [3], extended this notion to the topological entropy for finitely generated groups and pseudogroups of continuous transformations, as well as the geometric entropy of foliation on a compact foliated Riemannian manifold. The entropy of foliation has a more geometric nature, because it depends on a Riemannian metric chosen for a foliated manifold. On the other hand, the dynamics of Finsler spaces have become a subject of research for mathematicians in late 90's and recently. However, the research in this field is in the initial phase. Finsler geometry [5] is a generalization of Riemannian geometry and therefore, developments in Finsler geometry are important and deserve attention.

The aim of this paper is to extend the notion of the geometric entropy of foliations to the foliated manifolds equipped with leafwise Finsler structure. In Sections 2 and 3, one can find all necessary definitions and properties related to entropy and foliations with a leafwise Finsler metric. The next paragraph describes relations between geometric and topological entropy. The fifth part of the paper refers to foliations with a leafwise

Randers norm. The last paragraph describes the entropy of one dimensional foliations defined by a unit vector field with a leafwise Randers metric.

2. LEAFWISE FINSLER STRUCTURES

Let us recall that a *Minkowski norm* on a vector space V is a non-negative function $F : V \to [0, \infty)$ such that

1. F is C^∞ on $V \setminus \{0\}$,
2. $F(\lambda v) = \lambda F(v)$ for any $\lambda > 0$ and $v \in V$,
3. for every $y \in V \setminus \{0\}$, the symmetric bilinear form

$$g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial t \partial s} F^2(y + su + tv)|_{t=s=0}$$

is positively defined.

Now, let M be a smooth manifold. A function $F : TM \to [0, \infty)$ is called a *Finsler norm* if

1. F is C^∞ on the tangent bundle with removed the zero section $TM \setminus 0_M$,
2. for any $x \in M$ the restricted norm $F_x = F|_{T_x M}$ is a Minkowski norm.

The pair (M, F) is called a Finsler space.

Let (M, g) be a Riemannian manifold, and let $\beta : TM \to \mathbb{R}$ be a 1-form. Let $\alpha : TM \to [0, \infty)$ be the norm defined by g, that is, $\alpha(v) = \sqrt{g_x(v, v)}$ for all $v \in T_x M$. Suppose that the g-norm of β satisfies $\|\beta\|_g < 1$. We set

$$F(v) = \alpha(v) + \beta(v).$$

F is a Finsler norm and it is called a *Randers norm.*

Note that the Finsler norm induces a function $d : M \times M \to [0, \infty)$ by the formula

$$d(x, y) = \inf_\gamma \int_0^1 F(\dot{\gamma}(t)) dt,$$

where the infimum is taken over all curves $\gamma : [0, 1] \to M$ linking x and y. Function d is a quasi-metric, that is,

$$(d(x, y) = 0 \text{ iff } x = y) \text{ and } d(x, y) + d(y, z) \geq d(x, z).$$

Let $(M, \mathcal{F}, g)$ be a foliated Riemaniann manifold. Having g, we decompose the tangent bundle to the orthogonal sum of the bundle tangent to $\mathcal{F}$ and the orthogonal bundle, that is, $TM = T\mathcal{F} \oplus T\mathcal{F}^\perp$. We replace the norm induced in $T\mathcal{F}$ by the

Riemannian structure $g|_{T\mathcal{F}}$ by a Finsler norm $F_{\mathcal{F}}$. Denote by $\pi_1 : TM \to T\mathcal{F}$ and $\pi_2 : TM \to T\mathcal{F}^{\perp}$ the natural projections. We set

$$F(v) = \sqrt{F_{\mathcal{F}}^2(\pi_1(v)) + g(\pi_2(v), \pi_2(v))}.$$

F is a Finsler norm on TM and coincides with $\sqrt{g(v,v)}$ for $v \in T\mathcal{F}^{\perp}$ and with $F_{\mathcal{F}}$ for $v \in T\mathcal{F}$. We call F a *leafwise Finsler structure* on $(M, \mathcal{F})$.

3. GEOMETRIC ENTROPY OF FOLIATIONS WITH LEAFWISE FINSLER METRIC

Let $(M, \mathcal{F}, F)$ be a foliated manifold with leafwise Finsler structure. Let $\mathcal{U}$ be a *nice covering*, i.e., a covering by the domains D_φ of the charts of a nice foliated atlas $\mathcal{A}$, that is an atlas satisfying

1. the covering $\{D_\varphi : \varphi \in \mathcal{A}\}$ is locally finite,
2. for any $\varphi \in \mathcal{A}$, the set $R_\varphi = \varphi(D_\varphi) \subset \mathbb{R}^n$ is an open cube,
3. if $\varphi, \psi \in \mathcal{A}$, and $D_\varphi \cap D_\psi \neq \emptyset$, then there exists a chart $\chi = (\chi', \chi'')$ such that for any leaf L of $\mathcal{F}$ the connected components of $L \cap D_\chi$ are given by the equation $\chi'' = \text{const}$, and R_χ is an open cube, D_χ contains the closure of $D_\varphi \cup D_\psi$ and $\varphi = \chi|_{D_\varphi}$ and $\psi = \chi|_{D_\psi}$.

Let $U \in \mathcal{U}$. Equip the space of plaques $T_U = U/_{\mathcal{F}|U}$ with the quotient topology. The disjoint union $T = \bigsqcup\{T_U; U \in \mathcal{U}\}$ is called a *complete transversal* for $\mathcal{F}$. Note that each T_U can be mapped homeomorphically onto a C^r-submanifold $T'_U \subset U$ transverse to $\mathcal{F}$.

Following [3], let us recall that for a given nice covering $\mathcal{U}$ of $(M, \mathcal{F})$ there exists an $\varepsilon_0 > 0$ such that any point $x \in U, U \in \mathcal{U}$, can be projected orthogonally in a unique way to the plaque $P_y \subset U$ passing through a point $y \in U$ if only $\max\{d(x,y), d(y,x)\} < \varepsilon_0$.

