Opuscula Math. 36, no. 5 (2016), 563–574
http://dx.doi.org/10.7494/OpMath.2016.36.5.563 Opuscula Mathematica

CRITICALITY INDICES OF 2-RAINBOW DOMINATION OF PATHS AND CYCLES

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Communicated by Hao Li

Abstract. A 2-*rainbow dominating function* of a graph $G(V(G),E(G))$ is a function f that assigns to each vertex a set of colors chosen from the set $\{1,2\}$ so that for each vertex with $f(v)=\emptyset$ we have $\bigcup_{u\in N(v)} f(u)=\{1,2\}$. The *weight* of a 2RDF f is defined as $w(f)=\sum_{v\in V(G)}|f(v)|$. The minimum weight of a 2RDF is called the 2-*rainbow domination number* of G, denoted by $\gamma_{2r}(G)$. The vertex criticality index of a 2-*rainbow domination* of a graph G is defined as $ci_{2r}^{v}(G)=(\sum_{v\in V(G)}(\gamma_{2r}(G)-\gamma_{2r}(G-v)))/|V(G)|$, the edge removal criticality index of a 2-*rainbow domination* of a graph G is defined as $ci_{2r}^{-e}(G)=(\sum_{e\in E(G)}(\gamma_{2r}(G)-\gamma_{2r}(G-e)))/|E(G)|$ and the edge addition of a 2-*rainbow domination* criticality index of G is defined as $ci_{2r}^{+e}(G)=(\sum_{e\in E(\overline{G})}(\gamma_{2r}(G)-\gamma_{2r}(G+e)))/\left|E(\overline{G})\right|$, where $\overline{G}$ is the complement graph of G. In this paper, we determine the criticality indices of paths and cycles.

Keywords: 2-rainbow domination number, criticality index.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

Let $G=(V(G),E(G))$ be a simple graph of *order* $|V(G)|=|V|=n(G)$ and *size* $|E(G)|=m(G)$. The *complement* of G is the graph $\overline{G}=(V,E(\overline{G}))$, where $E(\overline{G})=\{uv \mid uv\notin E\}$. The *neighborhood* of a vertex $v\in V$ is $N_G(v)=\{u\in V \mid uv\in E\}$ and the *closed neighborhood* of v is $N_G[v]=N_G(v)\cup\{v\}$. The *degree* of a vertex v of G is $d_G(v)=|N_G(v)|$. The *maximum degree* of G is $\Delta(G)=\max\{d_G(v);\, v\in V\}$. The path (respectively, the cycle) of order n is denoted by P_n (respectively, C_n). We recall that a leaf in a graph G is a vertex of degree one.

A 2-*rainbow dominating function* (2RDF) of a graph G is a function f that assigns to each vertex a set of colors chosen from the set $\{1,2\}$ such that for each vertex with $f(v)=\emptyset$ we have $\bigcup_{u\in N(v)} f(u)=\{1,2\}$. The *weight* of a 2RDF f is defined as $w(f)=\sum_{v\in V(G)}|f(v)|$. The minimum weight of a 2RDF on a graph G is called the 2-*rainbow domination number* of G, and is denoted by $\gamma_{2r}(G)$. We also refer to a γ_{2r}-*function* in a graph G as a 2RDF with minimum weight. For a γ_{2r}-function f on a graph G and a subgraph H of G we denote by $f_{|H}$ the restriction of f on $V(H)$. For references on rainbow domination in graphs, see for example [2, 3, 11, 12].

For many graph parameters, the concept of criticality with respect to various operations on graphs has been studied for several domination parameters such as *domination*, *total domination*, *Roman domination* and 2-*rainbow domination*. Much has been written about graphs where a parameter increases or decreases whenever an edge or vertex is removed or added, by several authors. For references on the criticality concept on various domination parameters see [4, 7–10].

Since any 2RDF of a spanning graph of G is also a 2RDF of G, we have $\gamma_{2r}(G)\le\gamma_{2r}(G-e)$ for every $e\in E(G)$ and $\gamma_{2r}(G+e)\le\gamma_{2r}(G)$ for every $e\notin E(G)$. Note that the removal of a vertex in a graph G may decrease or increase the 2-rainbow domination number. On the other hand, it was shown in [7] that removing any edge from G can increase by at most one the 2-rainbow domination number of G. Also adding any edge to G can decrease by at most one the 2-rainbow domination number of G.

For a graph G, we define the *criticality index* of 2-*rainbow domination* of a vertex $v\in V$ as

$$ci^{v}_{2r}(v)=\gamma_{2r}(G)-\gamma_{2r}(G-v),$$

and the vertex *criticality index* of 2-*rainbow domination* of a graph G as

$$ci^{v}_{2r}(G)=\Big(\sum\nolimits_{v\in V(G)} ci^{v}_{2r}(v)\Big)/n(G).$$

Also we define the *edge removal* criticality index of a 2-*rainbow domination* of an edge $e\in E(G)$ as

$$ci^{-e}_{2r}(e)=\gamma_{2r}(G)-\gamma_{2r}(G-e),$$

and the edge removal criticality index of 2-rainbow domination of a graph G as

$$ci^{-e}_{2r}(G)=\Big(\sum\nolimits_{e\in E(G)} ci^{-e}_{2r}(e)\Big)/m(G).$$

Similarly, we define the *edge addition criticality index* of a 2-*rainbow domination* of an edge $e\in E(\overline{G})$ as

$$ci^{+e}_{2r}(e)=\gamma_{2r}(G)-\gamma_{2r}(G+e),$$

and the *edge addition criticality index* of a 2-*rainbow domination* of a graph G as

$$ci^{+e}_{2r}(G)=\Big(\sum\nolimits_{e\in E(\overline{G})} ci^{+e}_{2r}(e)\Big)/m(\overline{G}).$$

The criticality index was introduced in [5, 6] and [1] for the total domination number and Roman domination number, respectively.

