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EXISTENCE AND BOUNDARY BEHAVIOR OF POSITIVE SOLUTIONS FOR A STURM–LIOUVILLE PROBLEM

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Abstract. In this paper, we discuss existence, uniqueness and boundary behavior of a positive solution to the following nonlinear Sturm–Liouville problem

$$\begin{array}{l} \frac{1}{A}(Au')' + a(t)u^{\sigma} = 0 \ \text{ in } (0,1), \\ \lim\limits_{t\to 0} Au'(t) = 0, \quad u(1) = 0, \end{array}$$

where $\sigma < 1$, A is a positive differentiable function on $(0,1)$ and a is a positive measurable function in $(0,1)$ satisfying some appropriate assumptions related to the Karamata class. Our main result is obtained by means of fixed point methods combined with Karamata regular variation theory.

Keywords: nonlinear Sturm–Liouville problem, Green's function, positive solutions, Karamata regular variation theory.

Mathematics Subject Classification: 34B18, 34B27.

1. INTRODUCTION

Many physical situations are modelled by Sturm–Liouville equations of type

$$\frac{1}{A}(Au')' + f(t,u) = 0 \text{ in } (0,1), \tag{1.1}$$

and numerous existence results have been established for (1.1) with various boundary data (see [1, 3–5, 7, 10–14, 16, 20–24] and the reference therein). For example, the following problem

$$\begin{cases} u'' + a(t)u^{\sigma} = 0 \quad \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{1.2}$$

is the well-known Emden-Fowler equation with a Dirichlet boundary value condition. Several problems in nonlinear mechanics [14], gas and fluid dynamics [5, 14] result in a problem of the form (1.2) (usually with $a(t) = 1$ and $\sigma < 0$). The existence and uniqueness of positive solutions of (1.2) have been studied by Taliaferro in [21] for $\sigma < 0$ using the shooting method and by Zhang in [24] for $0 < \sigma < 1$ using the method of lower and upper solutions.

More recently, the authors in [20] examined the following problem

$$\begin{cases} \frac{1}{A}(Au')' + f(t,u) = 0 \quad \text{in } (0,1), \\ \alpha u(0) - \beta \lim\limits_{t\to 0} Au'(t) = 0, \\ \gamma u(1) + \delta \lim\limits_{t\to 1} Au'(t) = 0, \end{cases} \tag{1.3}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ such that $\beta\gamma + \alpha\gamma + \alpha\delta > 0$. Using some fixed point index theorems, they proved existence results for (1.3) provided that $0 < \int_0^1 dt/A(t) < \infty$.

If $(\alpha, \delta) \neq (0,0)$ most of existence results require the condition $\int_0^1 dt/A(t) < \infty$ (see [1, 4, 11–13, 20, 23]). However, if $\alpha = \delta = 0$ this condition seems to be too restrictive from an application viewpoint. Indeed, if $A(t) = t^{n-1}, n > 2$ and $\alpha = \delta = 0$, the problem (1.3) arises naturally when looking for radial solutions of elliptic Dirichlet problems in higher dimensions.

In [16], Mâagli and Masmoudi considered the equation (1.1) under the condition $u'(0) = 0$, $u(1) = 0$, where A satisfies the following assumption:

(H1) $A \in C([0,1))$, positive and differentiable on $(0,1)$ such that $\int_\varepsilon^1 dt/A(t) < \infty$, for some $\varepsilon > 0$.

Based on potential theory tools, Mâagli and Masmoudi proved in [16] an existence and a uniqueness result provided that the nonlinear term f satisfies

(H2) $f : [0,1) \times (0,+\infty) \to (0,+\infty)$ is a continuous function, nonincreasing with respect to the second variable such that

$$\int_0^1 A(s)\rho(s)f(s,c)ds < +\infty \quad \text{for all} \quad c > 0.$$

Here and throughout this paper, ρ denotes the function defined on $(0,1]$ by $\rho(t) := \int_t^1 \frac{ds}{A(s)}$.

The following result is due to ([16, Theorem 2]).

Theorem 1.1 ([16]). *Let A satisfies* (H1) *such that*

$$\lim_{t\to 0} \frac{1}{A(t)} \int_0^t A(s)ds < \infty \tag{1.4}$$

and assume that (H2) *holds. Then* (1.1) *has a unique positive solution* $u \in C([0,1]) \cap C^1([0,1)) \cap C^2((0,1))$ *satisfies* $u'(0) = 0$, $u(1) = 0$.

Example 1.2. As an example of f satisfying (H2), the authors in [16] quote the following: $f(t,u) = a(t)u^{\sigma}$, $\sigma < 0$ where a is a nonnegative continuous function on $[0,1)$ satisfying

$$\int_0^1 A(s)\rho(s)a(s)ds < +\infty. \tag{1.5}$$

Remark 1.3. One can see that in (H1) there are not any conditions on the integrabilty of $1/A$ near 0. This permits us to include both examples $A(t) = 1$ and $A(t) = t^{n-1}$. Also, we note that the boundary condition $u'(0) = 0$ considered in Theorem 1.1 can be replaced by $Au'(0) = 0$ and then we can omit the integrability condition (1.4) imposed on the function A in [16].