Let $\gamma : [0,1] \to L$ be a leafwise curve beginning at $x \in U$. For any $y \in U$ lying within the distance $\varepsilon < \varepsilon_0$, we can project orthogonally an initial part of the curve γ to the plaque P_y passing through y. Replacing x and y by the endpoints of the already projected piece and its image γ_1, we can continue this process as long as the distance between γ and γ' does not exceed ε_0. We will denote the projection of γ by $p_y\gamma$.

Let $\mathcal{U}$ be a nice covering and let T be the complete transversal for $\mathcal{U}$. Let $\varepsilon \in (0, \varepsilon_0)$, and let d denote the metric induced by the Finsler structure.

Definition 3.1. We say that $x, y \in T$ are (R, ε)-separated by $\mathcal{F}$ with respect to F if either

— $\max\{d(x,y), d(y,x)\} \geq \varepsilon_0$

or

— there exists a leaf curve $\gamma : [0,1] \to L_x$ such that $\gamma(0) = x$,

$$l(\gamma) = \int_0^1 F(\dot\gamma(t))dt \leq R$$

and

$$\max\{d(\gamma(1), p_y\gamma(1)), d(p_y\gamma(1), \gamma(1))\} \geq \varepsilon.$$

(or a leaf curve $\gamma : [0,1] \to L_y$ such that $\gamma(0) = y$, $l(\gamma) \leq R$, and

$$\max\{d(\gamma(1), p_x\gamma(1)), d(p_x\gamma(1), \gamma(1))\} \geq \varepsilon).$$

A subset $A \subset T$ is called (R, ε)-separated if any two points $x, y \in A$, $x \neq y$, are (R, ε)-separated. Let $s(R, \varepsilon, \mathcal{F})$ denote the maximum cardinality of a (R, ε)-separated subset of T. We set

$$s(\varepsilon, \mathcal{F}) = \limsup_{R\to\infty} \frac{1}{R} \log s(R, \varepsilon, \mathcal{F}),$$

and

$$h(\mathcal{F}, F) = \lim_{\varepsilon\to 0^+} s(\varepsilon, \mathcal{F}).$$

Remark 3.2. The number $h(\mathcal{F}, F)$ does not depend on the choice of the nice covering $\mathcal{U}$. Let $\mathcal{U}'$ and T' be another nice covering and complete transversal. Let $\varepsilon > 0$ be small enough, and let us denote by $d_{\mathcal{F}}$ the leafwise metric induced by the Finsler structure $F_{\mathcal{F}}$. Since M is compact, the geometry of $\mathcal{F}$ is bounded. Hence, one can project T onto T' in such a way that any (R, ε)-separated points $x, y \in T$ are projected to $x', y' \in T'$, respectively, which are $(R + R_0, \varepsilon)$-separated with R_0 being the maximum of the numbers $d_{\mathcal{F}}(x, x')$ and $d_{\mathcal{F}}(x', x)$, $x \in T \cap U$, $x' \in T' \cap U'$, $U \in \mathcal{U}$, $U' \in \mathcal{U}'$, and the plaques $P_x \subset U$ and $P_{x'} \subset U'$ intersects. Thus

$$s'(R - R_0, \varepsilon, \mathcal{F}) \leq s(R, \varepsilon, \mathcal{F}) \leq s'(R + R_0, \varepsilon, \mathcal{F}),$$

and both numbers $h(\mathcal{F}, F)$ and $h'(\mathcal{F}, F)$ are equal.

Remark 3.3. Since any two Riemannian structures g and g' on a compact manifold satisfies

$$c^{-2} g(v, w) \leq g'(v, w) \leq c^2 g(v, w)$$

for some constant $c > 1$, then the number $h(\mathcal{F}, F)$ does not depend on the choice of the Riemannian part of F. Indeed, there exists a constant $a > 1$ such that for any leaf curve γ and its orthogonal projections $p_y\gamma$ and $p'_y\gamma$, with respect to g and g' respectively, satisfy

$$d(\gamma(t), p_y\gamma(t)) \leq a \cdot d'(\gamma(t), p'_y\gamma(t)),$$

if $d(\gamma(t), p'_y\gamma(t)) < \varepsilon$ for sufficiently small $\varepsilon > 0$. Thus any two (R, ε)-separated points with respect to $F = \sqrt{F_{\mathcal{F}}^2 + g}$ are $(R, \frac{\varepsilon}{a})$-separated with respect to $F' = \sqrt{F_{\mathcal{F}}^2 + g'}$, and $h(\mathcal{F}, F) \leq h(\mathcal{F}, F')$. Analogously we show that $h(\mathcal{F}, F') \leq h(\mathcal{F}, F)$.

Since any two leafwise Finsler structures F and F' on a compact foliated manifold satisfies

$$\frac{1}{c} F(v) \leq F'(v) \leq c \cdot F(v)$$

for some constant $c \geq 1$, the geometric entropies $h(\mathcal{F}, F)$ and $h(\mathcal{F}, F')$ are both either equal to zero or not. The number $h(\mathcal{F}, F)$ is called the *geometric entropy of foliation with leafwise Finsler structure*. In further consideration we will denote by F both, the structure $F_{\mathcal{F}}$ and the leafwise Finsler structure $F = \sqrt{F_{\mathcal{F}}^2 + g}$.

4. RELATION BETWEEN GEOMETRIC ENTROPY AND TOPOLOGICAL ENTROPY OF A HOLONOMY PSEUDOGROUP

Let $(M, \mathcal{F})$ be a compact foliated manifold. Following [3] or [6], one can define the topological entropy of the holonomy pseudogroup $\mathcal{H}_{\mathcal{U}}$ defined by the nice covering $\mathcal{U}$. The symbol D_f denotes here the domain of a map f.