In this paper, we determine exact values of the criticality indices of cycles and paths.

2. PRELIMINARY RESULTS

The following results will be of use throughout the paper.

Proposition 2.1 ([7]). *Let G be a graph with maximum degree $\Delta(G)$. Then*

(i) $\gamma_{2r}(G) - 1 \le \gamma_{2r}(G - v) \le \gamma_{2r}(G) + \Delta(G) - 1$ *for any vertex v of G,*
(ii) $\gamma_{2r}(G) \le \gamma_{2r}(G - e) \le \gamma_{2r}(G) + 1$ *for any edge e of G,*
(iii) $\gamma_{2r}(G) - 1 \le \gamma_{2r}(G + e) \le \gamma_{2r}(G)$ *for any edge e of $\overline{G}$.*

From the above, we can see that $ci^v_{2r}(v) \in \{1 - \Delta(G), \dots, 0, 1\}$ for every $v \in V(G)$, $ci^{-e}_{2r}(e) \in \{-1, 0\}$ for every $e \in E(G)$ and $ci^{+e}_{2r}(e) \in \{0, 1\}$ for every $e \in E(\overline{G})$.

Proposition 2.2 ([3]). *For a cycle C_n with $n \ge 3$,*

$$\gamma_{2r}(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor = \begin{cases} \gamma_{2r}(P_n) - 1 & \text{if } n \equiv 0 \pmod 4, \\ \gamma_{2r}(P_n) & \text{otherwise.} \end{cases}$$

Proposition 2.3 ([2]). *For a path P_n,*

$$\gamma_{2r}(P_n) = \lfloor n/2 \rfloor + 1 = \lceil (n+1)/2 \rceil .$$

Observation 2.4. *For a cycle C_n with $n \ge 7$,*

$$\gamma_{2r}(C_{n-4}) = \gamma_{2r}(C_n) - 2.$$

3. THE VERTEX CRITICALITY INDEX OF A 2-RAINBOW DOMINATION OF A CYCLE AND A PATH

In this section we determine the exact value of the vertex criticality index of a 2-rainbow domination of a cycle and a path. Recall that $ci^v_{2r}(v) = \gamma_{2r}(G) - \gamma_{2r}(G - v)$ and $ci^v_{2r}(v) \in \{-1, 0, 1\}$, where $G = C_n$ or P_n, and $v \in V(G)$.

Theorem 3.1. *For every cycle C_n with $n \ge 3$,*

$$ci^v_{2r}(C_n) = \begin{cases} 0 & \text{if } n \equiv 0, 1, 3 \pmod 4, \\ 1 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. Since removing a vertex v of a cycle C_n produces a path P_{n-1}, by Propositions 2.2 and 2.3 we have

$$ci^v_{2r}(v) = \gamma_{2r}(C_n) - \gamma_{2r}(P_{n-1}) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor - \lfloor (n-1)/2 \rfloor - 1.$$

Therefore, we can easily see that $ci^v_{2r}(v) = 0$ for $n \equiv 0, 1, 3 \pmod 4$ and $ci^v_{2r}(v) = 1$ for $n \equiv 2 \pmod 4$, and so $ci^v_{2r}(C_n) = 0$ for $n \equiv 0, 1, 3 \pmod 4$ and $ci^v_{2r}(C_n) = 1$ for $n \equiv 2 \pmod 4$. □

Let P_n be a path whose vertices are labeled $v_1, v_2, \dots, v_n$. Note that when a vertex v_i is removed from the path P_n, we obtain two paths P_{i-1} and P_{n-i}.

Theorem 3.2. *For every nontrivial path P_n,*

$$ci_{2r}^{v}\left(P_n\right)=\begin{cases}2/n & \text{if } n\equiv 0 \pmod 2,\\ -(n-3)/2n & \text{if } n\equiv 1 \pmod 2.\end{cases}$$

Proof. If $P_n=v_1,v_2,\ldots,v_n$ is a path, then by Proposition 2.3, we have

$$\begin{aligned}\gamma_{2r}\left(P_n-v_i\right)&=\begin{cases}\gamma_{2r}\left(P_{i-1}\right)+\gamma_{2r}\left(P_{n-i}\right) & \text{if } i\neq 1 \text{ and } n,\\ \gamma_{2r}\left(P_{n-1}\right) & \text{if } i=1 \text{ or } n\end{cases}\\ &=\begin{cases}\lfloor (i-1)/2\rfloor+\lfloor (n-i)/2\rfloor+2 & \text{if } i\neq 1 \text{ and } n,\\ \lfloor (n-1)/2\rfloor+1 & \text{if } i=1 \text{ or } n.\end{cases}\end{aligned}$$

Four cases are distinguished with respect to the parity of i and n.