In this paper, we are interested in establishing the existence and the boundary behavior of a unique positive continuous solution to the following nonlinear Sturm–Liouville boundary value problem

$$\begin{cases} \frac{1}{A}(Au')' + a(t)u^{\sigma} = 0 \quad \text{in } (0,1), \\ \lim\limits_{t\to 0} Au'(t) = 0, \ \ u(1) = 0, \end{cases} \tag{1.6}$$

where $\sigma < 1$, A satisfies (H1) and the weight function a is a positive measurable function on $(0,1)$ and satisfies a suitable condition relying to a functional class $\mathcal{K}$ called the Karamata class and defined on $(0,\eta], \eta > 1$, by

$$\mathcal{K} := \left\{ t \mapsto L(t) := c\exp\Big(\int_t^{\eta} \frac{z(s)}{s}ds\Big) : c > 0 \ \ and \ \ z \in C([0,\eta]), \ \ z(0) = 0 \right\}.$$

The Karamata regular variation theory has been shown to be very useful in the study of boundary behavior of solutions for differential equations. The use of this theory in the asymptotic analysis of solutions of nonlinear elliptic equations is due to Cirstea and Radulescu and a series of very rich and significant information about the qualitative behavior of solutions are obtained (see for example [2,3,6,8,9,11,15,17,18] and the references therein). Based on this theory, we focus our study on the asymptotic behavior of the unique solution of (1.6). This is not given in the previous work.

We need the following notation: For two nonnegative functions f and g defined in a set S, the notation $f(x) \approx g(x)$, $x \in S$, means that there exists a constant $c > 0$ such that $\frac{1}{c}g(x) \le f(x) \le cg(x)$ for each $x \in S$.

Let us give our conditions. We assume without loss of generality, that $\rho(0) \in [1,\infty]$ and we require the following:

(H3) a is a measurable function on $(0,1)$ and satisfies for $t \in (0,1)$,

$$a(t) \approx \frac{1}{A(t)^2}\min(1,\rho(t))^{-\lambda}(1+\rho(t))^{-\mu}L(\min(1,\rho(t))),$$

where $\lambda \le 2, \mu > 2$ and $L \in \mathcal{K}$ defined on $(0,\eta]$ $(\eta > 1)$ such that $\int_0^{\eta} t^{1-\lambda}L(t)dt < \infty$.

Our main result is stated as follows:

Theorem 1.4. *Let $\sigma < 1$ and assume* (H1) *and* (H3) *hold. Then problem* (1.6) *has a unique positive solution $u \in C([0,1]) \cap C^1((0,1))$ satisfying*

$$u(t) \approx \min(1,\rho(t))^{\min\left(1,\frac{2-\lambda}{1-\sigma}\right)} \Psi_{L,\sigma}(\min(1,\rho(t))),\ t \in (0,1), \tag{1.7}$$

where $\Psi_{L,\sigma}$ is the function defined on $(0,\eta]$ by

$$\Psi_{L,\sigma}(t) := \begin{cases} \left(\int\limits_0^t \frac{L(s)}{s} ds\right)^{\frac{1}{1-\sigma}} & \text{if } \lambda = 2, \\ (L(t))^{\frac{1}{1-\sigma}} & \text{if } 1+\sigma < \lambda < 2, \\ \left(\int\limits_t^\eta \frac{L(s)}{s} ds\right)^{\frac{1}{1-\sigma}} & \text{if } \lambda = 1+\sigma, \\ 1 & \text{if } \lambda < 1+\sigma. \end{cases} \tag{1.8}$$

Remark 1.5. In hypothesis (H3) we need to verify condition $\int_0^\eta t^{1-\lambda} L(t)dt < \infty$ only for the case $\lambda = 2$. This is due to Karamata's theorem which we recall in Lemma 2.3 below.

Remark 1.6. Using Lemma 3.1 cited below, if the function a satisfies (H3), then (1.5) holds. So, we undertake the existence of a unique positive solution of (1.6) only when $\sigma > 0$. The case $\sigma < 0$ is deduced by Theorem 1.1.

The outline of the paper is as follows. In Section 2, some useful results on functions in $\mathcal{K}$ have been summarized. In Section 3, we establish the asymptotic behavior of some potential functions which will be needed in order to prove Theorem 1.4 in Section 4. To illustrate our main result, we give an example in Section 4.

2. KARAMATA CLASS

Our approach relies on Karamata regular variation theory. In this section, we collect some old and new properties of functions belonging to the class $\mathcal{K}$ which come from [19].

It is obvious to see that a function L is in $\mathcal{K}$ if and only if L is a positive function in $C^1\big((0,\eta]\big)$ such that

$$\lim_{t\to 0} \frac{tL'(t)}{L(t)} = 0. \tag{2.1}$$

A standard function belonging to the class $\mathcal{K}$ is given in the following example.

