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MULTIPLICITY RESULTS FOR AN IMPULSIVE BOUNDARY VALUE PROBLEM OF $p(t)$-KIRCHHOFF TYPE VIA CRITICAL POINT THEORY

A. Mokhtari, T. Moussaoui, D. O'Regan

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Abstract. In this paper we obtain existence results of k distinct pairs nontrivial solutions for an impulsive boundary value problem of $p(t)$-Kirchhoff type under certain conditions on the parameter λ.

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1. INTRODUCTION

We study the multiplicity of nontrivial solutions for the problem

$$\begin{cases} -\Big(a+b\int\limits_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\Big)\Big(|u'(t)|^{p(t)-2}\cdot u'(t)\Big)' = \lambda h(t,u(t)), \quad t\neq t_j,\ t\in[0,T], \\ -\Delta u'(t_j) = I_j(u(t_j)), \quad j=1,2,\ldots,l, \\ u(0)=u(T)=0, \end{cases} \tag{1.1}$$

where $0=t_0<t_1<\ldots<t_l<t_{l+1}=T$, $\lambda>0$ is a numerical parameter, h is a Carathéodory function, $I_j\in C(\mathbb{R},\mathbb{R})$, $j=1,2,\ldots,l$, $\Delta u'(t_j)=u'(t_j^+)-u'(t_j^-)$, $u'(t_j^+)$ and $u'(t_j^-)$ denote the right and left derivative of u at $t=t_j$, $j=1,2,\ldots,l$. Here p is a function in $C([0,T],\mathbb{R})$ with

$$1<p^-=\inf_{t\in[0,T]}p(t)\le p^+=\sup_{t\in[0,T]}p(t),$$

and a,b are positive constants.

Impulsive problems for the $p(t)$-Laplacian were introduced in [12] and [13]. In [4] the authors considered

$$\begin{cases} u''(t) + \lambda h(t, u(t)) = 0, & t \neq t_j,\, t \in [0, T], \\ -\Delta u'(t_j) = I_j(u(t_j)), & j = 1, 2, \ldots, l, \\ u(0) = u(T) = 0, \end{cases}$$

and the goal in this paper is to generalize the results so that (1.1) can be considered.

The variable exponent Lebesgue space $L^{p(t)}(0, T)$ is defined by

$$L^{p(t)}(0,T) = \left\{ u : (0,T) \to \mathbb{R} \text{ is mesurable}, \int_0^T |u(t)|^{p(t)}\, dt < +\infty \right\}$$

endowed with the norm

$$|u|_{p(t)} = \inf \left\{ \lambda > 0 : \int_0^T \left| \frac{u(t)}{\lambda} \right|^{p(t)} dt \leq 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(t)}(0, T)$ is defined by

$$W^{1,p(t)}(0,T) = \{ u \in L^{p(t)}(0,T) : u' \in L^{p(t)}(0,T) \}$$

endowed with the norm $\|u\|_{1,p(t)} = |u|_{p(t)} + |u'|_{p(t)}$.

Denote by $C([0, T])$ the space of continuous functions on $[0, T]$ endowed with the norm $|u|_\infty = \sup_{t \in [0,T]} |u(t)|$. Now $W_0^{1,p(t)}(0, T)$ denotes the closure of $C_0^\infty(0, T)$ in $W^{1,p(t)}(0, T)$.

Proposition 1.1 ([11]). *$L^{p(t)}(0, T)$, $W^{1,p(t)}(0, T)$ and $W_0^{1,p(t)}(0, T)$ are separable, reflexive and uniformly convex Banach spaces.*

Proposition 1.2 ([11]). *For any $u \in L^{p(t)}(0, T)$ and $v \in L^{q(t)}(0, T)$, where $\frac{1}{p(t)} + \frac{1}{q(t)} = 1$, we have*

$$\left| \int_0^T uv\, dt \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(t)} |v|_{q(t)}.$$

Proposition 1.3 ([11]). *Let $\rho(u) = \int_0^T |u(t)|^{p(t)}\, dt$. For any $u \in L^{p(t)}(0, T)$, the following assertions hold.*