To begin, let $\mathcal{G}$ be a pseudogroup (see [6]) on a metric (quasi-metric) space (X, d) generated by a good symmetric finite set $\mathcal{G}_1$, that is

1. for any $g \in \mathcal{G}$ and any $x \in D_g$ there exists $g_1, \dots, g_n \in \mathcal{G}_1$ and an open subset $U \subset D_g$ containing x such that

$$g|_U = g_1 \circ \cdots \circ g_n|_U,$$

2. for any $g \in \mathcal{G}_1$ the exits a compact set $K_g \subset D_g$ such that $g|_{\mathrm{intK_g}}$ generate $\mathcal{G}$.

We say that $x, y \in X$ are (n, ε)-separated by $\mathcal{G}$ if there exists

$$g \in \mathcal{G}_n^c := \{g_1|_{K_1} \circ \cdots \circ g_n|_{K_n}; g_i \in \mathcal{G}_1\}$$

such that $\{x, y\} \subset D_g$ and

$$\max\{d(g(x), g(y)), d(g(y), g(x))\} \geq \varepsilon.$$

A subset A of X is called (n, ε)-separated if any two distinct points of A are (n, ε)-separated. Let $s(n, \varepsilon, \mathcal{G}_1)$ be the maximal cardinality of an (n, ε)-separated subset of X. We set

$$s(\varepsilon, \mathcal{G}_1) = \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, \mathcal{G}_1).$$

The number $h(\mathcal{G}, \mathcal{G}_1) = \lim_{\varepsilon \to 0^+} s(\varepsilon, \mathcal{G}_1)$ is called the *topological entropy of the pseudogroup $\mathcal{G}$ with respect to $\mathcal{G}_1$*.

Let $\mathcal{U}$ be a nice covering of (M, f) and let T be a complete transversal. Given two sets $U, V \in \mathcal{U}$ such that $U \cap V \neq \emptyset$, one can define the *holonomy map* $h_{VU} : D_{VU} \to T_V$ with D_{VU} being the open subset of U consisting of all plaques $P \subset U$ such that $P \cap V \neq \emptyset$ by

$$h_{VU}(P) = P' \text{ iff } P \subset U \text{ and } P' \subset V \text{ insterect.}$$

The mappings h_{VU} generates the *holonomy pseudogroup* $\mathcal{H}_{\mathcal{U}}$ on T. We will denote by $\mathcal{H}_{\mathcal{U}}^1$ the set of $\{h_{VU}\}$ of the generators of $\mathcal{H}_{\mathcal{U}}$.

One of the main results of [3] is Theorem 3.4 about the relation between the geometric entropy $h(\mathcal{F}, g)$ of a foliation on a Riemannian manifold and the topological entropy of the holonomy pseudogroup $\mathcal{H}_{\mathcal{U}}$ defined by the nice covering $\mathcal{U}$. We extend this result to the class of foliations with leafwise Finsler structures.

Let $(M, \mathcal{F}, F)$ be a foliated manifold equipped with leafwise Finsler structure.

Theorem 4.1. *Let U be a nice covering, and let* $\mathrm{diam}(\mathcal{U})$ *be the diameter of the nice covering $\mathcal{U}$, that is,*

$$\mathrm{diam}(\mathcal{U}) = \max_{\mathrm{U} \in \mathcal{U}} \max_{\mathrm{P} \subset \mathrm{U}} \max_{\mathrm{x,y} \in \mathrm{P}} \mathrm{d}_{\mathcal{F}}(\mathrm{x}, \mathrm{y}),$$

where P denotes a plaque of a chart U, and $d_{\mathcal{F}}$ is the leafwise distance defined by F. Then

$$h(\mathcal{F}, F) = \sup_{\mathcal{U}} \left\{ \frac{1}{\operatorname{diam}(\mathcal{U})} h(\mathcal{H}_{\mathcal{U}}, \mathcal{H}^1_{\mathcal{U}}) \right\}.$$

We here repeat the proof of Theorem 3.4 of [3] with the necessary changes due to the fact that the metric induced by the Finsler structure is asymmetric. In the proof, $\Lambda = \max_{v \in T\mathcal{F} \setminus \{0\}} \frac{F(v)}{F(-v)}$.

Lemma 4.2. *For Δ and ρ small enough, there exists $\tilde{\beta} > 0$ such that $\rho > \tilde{\beta}$ and the following is satisfied: Let x_1, x_2 be two points lying on the same leaf L and such that $d_{\mathcal{F}}(x_1, x_2) = \frac{2\Delta}{\Lambda} - \alpha$ for some $\alpha > 0$. Let y_1, y_2 be two points of transversals T_1 and T_2 passing through x_1 and x_2, respectively, and lying on the same plaque with diameter not exceeding 4Δ. If*

$$\max\{d(x_1, y_1), d(y_1, x_1), d(x_2, y_2), d(y_2, x_2)\} < \tilde{\beta},$$

then $d_{\mathcal{F}}(y_1, y_2) \leq \frac{2\Delta}{\Lambda} - \frac{\alpha}{2}$.