Example 2.1. Let $p \in \mathbb{N}^*$. Let $(\lambda_1, \lambda_2, \ldots, \lambda_p) \in \mathbb{R}^p$ and ω be a sufficiently large positive real number such that the function

$$L(t) = \prod_{k=1}^{p} \left(\log_k\left(\frac{\omega}{t}\right)\right)^{-\lambda_k}$$

is defined and positive on $(0, \eta]$, for some $\eta > 1$, where $\log_k(t) = \log \circ \log \ldots \circ \log(t)$ (k times). Then $L \in \mathcal{K}$.

Lemma 2.2 ([19])**.**

(i) *Let* $L \in \mathcal{K}$ *and* $\varepsilon > 0$. *Then we have*

$$\lim_{t \to 0} t^{\varepsilon} L(t) = 0.$$

(ii) *Let* $L_1, L_2 \in \mathcal{K}, p \in \mathbb{R}$. *Then the functions* $L_1 + L_2$, $L_1 L_2$ *and* L_1^p *are in* $\mathcal{K}$.

Lemma 2.3 (Karamata's Theorem [19])**.** *Let* $L \in \mathcal{K}$ *be defined on* $(0, \eta]$ *and* $\gamma \in \mathbb{R}$. *Then we have*

(i) *If* $\gamma > -1$, *then* $\int_0^\eta s^\gamma L(s) ds$ *converges and*

$$\int_0^t s^\gamma L(s) ds \sim \frac{t^{1+\gamma} L(t)}{\gamma + 1} \text{ as } t \to 0^+.$$

(ii) *If* $\gamma < -1$, *then* $\int_0^\eta s^\gamma L(s) ds$ *diverges and*

$$\int_t^\eta s^\gamma L(s) ds \sim -\frac{t^{1+\gamma} L(t)}{\gamma + 1} \text{ as } t \to 0^+.$$

Lemma 2.4 ([6])**.** *Let* $L \in \mathcal{K}$ *be defined on* $(0, \eta]$. *Then we have*

$$\lim_{t \to 0} \frac{L(t)}{\int_t^\eta \frac{L(s)}{s} ds} = 0. \tag{2.2}$$

If further $\int_0^\eta \frac{L(t)}{t} dt$ *converges, then we have*

$$\lim_{t \to 0} \frac{L(t)}{\int_0^t \frac{L(s)}{s} ds} = 0. \tag{2.3}$$

Remark 2.5. Let $L \in \mathcal{K}$ be defined on $(0, \eta]$, by using (2.2) and (2.1) we deduce that

$$t \to \int_t^\eta \frac{L(s)}{s} ds \in \mathcal{K}.$$

If further $\int_0^\eta \frac{L(t)}{t} dt$ converges, then we get by (2.3) and (2.1) that

$$t \to \int_0^t \frac{L(s)}{s} ds \in \mathcal{K}.$$

This proves by Lemma 2.2 (ii) that the function $\Psi_{L,\sigma}$ defined by (1.8) is in $\mathcal{K}$.

Lemma 2.6. *Let L be a function in $\mathcal{K}$ defined on $(0,\eta]$ and $f,g : S \to (0,\eta]$ be two functions defined on a set S such that $f \approx g$ on S. Then*

$$L \circ f \approx L \circ g \text{ on } S.$$

Proof. Let $c_0 > 0$ and z be the function in $C([0,\eta])$ such that $z(0) = 0$ and $L(t) = c_0 \exp\left(\int_t^\eta \frac{z(s)}{s} ds\right)$. Put $m = \sup_{t\in[0,\eta]} |z(t)|$, then for each $t \in [0,\eta]$

$$-m \le z(t) \le m.$$

Moreover, let $c > 0$ such that for each $x \in S$

$$\frac{1}{c} g(x) \le f(x) \le c g(x).$$

It follows that for $x \in S$,

$$-m \log c \le \int_{f(x)}^{g(x)} \frac{z(t)}{t} dt \le m \log c.$$

This implies that

$$c^{-m} \le \frac{L(g(x))}{L(f(x))} \le c^m.$$

Then the result holds. □

3. SHARP ESTIMATES ON SOME POTENTIAL FUNCTIONS

In the sequel, we denote $G(t,s)$ the Green's function of the operator $u \mapsto -\frac{1}{A}(Au')'$ with Dirichlet condition $Au'(0) = 0$, $u(1) = 0$ given by

$$G(t,s) = A(s) \int_{\max(t,s)}^{1} \frac{dr}{A(r)} = A(s) \min(\rho(t), \rho(s)) \tag{3.1}$$

and we refer to Vf, the potential of a nonnegative measurable function f defined on $(0,1)$ by

$$Vf(t) := \int_0^1 G(t,s) f(s) ds.$$

We point out that if f is a nonnegative measurable function such that the mapping $t \to A(t)f(t)$ is integrable in $[0,1]$, then Vf is the solution of the problem

$$\begin{cases} \frac{1}{A}(Au')' + f = 0 \text{ in } (0,1), \\ \lim_{t\to 0} Au'(t) = 0, \quad u(1) = 0. \end{cases}$$

Let a be a function satisfying (H3). So, we are going to derive estimates on the potential function Va. Throughout this section, we fix $t_0 \in [0,1)$ such that $\rho(t) < 1$, for $t \in [t_0, 1]$.