Case 1. $n\equiv 0 \pmod 2$ and $i\equiv 1 \pmod 2$, then $\gamma_{2r}\left(P_n-v_i\right)=\lfloor n/2\rfloor+1$ for $i\neq 1$ and $\gamma_{2r}\left(P_n-v_i\right)=\lfloor n/2\rfloor$ for $i=1$. Therefore,

$$ci_{2r}^{v}(v_i)=\gamma_{2r}\left(P_n\right)-\gamma_{2r}\left(P_n-v_i\right)=\begin{cases}0 & \text{for } i\neq 1,\\ 1 & \text{for } i=1.\end{cases}$$

Case 2. $n\equiv 0 \pmod 2$ and $i\equiv 0 \pmod 2$, then $\gamma_{2r}\left(P_n-v_i\right)=\lfloor n/2\rfloor+1$ for $i\neq n$ and $\gamma_{2r}\left(P_n-v_i\right)=\lfloor n/2\rfloor$ for $i=n$. Therefore,

$$ci_{2r}^{v}(v_i)=\gamma_{2r}\left(P_n\right)-\gamma_{2r}\left(P_n-v_i\right)=\begin{cases}0 & \text{for } i\neq n,\\ 1 & \text{for } i=n.\end{cases}$$

Case 3. $n\equiv 1 \pmod 2$ and $i\equiv 1 \pmod 2$, then $\gamma_{2r}\left(P_n-v_i\right)=\lfloor n/2\rfloor+2$ for $i\neq 1$ and $i\neq n$, and $\gamma_{2r}\left(P_n-v_i\right)=\lfloor n/2\rfloor+1$ for $i=1$ or $i=n$. Therefore,

$$ci_{2r}^{v}(v_i)=\gamma_{2r}\left(P_n\right)-\gamma_{2r}\left(P_n-v_i\right)=\begin{cases}-1 & \text{for } i\neq 1 \text{ and } n,\\ 0 & \text{for } i=1 \text{ or } n.\end{cases}$$

Case 4. $n\equiv 1 \pmod 2$ and $i\equiv 0 \pmod 2$, then $\gamma_{2r}\left(P_n-v_i\right)=\lfloor n/2\rfloor+1$ for all i. Therefore,

$$ci_{2r}^{v}(v_i)=\gamma_{2r}\left(P_n\right)-\gamma_{2r}\left(P_n-v_i\right)=0 \text{ for all } i.$$

Now we can establish the patterns for $ci_{2r}^{v}\left(v_i\right)$, $1\leq i\leq n$.

$$ci_{2r}^{v}\left(v_i\right)=\begin{cases}1, & 0, & 0, & 0, & 0, & \ldots, & 0, & 1 & & \text{for } n\equiv 0 \pmod 2,\\ 0, & 0, & -1, & 0, & -1, & \ldots, & -1, & 0, & 0 & \text{for } n\equiv 1 \pmod 2,\end{cases}$$

which implies that if $n\equiv 0 \pmod 2$, then $ci_{2r}^{v}\left(P_n\right)=2/n$ and if $n\equiv 1 \pmod 2$, then $ci_{2r}^{v}\left(P_n\right)=-(n-3)/2n$. □

4. THE EDGE REMOVAL CRITICALITY INDEX OF 2-RAINBOW DOMINATION OF A CYCLE AND A PATH

In this section we determine the exact value of the edge removal criticality index of 2-rainbow domination of a cycle and a path. Recall that $ci_{2r}^{-e}(e) = \gamma_{2r}(G) - \gamma_{2r}(G-e)$ and $ci_{2r}^{-e}(e) \in \{-1, 0\}$, where $G = C_n$ or P_n, and $e \in E(G)$.

Theorem 4.1. *For every cycle C_n with $n \geq 3$,*

$$ci_{2r}^{-e}(C_n) = \begin{cases} -1 & \text{if } n \equiv 0 \pmod 4, \\ 0 & \text{if } n \equiv 1, 2, 3 \pmod 4. \end{cases}$$

Proof. Since removing any edge e of a cycle C_n produces a path P_n, by Propositions 2.2 and 2.3 we have

$$ci_{2r}^{-e}(e) = \gamma_{2r}(C_n) - \gamma_{2r}(P_n) = \lceil n/4 \rceil - \lfloor n/4 \rfloor - 1.$$

Therefore, we can see that $ci_{2r}^{-e}(e) = -1$ for $n \equiv 0 \pmod 4$ and $ci_{2r}^{-e}(e) = 0$ for $n \equiv 1, 2, 3 \pmod 4$, and so $ci_{2r}^{-e}(C_n) = -1$ for $n \equiv 0 \pmod 4$ and $ci_{2r}^{-e}(C_n) = 0$ for $n \equiv 1, 2, 3 \pmod 4$. □

Let P_n be a path whose vertices are labeled $v_1, v_2, \ldots, v_n$. Note that when an edge $v_i v_{i+1}$ is removed from the path P_n, we obtain two paths P_i and P_{n-i}.