Proof. Let $\gamma : [0, 1] \to L$ be a curve linking x_1 with x_2 such that $l(\gamma) = \frac{2\Delta}{\Lambda} - \alpha$. Let $p_{y_1}\gamma$ be the orthogonal projection of γ onto the plaque P_{y_1}. Since M is compact, there exists $\tilde{\beta} > 0$ such that $|l(\gamma) - l(p_{y_1}\gamma)| < \frac{\alpha}{4}$, and

$$\{d_{\mathcal{F}}(y_2, p_{y_1}\gamma(1)), d_{\mathcal{F}}(p_{y_1}\gamma(1), y_2)\} < \frac{\alpha}{4}.$$

Thus

$$d_{\mathcal{F}}(y_1, y_2) \leq l(p_{y_1}\gamma) + d_{\mathcal{F}}(y_2, p_{y_1}\gamma(1)) + d_{\mathcal{F}}(p_{y_1}\gamma(1), y_2) \leq \frac{2\Delta}{\Lambda} - \frac{\alpha}{2}.$$

This ends our proof. □

Let $\rho > 0$, and let $S_x = \exp B^{\perp}(0_x, 2\rho)$ be the image in the exponential map exp on M where $B^{\perp}$ is a ball centered in 0_x and contained in the orthogonal complement $T_x\mathcal{F}^{\perp}$ of $T_x\mathcal{F}$. Set

$$T_x = \exp B^{\perp}(0, \rho), \quad U_x = \bigcup_{y \in T_x} B_F\left(y, \frac{\Delta}{\Lambda}\right).$$

Lemma 4.3. *Let $\mathcal{Z} = \{z_1, \ldots, z_N\}$ be a β-dense subset of M, $\beta < \frac{\tilde{\beta}}{10}$. Let x_1 and x_2 be two points of the same leaf with $d_{\mathcal{F}}(x_1, x_2) < \frac{2\Delta}{\Lambda} - \alpha$. Let $z \in \mathcal{Z}$ (resp. $z' \in \mathcal{Z}$) be a β-close point of x_1 (resp. x_2). Then the subsets U_z and $U_{z'}$ have the following property: If $\xi_1 \in T_z$ and $\xi_2 \in T_{z'}$ lie on the same plaque with diameter not exceeding 4Δ and*

$$\max\{d(\xi_1, x_1), d(x_1, \xi_1), d(\xi_2, x_2), d(x_2, \xi_2)\} < \beta,$$

then the minimal leaf geodesic in $L_{\xi_1} = L_{\xi_2}$ linking ξ_1 and ξ_2 is contained in the sum

$$B_F\left(\xi_1, \frac{\Delta}{\Lambda}\right) \cup B_F(\xi_2, \Delta).$$

Proof. By Lemma 4.2, $d_{\mathcal{F}}(\xi_1, \xi_2) < \frac{2\Delta}{\Lambda} - \frac{\alpha}{2}$. So, there exists a curve $\gamma : [0,1] \to L_{\xi_1}$ such that $\gamma(0) = \xi_1$, $\gamma(0) = \xi_2$, and $l(\gamma) = d_{\mathcal{F}}(\xi_1, \xi_2)$. Let $t \in [0,1]$ be a number such that $d_{\mathcal{F}}(\xi_1, \gamma(t)) = \frac{\Delta}{\Lambda}$. Then $d_{\mathcal{F}}(\gamma(t), \xi_2) \le \frac{\Delta}{\Lambda}$. Since $d_{\mathcal{F}}$ is asymmetric then $d_{\mathcal{F}}(\xi_2, \gamma(t)) \le \Delta$. □

Proof of Theorem 4.1. To begin, let $\epsilon > 0$, and $x, y \in T_U$, $U \in \mathcal{U}$, be (n, ε)-separated with respect to $h(\mathcal{H}_{\mathcal{U}}, \mathcal{H}^1_{\mathcal{U}})$. Then there exists a chain of maps $(U_1, \dots, U_n)$ such that the corresponding chains of plaques $(P_1, \dots, P_n)$ and $(Q_1, \dots, Q_n)$ with $x \in P_1$, $y \in Q_1$, $P_i, Q_i \in U_i$, $P_i \cap P_{i+1} \ne \emptyset$, $Q_i \cap Q_{i+1} \ne \emptyset$ satisfy

$$\max\{d(x_n, y_n), d(y_n, x_n)\} \ge \varepsilon,$$

where $x_n \in P_n \cap T_{U_n}$ and $y_n \in Q_n \cap T_{U_n}$ are the images in the holonomy map determined by $(U_1, \dots, U_n)$ of x and y, respectively. Let $x_0 = x$ and let us choose points $x_i \in P_i \cap P_{i+1}$, $i = 1, \dots, n-1$. Link the points x_i and x_{i+1} by a leaf geodesic γ_i, $i = 1, \dots, n-1$. The length of every γ_i is smaller than $\mathrm{diam}(\mathcal{U})$, and the length of a curve γ built of γ_i's and linking x_0 with x_n is smaller than $n \cdot \mathrm{diam}(\mathcal{U})$. Shortening γ, if necessary, we can assume that the distance between γ and its orthogonal projection $p_y\gamma$ is always smaller than ε_0, and the whole γ can be projected to L_y.

Since $T = \bigsqcup T_U$ is compact, there exists a constant $C > 0$ such that

$$\frac{1}{C} d(z, w) \le d(z, p(z)) \le C d(z, w)$$

and

$$\frac{1}{C} d(w, z) \le d(p(z), z) \le C d(w, z),$$

if only $z, w \in T_U$, $U \in \mathcal{U}$, and $p(z)$ is the orthogonal projection of z to the plaque P_w passing through w. Hence $d(\gamma(1), p_y\gamma(1)) \ge \frac{\varepsilon}{C}$. This gives that x and y are $(n \cdot \mathrm{diam}(\mathcal{U}), \frac{\varepsilon}{\mathrm{C}})$-separated with respect to $\mathcal{F}$. Thus,

$$s(n, \varepsilon, \mathcal{H}^1_{\mathcal{U}}) \le s\left(n \cdot \mathrm{diam}(\mathcal{U}), \frac{\varepsilon}{\mathrm{C}}, \mathcal{F}\right)$$

for all $n \in \mathbb{N}$, and $\varepsilon \in (0, \varepsilon_0)$. Finally,

$$h(\mathcal{H}_{\mathcal{U}}, \mathcal{H}^1_{\mathcal{U}}) \le \mathrm{diam}(\mathcal{U}) \cdot \mathrm{h}(\mathcal{F}, \mathrm{F}).$$