First, we will prove the following lemma.

Lemma 3.1. *Let a be a function satisfying* (H3), *then*

$$Va\,(0) = \int_0^1 A(s)\rho(s)a(s)ds < \infty.$$

In particular, the function Va is positive and continuous on $[0,1)$.

Proof. Using (H3), it yields

$$\begin{aligned}\int_0^1 A(s)\rho(s)a(s)ds &\approx \int_0^1 \frac{1}{A(s)}\rho(s)\min(1,\rho(s))^{-\lambda}(1+\rho(s))^{-\mu}L(\min(1,\rho(s)))ds \\ &= \int_0^{\rho(0)} \xi \min(1,\xi)^{-\lambda}(1+\xi)^{-\mu}L(\min(1,\xi))d\xi \\ &\approx \int_0^1 \xi^{1-\lambda}L(\xi)d\xi + \int_1^{\rho(0)} \xi^{1-\mu}d\xi.\end{aligned}$$

Since $\mu > 2$ and $\int_0^\eta t^{1-\lambda}L(t)dt < \infty$, we deduce the result. □

Proposition 3.2. *Let a be a function satisfying* (H3). *Then we have for $t \in (0,1)$*

$$Va(t) \approx \min(1,\rho(t))^{\min(1,2-\lambda)}\Psi_{L,0}(\min(1,\rho(t))), \tag{3.2}$$

where $\Psi_{L,0}$ is the function defined on $(0,\eta]$ by (1.8).

Proof. Since the function $t \to \min(1,\rho(t))^{\min(1,2-\lambda)}\Psi_{L,0}(\min(1,\rho(t))$ is positive and continuous on $[0,1)$, then by Lemma 3.1 we have to prove the estimates (3.2) only on $[t_0,1)$.

Let $t \in [t_0,1)$, using (H3) it follows that

$$\begin{aligned}Va(t) &= \int_0^1 A(s)\min(\rho(t),\rho(s))a(s)ds \\ &\approx \int_0^1 \frac{\min(\rho(t),\rho(s))}{A(s)}\min(1,\rho(s))^{-\lambda}(1+\rho(s))^{-\mu}L(\min(1,\rho(s)))ds \\ &= \int_0^{\rho(0)} \min(\rho(t),\xi)\min(1,\xi)^{-\lambda}(1+\xi)^{-\mu}L(\min(1,\xi))d\xi := F(\rho(t)),\end{aligned}$$

where $F(r) = \int_0^{\rho(0)} \min(r,\xi)\min(1,\xi)^{-\lambda}(1+\xi)^{-\mu}L(\min(1,\xi))d\xi$ for $r \in (0,1]$.

We aim to estimate $F(r)$. Let $r \in (0,1]$, then we have

$$F(r) \approx \int_0^1 \min(r,\xi)\xi^{-\lambda}L(\xi)d\xi + rL(1)\int_1^{\rho(0)} (1+\xi)^{-\mu}\, d\xi := I(r) + J(r).$$

It is obvious to see that

$$J(r) \approx r. \tag{3.3}$$

On the other hand,

$$I(r) = \int_0^r \xi^{1-\lambda}L(\xi)d\xi + r\int_r^1 \xi^{-\lambda}L(\xi)d\xi := I_1(r) + I_2(r).$$

Using Lemma 2.3, we deduce that

$$I_1(r) \approx \begin{cases} \int_0^r \frac{L(\xi)}{\xi}d\xi & \text{if } \lambda = 2, \\ r^{2-\lambda}L(r) & \text{if } \lambda < 2 \end{cases}$$

and

$$I_2(r) \approx \begin{cases} r^{2-\lambda}L(r) & \text{if } 1 < \lambda \leq 2, \\ r\int_r^1 \frac{L(\xi)}{\xi}d\xi & \text{if } \lambda = 1, \\ r & \text{if } \lambda < 1. \end{cases}$$

It follows that

$$I(r) \approx \begin{cases} \int_0^r \frac{L(\xi)}{\xi}d\xi + L(r) & \text{if } \lambda = 2, \\ r^{2-\lambda}L(r) & \text{if } 1 < \lambda < 2, \\ r\left(L(r) + \int_r^\eta \frac{L(\xi)}{\xi}d\xi\right) & \text{if } \lambda = 1, \\ r(r^{1-\lambda}L(r) + 1) & \text{if } \lambda < 1. \end{cases}$$

Hence, we deduce by (2.2), (2.3) and Lemma 2.2 (i) that

$$I(r) \approx \begin{cases} \int_0^r \frac{L(\xi)}{\xi} d\xi & \text{if } \lambda = 2, \\ r^{2-\lambda} L(r) & \text{if } 1 < \lambda < 2, \\ r \int_r^\eta \frac{L(\xi)}{\xi} d\xi & \text{if } \lambda = 1, \\ r & \text{if } \lambda < 1. \end{cases} \tag{3.4}$$