Theorem 4.2. *For every nontrivial path P_n,*

$$ci_{2r}^{-e}(P_n) = \begin{cases} -(n-2)/2(n-1) & \text{if } n \equiv 0 \pmod 2, \\ -1 & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

Proof. Let $P_n = v_1 v_2 \ldots v_n$. Then by Proposition 2.3 we have

$$\gamma_{2r}(P_n - v_i v_{i+1}) = \gamma_{2r}(P_i) + \gamma_{2r}(P_{n-i}) = \lfloor i/2 \rfloor + \lfloor (n-i)/2 \rfloor + 2$$

for every i with $1 \leq i \leq n-1$. Two cases are distinguished with respect to the parity of i.

Case 1. $i \equiv 1 \pmod 2$. Then $\gamma_{2r}(P_n - v_i v_{i+1}) = \lfloor (n-1)/2 \rfloor + 2$, and so

$$ci_{2r}^{-e}(v_i v_{i+1}) = \gamma_{2r}(P_n) - \gamma_{2r}(P_n - v_i v_{i+1}) = \lfloor n/2 \rfloor - \lfloor (n-1)/2 \rfloor - 1.$$

Therefore, $ci_{2r}^{-e}(v_i v_{i+1}) = 0$ for $n \equiv 0 \pmod 2$ and $ci_{2r}^{-e}(v_i v_{i+1}) = -1$ for $n \equiv 1 \pmod 2$.

Case 2. $i \equiv 0 \pmod 2$. Then $\gamma_{r2}(P_n - v_i v_{i+1}) = \lfloor n/2 \rfloor + 2$, and so

$$ci_{2r}^{-e}(v_i v_{i+1}) = \gamma_{2r}(P_n) - \gamma_{2r}(P_n - v_i v_{i+1}) = \lfloor n/2 \rfloor - \lfloor n/2 \rfloor - 1,$$

Therefore, $ci_{2r}^{-e}(v_i v_{i+1}) = -1$ for every i such that $1 \leq i \leq n-1$.

Now we can establish the patterns for $ci_{2r}^{-e}(v_i v_{i+1})$, $1 \leq i \leq n-1$.

$$ci_{2r}^{-e}(v_i v_{i+1}) = \begin{cases} 0, & -1, & \ldots, & -1, & 0, & & \text{for } n \equiv 0 \pmod 2, \\ -1, & -1, & \ldots, & -1, & -1, & -1 & \text{for } n \equiv 1 \pmod 2 \end{cases}$$

which implies that if $n \equiv 0 \pmod 2$, then $ci_{2r}^{-e}(P_n) = -(n-2)/2(n-1)$ and if $n \equiv 1 \pmod 2$, then $ci_{2r}^{-e}(P_n) = -1$. □

5. THE EDGE ADDITION CRITICALITY INDEX OF 2-RAINBOW DOMINATION OF A CYCLE

In this section we give exact values of the edge addition criticality index of a 2-rainbow domination of a cycle. Let G be a graph obtained from a cycle C_n by adding a chord such that G is forming from two cycles C_p and C_q, where $n = p + q - 2$.

We first describe a procedure and give a lemma that are fundamental in determining the value $ci_{2r}^{+e}(C_n)$.

Procedure 5.1. *Let F_1 be the graph obtained from C_n by joining two non-adjacent vertices u and v with an edge. Suppose that F_1 has a cycle of length at least 7. Then F_1 has a subpath $P = w, u_1, u_2, u_3, u_4, v$ of the cycle, and we form the graph F_2 from F_1 by deleting vertices u_1, u_2, u_3 and u_4 and joining vertices w to v. We repeat this process until eventually we obtain a graph F_k having two cycles of order 3, 4, 5 or 6.*

Lemma 5.2. $\gamma_{2r}(F_{i+1}) = \gamma_{2r}(F_i) - 2.$

Proof. Let f be a γ_{2r}-function on F_{i+1} and $n_{i+1} = n(F_{i+1})$. If $f(u) = f(v) = \varnothing$, or $f(u) \neq \varnothing$ and $f(v) \neq \varnothing$, then f is a 2RDF of $C_{n_{i+1}}$ with $\gamma_{2r}(F_{i+1}) = w(f) \geq \gamma_{2r}(C_{n_{i+1}}) \geq \gamma_{2r}(F_{i+1})$,which implies that $\gamma_{2r}(C_{n_{i+1}}) = \gamma_{2r}(F_{i+1})$. By Observation 2.4, we have $\gamma_{2r}(F_{i+1}) = \gamma_{2r}(C_{n_{i+1}}) = \gamma_{2r}(C_{n_i}) - 2 \geq \gamma_{2r}(F_i) - 2$, since $\gamma_{2r}(C_{n_{i+1}}) = \gamma_{2r}(C_{n_i-4})$. Now, without loss of generality, suppose that $f(v) \neq \varnothing$ and $f(u) = \varnothing$. If $f(v) = \{1\}$ or $\{1,2\}$, then the extension g_1 of f on F_i, such that $g_1(x) = f(x)$ for all $x \in V(F_{i+1})$, $g_1(u_2) = g_1(u_4) = \varnothing$, $g_1(u_1) = \{1\}$ and $g_1(u_3) = \{2\}$, is a 2RDF on F_i. If $f(v) = \{2\}$, then the function g_2, such that $g_2(x) = f(x)$ for all $x \in V(F_{i+1})$, $g_2(u_2) = g_2(u_4) = \varnothing$, $g_2(u_1) = \{2\}$ and $g_2(u_3) = \{1\}$, is a 2RDF on F_i. So in all cases there is a 2RDF g on F_i with $\gamma_{2r}(F_i) \leq w(g) = \gamma_{2r}(F_{i+1}) + 2$.