Let $\eta > 0$, and $\Delta > 0$ be such that the leafwise exponential mapping $\exp^{\mathcal{F}}$ maps the balls $B^{\mathcal{F}}(0_x, 4\Delta)$, where $B^{\mathcal{F}}(0_x, r) = \{v \in T_x\mathcal{F} : F(v) < r\}$, diffeomorphically onto strictly convex balls

$$B_F(x, 4\Delta) = \{y \in L_x : d_{\mathcal{F}}(x, y) < 4\Delta\}, \quad x \in M.$$

Note that for small enough ρ and Δ, the sets U_x are the domains of distinguished charts, and for any plaque $P \subset U_x$, the diameter $\mathrm{diam}(\mathrm{P_x}) \le (1 + \frac{1}{\Lambda})\Delta$.

Let $\mathcal{U}_\Delta = \{U_z, z \in \mathcal{Z}\}$. We may assume that the closures $\bar{U}_z$ and $\bar{U}_{z'}$, $z, z' \in \mathcal{Z}$, overlap if only $\bar{U}_z$ and $\bar{U}_{z'}$ do. Thus $\mathcal{U}_\Delta$ is a nice covering of $(M, \mathcal{F})$. Moreover, $\mathrm{diam}(\mathcal{U}_\Delta) \le (1 + \frac{1}{\Lambda})\Delta$.

Let $\varepsilon > 0$, and let x, y be such that

$$\max\{d(x,y), d(y,x)\} \leq \varepsilon$$

and additionally they are (R,ε)-separated by $\mathcal{F}$ with respect to F. Hence, there exists a curve $\gamma : [0,R] \to L_x$ starting at x with $l(\gamma) \leq R$ and such that $p_y\gamma$ is well defined on $[0,r]$, $r < R$, and $\max\{d(\gamma(r), p_y\gamma(y)), d(p_y\gamma(y), \gamma(r))\} \geq \varepsilon$. Let us assume that $R = (1+\frac{1}{\Lambda})(1-\eta)n\Delta$, and let $x_k = \gamma(\frac{kr}{n})$, $k = 0, \ldots, n$. For each x_k let us find a point $z_k \in \mathcal{Z}$ which is β-close (see Lemma 4.3).

The charts $(U_{z_0}, \ldots, U_{z_n})$ form a chain along $\gamma|_{[0,r]}$, and the corresponding holonomy map $h \in \mathcal{H}_{\mathcal{U}_\Delta}$ is well defined on the plaques $P, Q \in U_{z_0}$ containing x and y, respectively. Moreover,

$$\max\{d(h(Q), h(P)), d(h(P), h(Q))\} \geq C \cdot \varepsilon,$$

where C is the constant from the first part of this proof. We deduce that

$$s\left(\left(1+\frac{1}{\Lambda}\right)(1-\eta)n\Delta, C\varepsilon, \mathcal{F}\right) \leq N(\varepsilon) \cdot s(n, \varepsilon, \mathcal{H}_{\mathcal{U}_\Delta}),$$

with $N(\varepsilon)$ being the minimal cardinality of a covering of M by balls of radius ε. Therefore,

$$s(C\varepsilon, \mathcal{F}) \leq \frac{1}{(1+\frac{1}{\Lambda})(1-\eta)\Delta} s(\varepsilon, \mathcal{H}_{\mathcal{U}_\Delta}).$$

Passing with η to zero, we obtain

$$h(\mathcal{F}, F) \leq \frac{1}{(1+\frac{1}{\Lambda})\Delta} h(\mathcal{H}_{\mathcal{U}_\Delta}, \mathcal{H}^1_{\mathcal{U}_\Delta}) \leq \frac{1}{\operatorname{diam}(\mathcal{U})} h(\mathcal{H}_{\mathcal{U}_\Delta}, \mathcal{H}^1_{\mathcal{U}_\Delta}).$$

This ends the proof. □

5. FOLIATIONS WITH LEAFWISE RANDERS NORM

Let $(M, \mathcal{F}, g)$ be a foliated Riemannian manifold. Let F be a leafwise Randers norm, that is the norm given on leaves by

$$F(v) = \sqrt{g(v,v)} + \beta(v), \quad v \in T\mathcal{F}.$$

Let $\|\beta\| = \max_{v \in T^1_g\mathcal{F}} \beta(v)$. Let us suppose, similarly as the Example in Section 2, that $\|\beta\| < 1$.

Theorem 5.1. *The following inequalities hold:*

$$\frac{1}{1+\|\beta\|} h(\mathcal{F}, g) \leq h(\mathcal{F}, F) \leq \frac{1}{1-\|\beta\|} h(\mathcal{F}, g).$$

Proof. Let $G(v) = \sqrt{g(v,v)}$. Since $F(v) = G(v) + \beta(v)$ then for any $v \in T\mathcal{F}$

$$F(v) \leq G(v) + \|\beta\| G(v) \text{ and } G(v) \leq F(v) + \|\beta\| G(v). \tag{5.1}$$

Let x, y be (R, ε)-separated with respect to g. So, there exists a curve $\gamma : [0,1] \to L_x$ such that $\gamma(0) = x$, $l_G(\gamma) \leq R$ and

$$d(\gamma(1), p_y\gamma(1)) \geq \varepsilon.$$

Using the first inequality in (5.1), we obtain

$$\begin{aligned} l_F(\gamma) &= \int_0^1 F(\dot\gamma(t))dt \leq \int_0^1 G(\dot\gamma(t))dt + \int_0^1 \|\beta\| G(\dot\gamma(t))dt \\ &\leq R + \|\beta\| R = (1 + \|\beta\|)R. \end{aligned}$$