Combining (3.4) with (3.3), we deduce by Lemma 2.2 (i) and Remark 2.5 that for $0 < r \leq 1$,

$$F(r) \approx \begin{cases} \int_0^r \frac{L(\xi)}{\xi} d\xi & \text{if } \lambda = 2, \\ r^{2-\lambda} L(r) & \text{if } 1 < \lambda < 2, \\ r \int_r^\eta \frac{L(\xi)}{\xi} d\xi & \text{if } \lambda = 1, \\ r & \text{if } \lambda < 1 \end{cases}$$
$$= r^{\min(1,2-\lambda)} \Psi_{L,0}(r).$$

So, we obtain the required result. □

The next result will play a crucial role in the proof of our main result. In what follows, let

$$\theta_\lambda(t) := \min(1, \rho(t))^{\min\left(1, \frac{2-\lambda}{1-\sigma}\right)} \Psi_{L,\sigma}(\min(1, \rho(t))), \ t \in (0,1). \tag{3.5}$$

Then using Proposition 3.2, we are going to derive estimates on the potential function $V(a\theta_\lambda^\sigma)$.

Proposition 3.3. *Assume* (H3) *holds and let* θ_λ *be the function given by* (3.5)*. Then we have for* $t \in (0,1)$

$$V(a\theta_\lambda^\sigma)(t) \approx \theta_\lambda(t).$$

Proof. Let $\alpha := \min\left(1, \frac{2-\lambda}{1-\sigma}\right)$. Using (H3) we have for $t \in (0,1)$,

$$a\theta_\lambda^\sigma(t) \approx \frac{1}{A(t)^2} \min(1, \rho(t))^{-\lambda+\sigma\alpha} (1 + \rho(t))^{-\mu} (L\Psi_{L,\sigma}^\sigma)(\min(1, \rho(t))).$$

Since

$$\alpha = \begin{cases} 0 & \text{if } \lambda = 2, \\ \frac{2-\lambda}{1-\sigma} & \text{if } 1 + \sigma < \lambda < 2, \\ 1 & \text{if } \lambda \leq 1 + \sigma, \end{cases}$$

it follows that $\lambda-\sigma\alpha\leq 2$. Moreover, by Lemma 2.2 (ii) and Remark 2.5, $L\Psi^{\sigma}_{L,\sigma}\in\mathcal{K}$ satisfying $\int_0^\eta t^{1-\lambda+\sigma\alpha}L(t)\Psi^{\sigma}_{L,\sigma}(t)dt<\infty$. Then the function $a\theta^{\sigma}_{\lambda}$ satisfies hypothesis (H3). Hence, proceeding in the same manner as in the proof of Proposition 3.2, we have to show the estimates on $V(a\theta^{\sigma}_{\lambda})$ only on $[t_0,1)$.

Moreover, applying Proposition 3.2, by replacing λ by $\lambda-\sigma\alpha$ and L by $L\Psi^{\sigma}_{L,\sigma}$ we deduce that for $t\in[t_0,1)$,

$$V(a\theta^{\sigma}_{\lambda})(t)\approx\begin{cases}\displaystyle\int_0^{\rho(t)}\frac{L(s)}{s}\left(\int_0^s\frac{L(r)}{r}dr\right)^{\frac{\sigma}{1-\sigma}}ds & \text{if } \lambda=2,\\ \rho(t)^{\frac{2-\lambda}{1-\sigma}}L(\rho(t))L(\rho(t))^{\frac{\sigma}{1-\sigma}} & \text{if } 1+\sigma<\lambda<2,\\ \displaystyle\rho(t)\int_{\rho(t)}^{\eta}\frac{L(s)}{s}\left(\int_s^\eta\frac{L(r)}{r}dr\right)^{\frac{\sigma}{1-\sigma}}ds & \text{if } \lambda=1+\sigma,\\ \rho(t) & \text{if } \lambda<1+\sigma.\end{cases}$$

So by a direct calculation, we have

$$V(a\theta^{\sigma}_{\lambda})(t)\approx\begin{cases}\displaystyle\left(\int_0^{\rho(t)}\frac{L(s)}{s}ds\right)^{\frac{1}{1-\sigma}} & \text{if } \lambda=2,\\ \rho(t)^{\frac{2-\lambda}{1-\sigma}}L(\rho(t))^{\frac{1}{1-\sigma}} & \text{if } 1+\sigma<\lambda<2,\\ \displaystyle\rho(t)\left(\int_{\rho(t)}^{\eta}\frac{L(s)}{s}ds\right)^{\frac{1}{1-\sigma}} & \text{if } \lambda=1+\sigma,\\ \rho(t) & \text{if } \lambda<1+\sigma.\end{cases}$$

That is $V(a\theta^{\sigma}_{\lambda})(t)\approx\theta_\lambda(t)$ for $t\in[t_0,1)$. This completes the proof. □

4. PROOF OF THEOREM 1.4

In order to prove Theorem 1.4, we need the following lemma.