1. *For $u \neq 0$, $|u|_{p(t)} = \lambda \Leftrightarrow \rho\left(\frac{u}{\lambda}\right) = 1$.*
2. *$|u|_{p(t)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$.*
3. *If $|u|_{p(t)} > 1$, then $|u|_{p(t)}^{p^-} \leq \rho(u) \leq |u|_{p(t)}^{p^+}$.*
4. *If $|u|_{p(t)} < 1$, then $|u|_{p(t)}^{p+} \leq \rho(u) \leq |u|_{p(t)}^{p^-}$.*

Proposition 1.4 ([11]). *If $u, u_k \in L^{p(t)}(0,T)$, $k = 1,2,\ldots$, then the following statements are equivalent:*

1. $\lim\limits_{k\to+\infty} |u_k - u|_{p(t)} = 0$ *(i.e. $u_k \to u$ in $L^{p(t)}(0,T)$),*
2. $\lim\limits_{k\to+\infty} \rho(u_k - u) = 0$,
3. *$u_k \to u$ in measure in $(0,T)$ and $\lim\limits_{k\to+\infty} \rho(u_k) = \rho(u)$.*

Proposition 1.5 ([9]). *The Poincaré-type inequality holds, that is, there exists a positive constant c such that*

$$|u|_{p(t)} \le c|u'|_{p(t)}, \quad \text{for all} \quad u \in W_0^{1,p(t)}(0,T).$$

Thus $|u'|_{p(t)}$ is an equivalent norm in $W_0^{1,p(t)}(0,T)$. We will use this equivalent norm in the following discussion and write $\|u\| = |u'|_{p(t)}$ for simplicity.

We now recall the Krasnoselskii genus and information on this may be found in [1,2,14,15]. Let E be a real Banach space. Let us denote by Σ the class of all closed subsets $A \subset E\backslash\{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Definition 1.6. Let $A \in \Sigma$. The Krasnoselskii genus $\gamma(A)$ is defined as being the least positive integer n such that there is an odd mapping $\varphi \in C(A, \mathbb{R}^n \setminus \{0\})$. If such n does not exist, we set $\gamma(A) = +\infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.

Theorem 1.7 ([14]). *Let $E = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Note $\gamma(S^{N-1}) = N$. If E is infinite dimension and separable and S is the unit sphere in E, then $\gamma(S) = +\infty$.

Proposition 1.8 ([14]). *Let $A, B \in \Sigma$. Then, if there exists an odd map $f \in C(A,B)$, then $\gamma(A) \le \gamma(B)$. Consequently, if there exists an odd homeomorphism $f : A \to B$, then $\gamma(A) = \gamma(B)$.*

Definition 1.9. Let $J \in C^1(E,\mathbb{R})$. If any sequence $(u_n) \subset E$ for which $(J(u_n))$ is bounded and $J'(u_n) \to 0$ when $n \to +\infty$ in E' possesses a convergent subsequence, then we say that J satisfies the Palais-Smale condition (the (PS) condition).

We now state a theorem due to Clarke.

Theorem 1.10 ([5,17]). *Let $J \in C^1(E,\mathbb{R})$ be a functional satisfying the Palais-Smale condition. Also suppose that:*

1) *J is bounded from below and even,*
2) *there is a compact set $K \in \Sigma$ such that $\gamma(K) = k$ and $\sup_{x\in K} J(x) < J(0)$.*

Then J possesses at least k pairs of distinct critical points and their corresponding critical values are less than $J(0)$.

Definition 1.11. We say that $u \in W_0^{1,p(t)}(0,T) = X$ is a weak solution of Problem (1.1) if and only if

$$\Big(a + b\int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\Big)\int_0^T |u'(t)|^{p(t)-2}\cdot u'(t)v'(t)\,dt = \lambda\int_0^T h(t,u(t))v(t)\,dt + \sum_{j=1}^{l} I_j(u(t_j))v(t_j)$$

for all $v \in X$.