Thus x, y are $((1 + \|\beta\|)R, \varepsilon)$-separated with respect to F. Hence,

$$\begin{aligned} &s(R, \varepsilon, g) \leq s((1 + \|\beta\|)R, \varepsilon, F), \\ &\frac{1}{R} \log s(R, \varepsilon, g) \leq \frac{1 + \|\beta\|}{1 + \|\beta\|} \frac{1}{R} \log s((1 + \|\beta\|)R, \varepsilon, F), \\ &\limsup_{R\to\infty} \frac{1}{R} \log s(R, \varepsilon, g) \\ &\qquad \leq (1 + \|\beta\|) \limsup_{R\to\infty} \frac{1}{(1 + \|\beta\|)R} \log s((1 + \|\beta\|)R, \varepsilon, F), \\ &s(\varepsilon, \mathcal{F}, g) \leq (1 + \|\beta\|) s(\varepsilon, \mathcal{F}, F). \end{aligned}$$

Finally,

$$h(\mathcal{F}, g) \leq (1 + \|\beta\|) h(\mathcal{F}, F).$$

The second inequality follows directly from the second inequality in (5.1) and from the fact that every two points which are (R, ε)-separated with respect to F are $(\frac{R}{1-\|\beta\|}, \varepsilon)$-separated with respect to g. □

6. TOPOLOGICAL ENTROPY OF ONE DIMENSIONAL FOLIATION

We will now recall the definition (following [2] and [4]) of the topological entropy of a uniformly continuous map on a quasi-metric space.

Let $f : X \to X$ be a uniformly continuous transformation of a quasi-metric space X, that is, for any $\varepsilon > 0$ and any $x \in X$ there exists $\delta > 0$ such that for any $y \in X$

$$\max\{d(x, y), d(y, x)\} < \delta \Rightarrow \max\{(d(f(x), f(y)), d(f(y), f(x))\} < \epsilon.$$

For any $n \in \mathbb{N}$ and $x, y \in X$ let

$$d_n(x, y) = \max_{0\leq k\leq n-1} \{\max\{d(f^k(x), f^k(y)), d(f^k(y), f^k(x))\}\}, \ k \in \mathbb{N}.$$

Let $n \in \mathbb{N}$ and $\varepsilon > 0$. A subset A of X is said to be (n, ε)*-separated* if $d_n(x, y) > \varepsilon$ for every $x, y \in A, x \neq y$. A set $B \subset X$ is said to (n, ε)*-span* another set K if for every $x \in K$ there is $y \in B$ such that $d_n(x, y) \leq \varepsilon$.

We set

$$s(n, \varepsilon, K) = \max\{\#A : A \subset K \text{ is } (n, \varepsilon)\text{-separated}\},$$
$$r(n, \varepsilon, K) = \min\{\#A : A \subset X \text{ is } (n, \varepsilon)\text{-spanning } K\}.$$

Lemma 6.1. *The following inequalities hold*

1. $r(n, \varepsilon, K) \leq s(n, \varepsilon, K) \leq r(n, \frac{\varepsilon}{2}, K) < \infty$,
2. *for* $\varepsilon' < \varepsilon$
$$r(\varepsilon', K) \geq r(\varepsilon, K) \text{ and } s(\varepsilon', K) \geq s(\varepsilon, K).$$

Proof. If A is a maximal (n, ε)-separated subset of K, then A also (n, ε)-spans K. Thus $r(n, \varepsilon, K) \leq s(n, \varepsilon, K)$.

Let $A \subset K$ be a (n, ε)-separated set and let B $(n, \frac{1}{2}\varepsilon)$-spans K. For any $x \in K$, there exists $g(x) \in B$ such that $d_n(x, g(x)) < \frac{\varepsilon}{2}$. Moreover, if $g(x) = g(y)$ then $d_n(x, y) < \varepsilon$. Thus g is injective on A (since A is (n, ε)-separated), and $s(n, \varepsilon, K) \leq r(n, \frac{\varepsilon}{2}, K)$.

As f is uniformly continuous on (X, d) there is a $\delta > 0$ such that $d_n(x, y) < \frac{\varepsilon}{2}$ if only $d(x, y) < \delta$ and $d(y, x) < \delta$. Thus $r(n, \frac{\varepsilon}{2}, K)$ does not exceed the number of δ-balls $B_\delta(z) = \{z' \in X : d(z, z') < \delta \text{ and } d(z', z) < \delta\}$ needed to cover K. So, $r(n, \frac{\varepsilon}{2}, K)$ is finite, as K is compact.

The inequalities in (2) are obvious. □

Finally, we define

$$s(\varepsilon, K) = \lim_{n \to \infty} \sup \frac{1}{n} \log s(n, \varepsilon, K),$$
$$r(\varepsilon, K) = \lim_{n \to \infty} \sup \frac{1}{n} \log r(n, \varepsilon, K).$$

Definition 6.2. For any uniformly continuous map $f : X \to X$ on a quasi-metric space (X, d) and any compact set $K \subset X$ define

$$h_{\text{top}}(f, K) = \lim_{\varepsilon \to 0^+} s(\varepsilon, K) = \lim_{\varepsilon \to 0^+} r(\varepsilon, K)$$

and

$$h_{\text{top}}(f) = \sup_{K \text{ compact}} h_{\text{top}}(f, K).$$

The number $h_{\text{top}}(f)$ is called the *topological entropy of* f.