Lemma 4.1. *Let $\sigma<0$ and $u_1,u_2\in C^1((0,1))\cap C([0,1])$ be two positive functions in $(0,1)$ such that*

$$\begin{cases}-\frac{1}{A}(Au_1')'\leq a(t)u_1^{\sigma} & \text{in } (0,1),\\ \lim_{t\to 0}Au_1'(t)=0, & u_1(1)=0,\end{cases}\tag{4.1}$$

and

$$\begin{cases}-\frac{1}{A}(Au_2')'\geq a(t)u_2^{\sigma} & \text{in } (0,1),\\ \lim_{t\to 0}Au_2'(t)=0, & u_2(1)=0.\end{cases}\tag{4.2}$$

Then $u_1\leq u_2$.

Proof. Let $v(t) := u_1(t) - u_2(t)$ for $t \in (0,1)$. Suppose that there exists $t_0 \in (0,1)$ such that $v(t_0) > 0$. Then there exists an interval $(t_1, t_2) \subset [0,1]$ containing t_0 such that

$$v(t) > 0 \text{ for each } t \in (t_1, t_2)$$

with $v(t_2) = 0$ and $v(t_1) = 0$ or $t_1 = 0$.

Since $\sigma < 0$, we have $u_1^\sigma(t) < u_2^\sigma(t)$ for $t \in (t_1, t_2)$. This yields

$$\frac{1}{A}(Av')' = \frac{1}{A}(Au_1')' - \frac{1}{A}(Au_2')' \geq a(u_2^\sigma - u_1^\sigma) > 0 \text{ on } (t_1, t_2).$$

Then, we deduce that the function Av' is nondecreasing on (t_1, t_2) with $Av'(t_1) \geq 0$. Hence, we obtain that v is nondecreasing on (t_1, t_2) with $v(t_0) > 0$ and $v(t_2) = 0$. This yields to a contradiction. Hence $u_1 \leq u_2$. □

Proof of Theorem 1.4. Assume (H3) holds and let θ_λ be the function given by (3.5). By Proposition 3.3, there exists a constant $m \geq 1$ such that for each $t \in [0,1)$,

$$\frac{1}{m}\theta_\lambda(t) \leq V(a\theta_\lambda^\sigma)(t) \leq m\theta_\lambda(t). \tag{4.3}$$

1. Existence and asymptotic behavior. We look at the existence of a positive solution u of problem (1.6) satisfying (1.7). So, we distinguish the following cases.

Case 1. $\sigma < 0$. By Lemma 3.1, (1.5) is satisfied. This implies, by [16], that problem (1.6) has a unique solution $u \in C([0,1]) \cap C^1((0,1))$ for the case $\sigma < 0$. So, it remains to prove the boundary behavior (1.7) of the solution u.

Put $c = m^{-\frac{\sigma}{1-\sigma}}$ and $\varphi := V(a\theta_\lambda^\sigma)$. It follows that the function φ satisfies

$$-\frac{1}{A}(A\varphi')' = a\theta_\lambda^\sigma \text{ in } (0,1).$$

This together with (4.3), we obtain by simple calculus that $\frac{1}{c}\varphi$ and $c\varphi$ satisfy respectively (4.1) and (4.2). Thus, we deduce by Lemma 4.1 that the solution u of problem (1.6) satisfies

$$\frac{1}{c}\varphi(t) \leq u(t) \leq c\varphi(t), \ t \in (0,1).$$

Using again (4.3), we prove (1.7). So the result holds.

Case 2. $0 \leq \sigma < 1$. Put $c_0 = m^{\frac{1}{1-\sigma}}$, where the constant m is given in (4.3) and let

$$\Gamma := \Big\{u \in C([0,1]) : \ \frac{\theta_\lambda(t)}{c_0} \leq u(t) \leq c_0\theta_\lambda(t), \ t \in (0,1)\Big\}.$$

Obviously, the function $\theta_\lambda \in C([0,1])$ and so Γ is non-empty.

We consider the integral operator T on Γ defined by

$$Tu(t) := \int_0^1 G(t,s)a(s)u^\sigma(s)ds, \ t \in [0,1].$$

We shall prove that T has a fixed point in Γ. For this aim, first, we show that T leaves invariant the convex Γ. Let $u \in \Gamma$ then for $t, s > 0$,

$$G(t,s)a(s)u^{\sigma}(s) \leq c_0^{\sigma} A(s)\rho(s)(a\theta_\lambda^{\sigma})(s). \tag{4.4}$$

By (4.3), it is clear that

$$V(a\theta_\lambda^{\sigma})(0) = \int_0^1 A(s)\rho(s)a(s)\theta_\lambda^{\sigma}(s)ds < \infty. \tag{4.5}$$

Since for each $s > 0$, the function $t \mapsto G(t,s)$ is in $C([0,1])$, it follows by (4.4), (4.5) and the convergence dominated theorem that $T\Gamma \subset C([0,1])$.