Next, let f be a γ_{2r}-function on F_i. If $f(u) = f(v) = \varnothing$, or $f(u) \neq \varnothing$ and $f(v) \neq \varnothing$, then, by the same argument above, $\gamma_{2r}(F_i) \geq \gamma_{r2}(F_{i+1}) + 2$. Now, without loss of generality, suppose that $f(v) \neq \varnothing$ and $f(u) = \varnothing$. If $f(v) = \{1\}$ or $\{2\}$, then there exists a γ_{2r}-function on F_i such that $f(u_2) = f(u_4) = \varnothing$ and $(f(u_1), f(u_3)) = (\{1\},\{2\})$ or $(\{2\},\{1\})$, respectively. Finally, If $f(v) = \{1,2\}$, then there exists a γ_{2r}-function on F_i such that $\sum_{j=1}^{4} |f(u_j)| = 2$. So in all cases the restriction of f on F_{i+1}, is a 2RDF on F_{i+1} with $\gamma_{2r}(F_{i+1}) \leq w\left(f_{|F_{i+1}}\right) = \gamma_{2r}(F_i) - 2$. Hence, $\gamma_{2r}(F_{i+1}) = \gamma_{2r}(F_i) - 2$. □

Now we are ready to present the exact value $ci_{2r}^{+e}(C_n)$. Recall that $ci_{2r}^{+e}(e) = \gamma_{2r}(C_n) - \gamma_{2r}(C_n + e)$ and $ci_{2r}^{+e}(e) \in \{0,1\}$ for every $e \in E(\overline{G})$.

Theorem 5.3. *For a cycle C_n with $n \geq 3$,*

$$ci_{2r}^{+e}(C_n) = \begin{cases} 0 & \text{for } n \equiv 0,1,3 \pmod 4, \\ (n-2)/4(n-3) & \text{for } n \equiv 2 \pmod 4. \end{cases}$$

Proof. Let $F(n_1, n_2)$, where $n_1, n_2 \in \{3,4,5,6\}$, be the graph obtained from the cycle $C_{n_1+n_2-2}$ by adding a chord such that $F(n_1, n_2)$ is formed from two cycles C_{n_1} and C_{n_2}. The graph $F(n_1, n_2)$ will be called an elementary bicyclic graph.

By applying Procedure 5.1 on a $C_n + e$, where $e \in E\left(\overline{C_n}\right)$ on the resulting graphs as much as possible, at the end we obtain an elementary bicyclic graph $F(n_1, n_2)$ of order $n_1 + n_2 - 2$.

Let k_1 and k_2 denote the number of groups of four vertices that were removed from $C_n + e$ to obtain the cycles C_{n_1}, C_{n_2}, respectively, of the elementary bicyclic graph $F = F(n_1, n_2)$. Thus

$$k_1 + k_2 = (n - n(F))/4. \tag{5.1}$$

The number of nonnegative integer solutions of Equation (5.1) equals to

$$\mathtt{C}^1_{(n-n(F))/4+1} = (n - n(F) + 4)/4.$$

By the symmetry of the vertices of C_n and since every edge is computed two times for $n_1 = n_2$, the number of graphs $C_n + e$ corresponding to the elementary bicyclic graph F equals to

$$\begin{cases} \frac{n}{2}(n - n(G) + 4)/4 & \text{if } n_1 = n_2, \\ n(n - n(G) + 4)/4 & \text{if } n_1 \neq n_2. \end{cases}$$

By Observation 2.4 and Lemma 5.2, we have that

$$ci^{+e}_{2r}(e) = \gamma_{2r}(C_n) - \gamma_{2r}(C_n + e) = \gamma_{2r}(C_{n_1+n_2-2}) - \gamma_{2r}(F)$$

for some $e \in E\left(\overline{C_n}\right)$.

Let $\mathcal{F}_i$, for $i = 0, 1$, be the set of all elementary bicyclic graphs $F = F(n_1, n_2)$ for which $ci^{+e}_{2r}(e) = i$ and set $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$. Therefore,

$$\begin{aligned} ci^{+e}_{2r}(C_n) &= \left(\sum\nolimits_{e\in E(\overline{C_n})} ci^{+e}_{2r}(e)\right)/m\left(\overline{C_n}\right) \\ &= \left(\sum\nolimits_{F\in\mathcal{F}_1} (\#\text{ of graphs } C_n + e \text{ corresponding to } F)\right)/m\left(\overline{C_n}\right) \\ &= \left(\sum\nolimits_{F\in\mathcal{F}_1} n(n - n(F) + 4)/8\right)/m\left(\overline{C_n}\right). \end{aligned}$$

Note that $m\left(\overline{C_n}\right) = n(n-3)/2$, so

$$ci^{+e}_{2r}(C_n) = \left(\sum\nolimits_{F\in\mathcal{F}_1} (n - n(F) + 4)/4(n-3)\right). \tag{5.2}$$

Then by applying Procedure 5.1, we consider four cases with respect to n.