In Section 2 we will use the following elementary inequalities (see [16]): for all $x, y \in \mathbb{R}$, we have

$$(|x|^{p(\cdot)-2}x - |y|^{p(\cdot)-2}y)(x-y) \geq \frac{1}{2^{p(\cdot)}}|x-y|^{p(\cdot)} \quad \text{if } p(\cdot) \geq 2, \tag{1.2}$$

and

$$(|x|^{p(\cdot)-2}x - |y|^{p(\cdot)-2}y)(x-y) \geq \big(p(\cdot)-1\big)\frac{|x-y|^2}{(|x|+|y|)^{2-p(\cdot)}} \quad \text{if } 1 < p(\cdot) < 2. \tag{1.3}$$

Remark 1.12. Note (1.2) implies

$$(|x|^{p(\cdot)-2}x - |y|^{p(\cdot)-2}y)(x-y) \geq \frac{1}{2^{p^+}}|x-y|^{p(\cdot)} \quad \text{if } p(\cdot) \geq 2. \tag{1.4}$$

Also (1.3) implies

$$\Big[(|x|^{p(\cdot)-2}x - |y|^{p(\cdot)-2}y)(x-y)\Big]^{\frac{p(\cdot)}{2}} \geq \frac{p^- - 1}{\sqrt{2}}\frac{|x-y|^{p(\cdot)}}{\big(|x|^{p(\cdot)}+|y|^{p(\cdot)}\big)^{\frac{2-p(\cdot)}{2}}} \tag{1.5}$$

if $1 < p(\cdot) < 2$. To see this note for any $x, y \in \mathbb{R}$ and $1 < p(\cdot) < 2$, from (1.3) we have

$$\Big[(|x|^{p(\cdot)-2}x - |y|^{p(\cdot)-2}y)(x-y)\Big]^{\frac{p(\cdot)}{2}} \geq (p^- - 1)\frac{|x-y|^{p(\cdot)}}{(|x|+|y|)^{p(\cdot)\frac{2-p(\cdot)}{2}}}$$

and now using

$$(|x|+|y|)^{p(\cdot)} \leq 2^{p(\cdot)-1}(|x|^{p(\cdot)}+|y|^{p(\cdot)}) \leq 2(|x|^{p(\cdot)}+|y|^{p(\cdot)}),$$

we obtain

$$\begin{aligned}\Big[(|x|^{p(\cdot)-2}x - |y|^{p(\cdot)-2}y)(x-y)\Big]^{\frac{p(\cdot)}{2}} &\geq (p^- - 1)\frac{1}{2^{\frac{2-p(\cdot)}{2}}}\frac{|x-y|^{p(\cdot)}}{\big(|x|^{p(\cdot)}+|y|^{p(\cdot)}\big)^{\frac{2-p(\cdot)}{2}}} \\ &\geq \frac{p^- - 1}{\sqrt{2}}\frac{|x-y|^{p(\cdot)}}{\big(|x|^{p(\cdot)}+|y|^{p(\cdot)}\big)^{\frac{2-p(\cdot)}{2}}}.\end{aligned}$$

2. MAIN RESULTS

Theorem 2.1. *Assume the following are satisfied.*

(h_1) *There exist $\alpha, \beta \in L^1(0,T)$ and a continuous function $\gamma : [0,T] \to \mathbb{R}$ such that $0 \le \gamma^+ = \sup_{t\in[0,T]} \gamma(t) < 2p^- - 1$ with*

$$|h(t,u)| \le \alpha(t) + \beta(t)|u|^{\gamma(t)} \quad \text{for any } (t,u) \in [0,T] \times \mathbb{R}.$$

(h_2) *$h(t,u)$ is odd with respect to u and $H(t,u) = \int_0^u h(t,\xi)\, d\xi > 0$ for every $(t,u) \in [0,T] \times \mathbb{R} \setminus \{0\}$.*

(h_3) *$I_j(u)$ $(j = 1,2,\ldots,l)$ are odd and $\int_0^u I_j(s)\, ds \le 0$ for any $u \in \mathbb{R}$ $(j = 1,\ldots,l)$.*

Then for any $k \in \mathbb{N}$, there exists a λ_k such that when $\lambda > \lambda_k$, Problem (1.1) *has at least k distinct pairs of nontrivial solutions.*