Let us now study the geometrical entropy of a foliation given by the integral curves of a vector field X on a compact manifold M. Note that any Finsler norm on a 1-dimensional vector space is a Randers norm. Indeed, let

$$G(v) = \frac{1}{2}(F(v) + F(-v)), \quad \beta(v) = \frac{1}{2}(F(v) - F(-v)).$$

Then G is a norm associated with an inner product g, while β is a 1-form. Moreover, $F(v) = G(v) + \beta(v)$ and $\|\beta\|_G < 1$.

Let $F = G + \beta$ be a leafwise Randers norm for which X is a G-unit vector field, that is, $G(X(p)) = 1$ for all $p \in M$. Let $\varphi = (\varphi_t : M \to M)_{t\in\mathbb{R}}$ denote the flow of X. We recall [6] that the topological entropy of a flow is equal to $h_{\mathrm{top}}(\varphi_1)$.

Theorem 6.3. *If* $\Lambda = \max_{v\in T\mathcal{F}\setminus\{0\}} \frac{F(v)}{F(-v)}$, *then*

$$\left(1 + \frac{1}{\Lambda}\right) h_{\mathrm{top}}(\varphi) \le h(\mathcal{F}, F) \le (1+\Lambda) h_{\mathrm{top}}(\varphi).$$

Proof. By Theorem 3.4.3 in [6], $h(\mathcal{F}, g) = 2h_{\mathrm{top}}(\varphi)$. We have

$$h(\mathcal{F}, F) \le \frac{1}{1-\|\beta\|} h(\mathcal{F}, g) \le \frac{2}{1-\|\beta\|} h_{\mathrm{top}}(\varphi) \le \left(1 + \frac{1+\|\beta\|}{1-\|\beta\|}\right) h_{\mathrm{top}}(\varphi).$$

However, $\Lambda = \frac{1+\|\beta\|}{1-\|\beta\|}$. This gives the second inequality.

To prove the first inequality, it is enough to observe that

$$h(\mathcal{F}, F) \ge \frac{1}{1+\|\beta\|} h(\mathcal{F}, g) \ge \frac{2}{1+\|\beta\|} h_{\mathrm{top}}(\varphi) \ge \left(1 + \frac{1-\|\beta\|}{1+\|\beta\|}\right) h_{\mathrm{top}}(\varphi).$$

This ends our proof. □

Theorem 6.4. *Let* $F = G + \beta$ *be a leafwise Randers metric along a one dimensional foliation defined by a* G*-unit vector field* X*. Suppose that* $\beta(X) = \mathrm{const}$*. Then*

$$h(\mathcal{F}, F) = \frac{2}{1-\beta^2(X)} h_{\mathrm{top}}(\varphi),$$

where $(\varphi_t)_{t\in\mathbb{R}}$ *denotes the flow of* X*.*

Proof. Let $a = \beta(X)$. Since $\|\beta\|_G < 1$ then $a \in (-1, 1)$. Let A be a (n, ϵ)-separated set by $\mathcal{F}$ with respect to F. Then the set $B = \varphi_{-\frac{n}{1-a}}(A)$ is $(\frac{2n}{1-a^2}, \epsilon)$-separated with respect to φ. Hence

$$s(n, \epsilon, F) \le s\left(\frac{2n}{1-a^2}, \epsilon, \varphi\right).$$

This gives

$$h(\mathcal{F}, F) \le \frac{2}{1-\beta^2(X)} h_{\mathrm{top}}(\varphi).$$

Let $\eta > 0$. Let us consider the fiber bundle $\pi : \tilde{M}_\eta \to M$ built of orthogonal balls $B^\perp(0_x, \eta) \subset T_x\mathcal{F}^\perp$, $x \in M$. For every $x \in M$, let $\mathrm{Tub}_\eta(\mathrm{x}) = \varphi_{\mathrm{x}*}^{-1}\tilde{\mathrm{M}}_\eta$ be the bundle over $\mathbb{R}$ induced by a map $\varphi_x : t \mapsto \varphi_t(x)$. It is known, that for η small enough, the exponential map on M defines a natural immersion $\iota_x : \mathrm{Tub}_\eta(\mathrm{x}) \to \mathrm{M}$, and one can equip $\mathrm{Tub}_\eta(\mathrm{x})$ with the induced leafwise Randers structure and with the induced vector field $\tilde{X}$, which generates a local flow $(\tilde{\varphi}_t)$. As mentioned in [6], the family $\pi_x^{-1}(s)$, where π_x is a fiber bundle projection in $\mathrm{Tub}_\eta(\mathrm{x})$ and $s \in \mathbb{R}$, of fibers of $\mathrm{Tub}_\eta(\mathrm{x})$ is not invariant under the flow $(\tilde{\varphi}_t)$.

Let us fix $\varepsilon > 0$. Since M is compact, the family $\pi^{-1}(s)$, $s \in \mathbb{R}$, of fibers of $\mathrm{Tub}_\eta(\mathrm{x})$ satisfies the following:

For any $\tau \in (0,1)$ there exists $\eta > 0$ such that for any $x \in M$ and $y \in \mathrm{Tub}_\eta(\mathrm{x}) \cap \pi^{-1}(0)$ with defined local flow $(\tilde{\varphi}_t)$ and $\pi(\tilde{\varphi}_t(y)) = 1$ (respectively $\pi(\tilde{\varphi}_t(y)) = -1$) we have $t > \tau$ $(t < -\tau)$. Moreover, if τ and η are as above, then $t \geq n\tau$ $(t \leq -n\tau)$ whenever $(\tilde{\varphi}_t)$ is defined and $\pi(\tilde{\varphi}_t(y)) = n$ (respectively, $-n$), $n \in \mathbb{N}$.