Case 1. $n \equiv 0 \pmod 4$. We have $n(F) \equiv 0 \pmod 4$. Note that $n(F) = n_1 + n_2 - 2 = 4$ or 8 for each $F \in \mathcal{F}$. So,

$$\mathcal{F} = \{F(3,3), F(4,6), F(5,5)\}.$$

It is a routine matter to check that $\mathcal{F}_1 = \varnothing$ and $\mathcal{F}_0 = \mathcal{F}$. So, by Equation (5.2), we have $ci^{+e}_{2r}(C_n) = 0$.

Case 2. $n \equiv 1 \pmod 4$. We have $n(F) \equiv 1 \pmod 4$. Note that $n(F) = n_1 + n_2 - 2 = 5$ or 9 for each $F \in \mathcal{F}$. So,

$$\mathcal{F} = \{F(3,4), F(5,6)\}.$$

We can easily check that $\mathcal{F}_1 = \varnothing$ and $\mathcal{F}_0 = \mathcal{F}$. So, by Equation (5.2), we have $ci_{2r}^{+e}(C_n) = 0$.

Case 3. $n \equiv 2 \pmod 4$. We have $n(F) \equiv 2 \pmod 4$. Note that $n(F) = n_1 + n_2 - 2 = 6$ or 10 for each $F \in \mathcal{F}$. So,

$$\mathcal{F} = \{F(3,5), F(4,4), F(6,6)\}.$$

It is easy to see that $\mathcal{F}_1 = \{F(4,4)\}$ and $\mathcal{F}_0 = \{F(3,5), F(6,6)\}$. So, by Equation (5.2), we have

$$ci_{2r}^{+e}(C_n) = (n - n(F(4,4)) + 4)/4(n-3) = (n - 6 + 4)/4(n-3) = (n-2)/4(n-3).$$

Case 4. $n \equiv 3 \pmod 4$. We have $n(F) \equiv 3 \pmod 4$. Note that $n(F) = n_1 + n_2 - 2 = 7$ for each $F \in \mathcal{F}$. So,

$$\mathcal{F} = \{F(3,6), F(4,5)\}.$$

Again it is easy to see that $\mathcal{F}_1 = \varnothing$ and $\mathcal{F}_0 = \mathcal{F}$. So, by Equation (5.2), we have $ci_{2r}^{+e}(C_n) = 0$, and the proof is complete. □

6. THE EDGE ADDITION CRITICALITY INDEX OF A 2-RAINBOW DOMINATION OF A PATH

In this section we give exact values of the edge addition criticality index of a 2-rainbow domination of a path P_n.

We first give a lemma that is fundamental in determining the value $ci_{2r}^{+e}(P_n)$.

Lemma 6.1. *Let $G = P_n + uv$ be a graph obtained from a path P_n of order $n \geq 3$ by adding a chord (u,v) forming two paths P_p, P_q and a cycle C_t, where $n = p + q + t$. Then $\gamma_{2r}(P_n + uv) = \gamma_{2r}(P_n) - 1$ if and only if either*

1. *$n = 4$ and $uv \in E\left(\overline{P_4}\right)$, or*
2. *$n \neq 4$ and $uv \in \mathcal{E} = \{e \in E\left(\overline{P_n}\right) \mid n \equiv 0 \pmod 2,\ pq = 0$ and $t \equiv 0 \pmod 4\}$.*

Proof. If $n = 4$, then it is easy to see that $G = K_{1,3} + e$ or $G = C_4$, and so $\gamma_{2r}(G) = \gamma_{2r}(P_4) - 1$ for all edge uv of $E\left(\overline{P_4}\right)$. Now assume that $n \geq 3$ and $n \neq 4$. If G is a cycle, then $p = q = 0$ and $t = n$. By Proposition 2.2, $uv \notin \mathcal{E}$ and $\gamma_{2r}(G) = \gamma_{2r}(P_n)$ for $n \equiv 1, 2, 3 \pmod 4$, and $uv \in \mathcal{E}$ and $\gamma_{2r}(G) = \gamma_{2r}(P_n) - 1$ for $n \equiv 0 \pmod 4$. Now we suppose that G is not a cycle, then G is obtained from the graph $G' = C_n + uv$ by removing an edge $e \neq uv$. In this case $p \neq 0$ or $q \neq 0$. We suppose, without loss of generality, that $p \neq 0$. Let f be a γ_{2r}-function on G. We consider two cases:

Case 1. $n \equiv 1 \pmod 2$. Then $uv \notin \mathcal{E}$, and by Proposition 2.1 (ii), we have $\gamma_{2r}(G) \geq \gamma_{2r}(G')$, and so from Theorem 5.3 and Proposition 2.2, we obtain that $\gamma_{2r}(G) \geq \gamma_{2r}(G') = \gamma_{2r}(C_n) = \gamma_{2r}(P_n)$. Since $\gamma_{2r}(G) \leq \gamma_{2r}(P_n)$ (see Proposition 2.1 (iii)), we deduce that $\gamma_{2r}(G) = \gamma_{2r}(P_n)$.