On the other hand, we observe that for $u \in \Gamma$ and $t \in (0,1)$,

$$c_0^{-\sigma} V(a\theta_\lambda^{\sigma})(t) \leq Tu(t) \leq c_0^{\sigma} V(a\theta_\lambda^{\sigma})(t).$$

Combining with (4.3), we obtain for $t \in (0,1)$ that

$$\frac{\theta_\lambda(t)}{mc_o^{\sigma}} \leq Tu(t) \leq mc_o^{\sigma}\theta_\lambda(t).$$

Since $mc_o^{\sigma} = c_0$, then T leaves invariant the convex Γ.

Now, let $(u_n)_n$ be a sequence of functions in $C([0,1])$ defined by

$$u_0 = \frac{\theta_\lambda}{c_0} \text{ and } u_{n+1} = Tu_n \text{ for } n \in \mathbb{N}.$$

Since the operator T is nondecreasing and $T(\Gamma) \subset \Gamma$, we deduce that

$$u_0 \leq u_1 \leq u_2 \leq \ldots \leq u_n \leq u_{n+1} \leq c_0\theta_\lambda.$$

Therefore, using the convergence monotone theorem, the sequence $(u_n)_n$ converges to a function u satisfying for each $t \in (0,1)$,

$$\frac{\theta_\lambda(t)}{c_0} \leq u(t) \leq c_0\theta_\lambda(t) \text{ and } u(t) = \int_0^1 G(t,s)a(s)u^{\sigma}(s)ds.$$

Using (4.4) and (4.5), we prove that u is a continuous function on $[0,1]$ satisfying

$$u = V(au^{\sigma}).$$

Hence, it follows that $u \in C([0,1]) \cap C^1((0,1))$ is a solution of problem (1.6) satisfying (1.7).

2. *Uniqueness.* As is mentioned above, if $\sigma < 0$ the uniqueness follows by [16]. Let $0 \leq \sigma < 1$ and let u and v be two solutions of (1.6) in Γ. Then, there exists a constant $M > 1$ such that

$$\frac{1}{M} \leq \frac{u}{v} \leq M.$$

This implies that the set

$$J = \left\{ t \in (1,\infty) : \frac{1}{t} u \le v \le tu \right\}$$

is non-empty. Now, put $c := \inf J$, then we aim to show that $c = 1$.

Suppose that $c > 1$, then the function $w := c^\sigma v - u$ satisfies

$$\begin{cases} -\dfrac{1}{A}(Aw')' = a\,(t)\,(c^\sigma v^\sigma - u^\sigma) \ge 0 \quad \text{in } (0,1), \\ \lim\limits_{t\to 0} Aw'(t) = 0, \;\; w(1) = 0. \end{cases}$$

This implies that the function Aw' is nonincreasing on $(0,1)$ with $\lim_{t\to 0} Aw'(t) = 0$. So, we obtain that the function $c^\sigma v - u$ is nonincreasing on $(0,1)$ satisfying $(c^\sigma v - u)(1) = 0$. Then we have that $c^\sigma v \ge u$. Similarly, we prove that $v \le c^\sigma u$. Hence, $c^\sigma \in J$. Now, since $\sigma < 1$, it follows that $c^\sigma < c$ and this yields a contradiction with the definition of c. Hence, $c = 1$ and then $u = v$. This completes the proof of Theorem 1.4. $\square$

Example 4.2. Let $A(t) = t^\alpha (1-t)^\beta$ with $\beta < 1 < \alpha$ and a be a function satisfying

$$a(t) \approx (1-t)^{-\tilde{\lambda}} \widetilde{L}(1-t),$$

where $\tilde{\lambda} \le 2$ and $\widetilde{L} \in \mathcal{K}$ defined on $(0,\eta]$ $(\eta > 1)$ satisfying $\int_0^\eta t^{1-\tilde{\lambda}} \widetilde{L}(t)dt < \infty$. Then problem (1.6) has a unique solution u satisfying for each $t \in (0,1)$,

$$u(t) \approx (1-t)^{\min(1-\beta, \frac{2-\tilde{\lambda}}{1-\sigma})} \Psi_{\widetilde{L},\sigma}(1-t).$$

Indeed, we claim that a satisfies (H3) with $\lambda = \frac{\tilde{\lambda}-2\beta}{1-\beta}$, $\mu = \frac{2\alpha}{\alpha-1}$ and $L(t) = \tilde{L}(t^{\frac{1}{1-\beta}})$. Let $t_0 \in (0,1)$. To prove the claim, we take two cases:

Case 1. If $t \in [t_0, 1)$, we have $A(t) \approx (1-t)^\beta$ and then

$$\rho(t) \approx \int_t^1 \frac{dr}{(1-r)^\beta} \approx (1-t)^{1-\beta}.$$

Since

$$a(t) \approx (1-t)^{-[(1-\beta)\lambda + 2\beta]} L((1-t)^{1-\beta}) \approx \frac{(1-t)^{-(1-\beta)\lambda}}{(1-t)^{2\beta}} L((1-t)^{1-\beta}),$$

it follows by Lemma 2.6, that for $t \in [t_0, 1)$

$$a(t) \approx \frac{\rho(t)^{-\lambda}}{A(t)^2} L(\rho(t)). \tag{4.6}$$

Case 2. If $t \in (0, t_0]$, we have $A(t) \approx t^\alpha$ and then

$$\rho(t) \approx \int_t^1 \frac{dr}{r^\alpha} \approx t^{1-\alpha}.$$

Hence, it yields that for $t \in (0, t_0]$

$$a(t) \approx 1 = t^{-\mu(1-\alpha)-2\alpha} \approx \frac{\rho(t)^{-\mu}}{A(t)^2}.$$

Combining with (4.6), we obtain for $t \in (0,1)$

$$a(t) \approx \frac{1}{A(t)^2} \min(1, \rho(t))^{-\lambda}(1+\rho(t))^{-\mu} L(\min(1, \rho(t))).$$

Moreover, we have

$$\begin{aligned}\int_0^{\eta^{1-\beta}} s^{1-\lambda} L(s)ds &= (1-\beta)\int_0^{\eta} t^{1-\lambda+\beta(\lambda-2)} L(t^{1-\beta})dt \\ &= (1-\beta)\int_0^{\eta} t^{1-\tilde{\lambda}} \widetilde{L}(t)dt < \infty.\end{aligned}$$

The claim is shown. Hence by Theorem 1.4, it follows that problem (1.6) has a unique continuous solution u satisfying for each $t \in (0,1)$,

$$u(t) \approx \min(1, \rho(t))^{\min\left(1, \frac{2-\lambda}{1-\sigma}\right)} \Psi_{L,\sigma}(\min(1, \rho(t))).$$

Since $\min(1, \rho(t)) \approx (1-t)^{1-\beta}$, we obtain

$$u(t) \approx \begin{cases} \left(\displaystyle\int_0^{(1-t)^{1-\beta}} \frac{L(s)}{s} ds \right)^{\frac{1}{1-\sigma}} & \text{if } \lambda = 2, \\ (1-t)^{(1-\beta)\frac{2-\lambda}{1-\sigma}} L((1-t)^{1-\beta})^{\frac{1}{1-\sigma}} & \text{if } 1+\sigma < \lambda < 2, \\ (1-t)^{1-\beta} \left(\displaystyle\int_{(1-t)^{1-\beta}}^{\eta} \frac{L(s)}{s} ds \right)^{\frac{1}{1-\sigma}} & \text{if } \lambda = 1+\sigma, \\ (1-t)^{1-\beta} & \text{if } \lambda < 1+\sigma. \end{cases}$$

Then by calculus, it follows that

$$u(t) \approx \begin{cases} \left(\int\limits_0^{1-t} \frac{L(\xi^{1-\beta})}{\xi} d\xi \right)^{\frac{1}{1-\sigma}} & \text{if } \tilde{\lambda} = 2, \\ (1-t)^{\frac{2-\tilde{\lambda}}{1-\sigma}} L((1-t)^{1-\beta})^{\frac{1}{1-\sigma}} & \text{if } (1+\beta)(1-\sigma)+2\sigma < \tilde{\lambda} < 2, \\ (1-t)^{1-\beta} \left(\int\limits_{1-t}^{\eta} \frac{L(\xi^{1-\beta})}{\xi} d\xi \right)^{\frac{1}{1-\sigma}} & \text{if } \tilde{\lambda} = (1+\beta)(1-\sigma)+2\sigma, \\ (1-t)^{1-\beta} & \text{if } \tilde{\lambda} < (1+\beta)(1-\sigma)+2\sigma, \end{cases}$$
$$= (1-t)^{\min(1-\beta, \frac{2-\tilde{\lambda}}{1-\sigma})} \Psi_{\widetilde{L},\sigma}(1-t).$$

Remark 4.3. We note that by Example 4.2, we find again the result obtained in [15] for the radial case of the following elliptic problem

$$\begin{cases} \triangle u + a(|x|)u^\sigma = 0 \quad \text{in } B, \\ u = 0 \ \text{ on } \partial B. \end{cases} \tag{4.7}$$

Here B is the unit ball of $\mathbb{R}^n$ and a is a measurable function satisfying

$$a(t) \approx (1-t)^{-\lambda} L(1-t), \quad t \in (0,1)$$

where $\lambda \leq 2$ and $L \in \mathcal{K}$ defined on $(0,\eta]$ $(\eta > 1)$ satisfying $\int\limits_0^\eta t^{1-\lambda} L(t)dt < \infty$. Indeed, taking $A(t) = t^{n-1}$ $(n \geq 2)$ in Example 4.2, we prove that problem (4.7) has a unique radial continuous solution u satisfying for $x \in B$

$$u(x) \approx (1-|x|)^{\min\left(1, \frac{2-\lambda}{1-\sigma}\right)} \Psi_{L,\sigma}\left(1-|x|\right).$$

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