Let us decompose $\mathrm{Tub}_\eta(\mathrm{x})$ into the cylinders $C_n(x) = \pi_x^{-1}([(2n-1)\varepsilon, (2n+1)\varepsilon])$, $n \in \mathbb{Z}$. Since ε is fixed, there exists η independent of $x \in M$ such that the sets $\tilde{\varphi}_1(C_0(x))$ and $\tilde{\varphi}_{-1}(C_0(x))$ intersect at most three cylinders of the form $C_n(x)$. For every $y \in C_0(x)$, we consider the sequences $(n_k)_{k\in\mathbb{N}}$ of integers such that $\tilde{\varphi}_k(y) \in C_{n_k}(x)$ (we set ∞ if $\tilde{\varphi}_k(y)$ is undefined). The number of such sequences of length $[2n/(1-a^2)] - 1$ do not exceed $3^{k_a n}$ for some natural number k_a. So, we can decompose all cylinders $C_0(x)$ into the unions of sets $C_0(x) = K_1(x) \cup \cdots \cup K_{m(x)}(x)$, $m(x) \leq 3^{k_a n+2}$ satisfying the following:

$(*)$ If $y, z \in K_j(x)$, $[\frac{n}{1-a}] \leq k \leq [\frac{n}{1+a}]$ and $\tilde{\varphi}_k(y)$ and $\tilde{\varphi}_k(z)$ are defined, then $\tilde{\varphi}_k(z)$ and $\tilde{\varphi}_k(y)$ belongs to the same cylinder $C_{n_k}(x)$.

Let $A \subset T$ be a maximal $(n, \frac{\eta}{3})$-separated by $\mathcal{F}$ with respect to F, that is, $\sharp A = s(n, \frac{\eta}{3}, F)$. Since A is maximal, then it is $(n, \frac{\eta}{3})$-spanning for T, and the sets

$$A(x) = \Big\{ y \in T : \sup_{-[\frac{n}{1-a}] \leq t \leq [\frac{n}{1+a}]} d(\varphi_t(x), p_y\varphi_t(x)) \leq \frac{\eta}{3}$$

$$\text{and} \sup_{-[\frac{n}{1-a}] \leq t \leq [\frac{n}{1+a}]} d(p_y\varphi_t(x), \varphi_t(x)) \leq \frac{\eta}{3} \Big\}, \quad x \in A$$

cover T. Moreover, $\max_{x\in A} \mathrm{diam} A(x) \leq \frac{2\eta}{3}$. Therefore, $A(x) \subset \iota_x(C_0(x))$, and we can decompose $C_0(x)$ into $K_j(x)$ and choose one point y_j^x in each nonempty piece of $A(x) \cap K_j(x)$. Let $B = \{y_j^x\}$. We have

$$\sharp B \leq 3^{k_a n+2} s\Big(n, \frac{\eta}{3}, F\Big).$$

Finally, let $y \in M$. There exists $R_0 > 0$ independent of y such that $\varphi_t(y) \in T$ for some $t \in (-R_0, R_0)$. So there exists $x \in A$ and $j \leq m(x)$ for which $\varphi_t(y) \in A(x) \cap \iota_x K_j(x)$. Thus, by $(*)$, $\tilde{\varphi}_{t+i}(y)$ and $\varphi_i(y_j^x)$ belong to the same cylinder $C_{n(i)}(x)$, and

$$d(\varphi_{t+i}(y), \varphi_i(y_j^x)) \leq \frac{2\varepsilon}{1-a^2} + 2\eta \quad \text{and} \quad d(\varphi_i(y_j^x), \varphi_{t+i}(y)) \leq 2\varepsilon + \frac{2\varepsilon}{1-a^2},$$

for all $-[\frac{n\tau}{1-a}] \leq i \leq [\frac{n\tau}{1+a}]$. Moreover, there exists a constant ω such that for small η and any $z, z' \in M$ the inequalities $\max\{d(z,z') < \eta, d(z',z) < \eta\}$ implies the relations $\max\{d(\varphi_t(z), \varphi_t(z')), d(\varphi_t(z'), \varphi_t(z))\} \leq \omega\eta$ for all $t \in [-R_0, R_0]$. Therefore, the set $\varphi_{-[\frac{\tau n}{1-a}]} B$ is $(\frac{2\tau n}{1-a^2}, 2\omega(\eta + \frac{\epsilon}{1-a^2}))$-spanning with respect to φ. Next,

$$r\left(\frac{2n\tau}{1-a^2}, 2\omega\left(\eta + \frac{\epsilon}{1-a^2}\right), \phi\right) \leq 3^{k_a n+2} s\left(n, \frac{\eta}{3}, F\right).$$

Thus,

$$\frac{2\tau}{1-a^2} r\left(2\omega\left(\eta+\frac{\epsilon}{1-a^2}\right),\phi\right) \leq k_a \log 3 + s\left(\frac{\eta}{3},F\right).$$

This gives

$$\frac{2}{1-a^2} h_{\mathrm{top}}(\varphi) \leq k_a \cdot \log 3 + h(\mathcal{F},F),$$

when we tend with η and ϵ to zero, and choose τ arbitrarily close to 1. Replacing F by λF, $\lambda > 0$ we must replace X by $\lambda^{-1}X$, and (φ_t) by $(\tilde{\varphi}_t) = (\varphi_{t/\lambda})$. Hence

$$\frac{2}{1-a^2} h_{\mathrm{top}}(\varphi) \leq \lambda \log 3 + h(\mathcal{F},F).$$

Since λ can be arbitrarily small, we get the equality. □

Remark 6.5. Let $\mathcal{F}$ be a one-dimensional foliation given by a vector field X. If F is Riemannian, then we get the exact result as in Theorem 3.4.3 of [6], that is,

$$h(\mathcal{F},F) = 2h_{\mathrm{top}}(\varphi).$$

Remark 6.6. The 1-form β in the Randers metric is commonly understood as a mild wind blowing along the leaves of foliation. The direct conclusion of Theorem 6.4 is that increasing the wind along the leaves increases the entropy $h(\mathcal{F},F)$.

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