Case 2. $n \equiv 0 \pmod 2$. We have to examine three possibilities:

Subcase 2.1. $q \neq 0$. Then $uv \notin \mathcal{E}$. If $f(u) = f(v) = \emptyset$, or $f(u) \neq \emptyset$ and $f(v) \neq \emptyset$, then $\gamma_{2r}(P_n) \leq \gamma_{2r}(G)$ and by Proposition 2.1 (iii), $\gamma_{2r}(G) = \gamma_{2r}(P_n)$. Now we suppose, without loss of generality, that $f(u) \neq \emptyset$ and $f(v) = \emptyset$. Let P_{p+t-1} be the subpath of G defined by the vertices $V(P_p) \cup (V(C_t) - \{v\})$. It is clear that the restriction of f on $V(P_{p+t-1})$ is a 2RDF on P_{p+t-1} and the restriction of f on $V(P_q)$ is a 2RDF on P_q. Thus, by Proposition 2.3,

$$\begin{aligned}\gamma_{2r}(G) &= w(f_{|P_{p+t-1}}) + w\left(f_{|P_q}\right) \geq \gamma_{2r}(P_{p+t-1}) + \gamma_{2r}(P_q)\\ &= \lceil (p+t)/2 \rceil + \lceil (q+1)/2 \rceil \geq (p+t)/2 + (q+1)/2 = (n+1)/2.\end{aligned}$$

Hence $\gamma_{2r}(G) \geq \lceil (n+1)/2 \rceil = \gamma_{2r}(P_n)$, and so $\gamma_{2r}(G) = \gamma_{2r}(P_n)$.

Subcase 2.2. $q = 0$ and $t \equiv 1, 2, 3 \pmod 4$. Then $uv \notin \mathcal{E}$. If $f(u) = f(v) = \emptyset$, or $f(u) \neq \emptyset$ and $f(v) \neq \emptyset$, then similarly to Subcase 2.1, we have $\gamma_{2r}(G) = \gamma_{2r}(P_n)$. Now we suppose that $f(u) = \emptyset$ and $f(v) \neq \emptyset$, or $f(u) \neq \emptyset$ and $f(v) = \emptyset$.

If $f(u) = \emptyset$ and $f(v) \neq \emptyset$, then the restriction of f on $V(P_p)$ is a 2RDF on P_p and the restriction of f on $V(C_t) - \{u\}$ is a 2RDF on P_{t-1}. Thus, by Proposition 2.3,

$$\begin{aligned}\gamma_{2r}(G) &= w\left(f_{|P_p}\right) + w\left(f_{|P_{t-1}}\right) \geq \gamma_{2r}(P_p) + \gamma_{2r}(P_{t-1})\\ &= \lceil (p+1)/2 \rceil + \lceil t/2 \rceil \geq (p+1)/2 + t/2 = (n+1)/2.\end{aligned}$$

Hence, $\gamma_{2r}(G) \geq \lceil (n+1)/2 \rceil = \gamma_{2r}(P_n)$ and so $\gamma_{2r}(G) = \gamma_{2r}(P_n)$.

If $f(u) \neq \emptyset$, $f(v) = \emptyset$ and $p \geq 2$, then there is a γ_{2r}-function on G such that $f(x) = \emptyset$, where $x \in N(u) \cap V(P_p)$, and so the restriction of f on $V(P_p) - \{x\}$ is a 2RDF on the subpath P_{p-1} and the restriction of f on $V(C_t)$ is a 2RDF on C_t. Thus, by Propositions 2.3 and 2.2,

$$\begin{aligned}\gamma_{2r}(G) &= w(f_{|P_{p-1}}) + w\left(f_{|C_t}\right) \geq \gamma_{2r}(P_{p-1}) + \gamma_{2r}(C_t)\\ &= \lceil p/2 \rceil + \lceil (t+1)/2 \rceil \geq p/2 + (t+1)/2 = (n+1)/2.\end{aligned}$$

Hence, $\gamma_{2r}(G) \geq \lceil (n+1)/2 \rceil = \gamma_{2r}(P_n)$ and so $\gamma_{2r}(G) = \gamma_{2r}(P_n)$.

If $f(u) \neq \emptyset$, $f(v) = \emptyset$ and $p = 1$, then $t \equiv 1, 3 \pmod 4$ and $t \neq 3$, since $n \equiv 0 \pmod 2$ and $n \neq 4$. Let $x, v \in N(u) \cap V(C_t)$ and z be the unique leaf in G. We have to examine possibilities for f depending on whether $|f(u)| = 2$ or $|f(u)| = 1$.

If $|f(u)| = 2$, then there exists a γ_{2r}-function on G such that the restriction of f on $\{u, z\}$ is a 2RDF on the subpath P_2, the restriction of f on $V(C_t - \{x, v, u\})$ is a 2RDF on the subpath P_{t-3} and $f(x) = \varnothing$. Thus, by Proposition 2.3,

$$\begin{aligned}\gamma_{2r}(G) &= w\left(f_{|P_2}\right) + w\left(f_{|P_{t-3}}\right) \geq \gamma_{2r}(P_2) + \gamma_{2r}(P_{t-3})\\ &= 2 + \lceil (t-2)/2 \rceil = 1 + (t+1)/2 = n/2 + 1.\end{aligned}$$

Hence, $\gamma_{2r}(G) \geq n/2 + 1 = \gamma_{2r}(P_n)$ and so $\gamma_{2r}(G) = \gamma_{2r}(P_n)$.

If $|f(u)| = 1$, then the restriction of f on $V(C_t)$ is a 2RDF on C_t and $|f(z)| = 1$. Thus, by Propositions 2.3 and 2.2,

$$\begin{aligned}\gamma_{2r}(G) &= |f(z)| + w\left(f_{|C_t}\right) \geq 1 + \gamma_{2r}(C_t)\\ &= 1 + \lceil (t+1)/2 \rceil = 1 + (t+1)/2 = n/2 + 1.\end{aligned}$$

Hence, $\gamma_{2r}(G) \geq n/2+1 = \gamma_{2r}(P_n)$ and so $\gamma_{2r}(G) = \gamma_{2r}(P_n)$.
Subcase 2.3. $q = 0$ and $t \equiv 0 \pmod 4$. Then $p \equiv 0 \pmod 2$ and so $uv \in \mathcal{E}$. Since C_t is vertex transitive, there exists a γ_{2r}-function h_1 on C_t, with $h_1(u) = \{1\}$. And there exists a γ_{2r}-function h_2 on the subpath of G defined by the vertices $V(P_p) \cup \{u\}$, with $h_2(u) = \{1\}$. Let h be a function on G defined as follows,

$$h(x) = \begin{cases} h_1(u) & \text{if } x \in V(C_t), \\ h_2(x) & \text{if } x \in V(P_p). \end{cases}$$

It is easy to see that h is a 2RDF on G. Thus, by Propositions 2.3 and 2.2,

$$\begin{aligned} \gamma_{2r}(G) \leq w(h) &= w(h_1) + w(h_2) - 1 = \gamma_{2r}(C_t) + \gamma_{2r}(P_{p+1}) - 1 \\ &= t/2 + \lfloor (p+1)/2 \rfloor + 1 - 1 = t/2 + p/2 = n/2. \end{aligned}$$

Hence, $\gamma_{2r}(G) \leq \gamma_{2r}(P_n) - 1$, and so by Proposition 2.1 (iii), $\gamma_{2r}(G) = \gamma_{2r}(P_n) - 1$. $\square$

Now we are ready to present the exact value $ci_{2r}^{+e}(P_n)$. Recall that $ci_{2r}^{+e}(e) = \gamma_{2r}(P_n) - \gamma_{2r}(P_n + e)$ and $ci_{2r}^{+e}(e) \in \{0,1\}$ for every $e \in E(\overline{G})$.

Theorem 6.2. *For a path P_n,*

$$ci_{2r}^{+e}(P_n) = \begin{cases} 1/(n-1) & \textit{for } n \geq 5 \textit{ and } n \equiv 0 \pmod 2, \\ 0 & \textit{for } n \geq 3 \textit{ and } n \equiv 1 \pmod 2, \\ 1 & \textit{for } n = 4. \end{cases}$$

Proof. If $n = 4$, then $G = K_{1,3} + e$ or C_4, and it is easy to see that $ci_{2r}^{+e}(e) = 1$ for all edge e of $E(\overline{P_4})$. Hence $ci_{2r}^{+e}(P_n) = 1$.

Now assume that $n \geq 3$ and $n \neq 4$. Two cases are distinguished with respect to the parity of n.
Case 1. $n \equiv 1 \pmod 2$. Then $e \notin \mathcal{E}$ for all edge e of $E(\overline{P_n})$, and from Lemma 6.1, $ci_{2r}^{+e}(e) = 0$ which implies that $ci_{2r}^{+e}(P_n) = 0$.
Case 2. $n \equiv 0 \pmod 2$. Then by Lemma 6.1, $ci_{2r}^{+e}(e) = 1$ for $e \in \mathcal{E}$, and $ci_{2r}^{+e}(e) = 0$ for $e \in E(\overline{P_n}) - \mathcal{E}$. So

$$\begin{aligned} ci_{2r}^{+e}(P_n) &= \left(\sum\nolimits_{e \in E(\overline{P_n})} ci_{2r}^{+e}(e)\right) / m(\overline{P_n}) \\ &= \left(\sum\nolimits_{e \in \mathcal{E}} (\#\text{ of graphs } P_n + e \text{ corresponding to } e)\right) / m(\overline{P_n}). \\ &= |\mathcal{E}| / m(\overline{P_n}). \end{aligned}$$

Therefore,

$$ci_{2r}^{+e}(P_n) = |\mathcal{E}| / m(\overline{P_n}). \tag{6.1}$$

Note that $m(\overline{P_n}) = (n-1)(n-2)/2$, and the number of edges of $\mathcal{E}$ is

$$\begin{aligned} |\mathcal{E}| &= \begin{cases} 2(n/4) - 1 & \text{for } n \equiv 0 \pmod 4, \\ 2(n-2)/4 & \text{for } n \equiv 2 \pmod 4 \end{cases} \\ &= n/2 - 1. \end{aligned}$$

Hence, by Equation (6.1), we obtain that

$$ci_{2r}^{+e}(P_n) = 2(n/2-1)/(n-1)(n-2) = 1/(n-1),$$

and the proof is complete. □

Acknowledgments

This work was supported by 'PNR: Code 8/u09/510'.

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Received: December 6, 2014.
Revised: February 15, 2016.
Accepted: March 2, 2016.