Proof. The corresponding functional to Problem (1.1) is defined as follows:

$$\begin{aligned}\varphi(u) = a \int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\, dt + \frac{b}{2}\Big(\int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\, dt\Big)^2 \\ - \sum_{j=1}^{l} \int_0^{u(t_j)} I_j(s)\, ds - \lambda \int_0^T H(t,u(t))\, dt.\end{aligned} \tag{2.1}$$

From (h_1) and the fact that $I_j \in C(\mathbb{R},\mathbb{R})$ it is easy to see that $\varphi \in C^1(X,\mathbb{R})$ and for all $u, v \in X$

$$\begin{aligned}\varphi'(u) \cdot v = \Big(a + b\int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\, dt\Big) \int_0^T |u'(t)|^{p(t)-2}u'(t)v'(t)\, dt \\ - \sum_{j=1}^{l} I_j(u(t_j))v(t_j) - \lambda \int_0^T h(t,u(t))v(t)\, dt.\end{aligned} \tag{2.2}$$

Thus the critical points of φ are the weak solutions of (1.1).

First we show that φ is bounded from below. From the continuous embedding of X in $C([0,T])$, for any $u \in X$ with $\|u\| > 1$, we have that

$$\int_0^T |u(t)|^{\gamma(t)+1}\, dt \le \int_0^T |u|_\infty^{\gamma(t)+1}\, dt \le cT\|u\|^{\gamma^+ + 1}, \tag{2.3}$$

where $c = \max_{t\in[0,T]} c_0^{\gamma(t)+1}$ and c_0 is the best constant of the continuous embedding. It follows from conditions (h_1), (h_2), (h_3) and (2.3) that

$$\begin{aligned}\varphi(u) &= a\int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt + \frac{b}{2}\Big(\int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\Big)^2\\ &\quad - \sum_{j=1}^{l}\int_0^{u(t_j)} I_j(s)\,ds - \lambda\int_0^T H(t,u(t))\,dt\\ &\geq \frac{a}{p^+}\int_0^T |u'(t)|^{p(t)}\,dt + \frac{b}{2(p^+)^2}\Big(\int_0^T |u'(t)|^{p(t)}\,dt\Big)^2\\ &\quad - \lambda\int_0^T \alpha(t)|u(t)| + \beta(t)|u(t)|^{\gamma(t)+1}\,dt\\ &\geq \frac{a}{p^+}\|u\|^{p^-} + \frac{b}{2(p^+)^2}\|u\|^{2p^-} - \lambda|\alpha|_{L^1}c_0\|u\| - \lambda|\beta|_{L^1}c\|u\|^{\gamma^++1},\end{aligned} \tag{2.4}$$

for any $u \in X$ with $\|u\| \geq 1$. Since $\gamma^+ < 2p^- - 1$, then $\lim_{\|u\|\to+\infty}\varphi(u) = +\infty$ and consequently, φ is bounded from below.

Next, we show that the functional φ satisfies the (PS) condition. Now for any $u \in X$ with $\|u\| \leq 1$, it is easy to see that

$$\varphi(u) \geq \frac{a}{p^+}\|u\|^{p^+} + \frac{b}{2(p^+)^2}\|u\|^{2p^+} - \lambda|\alpha|_{L^1}c_0\|u\| - \lambda|\beta|_{L^1}c\|u\|^{\gamma^-+1}. \tag{2.5}$$

Let $(u_n) \subset X$ be a Palais-Smale sequence for φ, i.e. $(\varphi(u_n))$ is a bounded sequence and $\lim_{n\to+\infty}\varphi'(u_n) = 0$. Thus there exists a positive constant B such that

$$\varphi(u_n) \leq B. \tag{2.6}$$

From (2.4), (2.5), (2.6) and since $\gamma^+ < 2p^- - 1$, in all cases we deduce that the sequence (u_n) is bounded in X. Thus, passing to a subsequence if necessary, there exists $u \in X$ such that $u_n \rightharpoonup u$ weakly in X. Moreover, by (2.1) and (2.2), we have

$$
\begin{aligned}
&\big(\varphi'(u_n) - \varphi'(u)\big) \cdot \big(u_n - u\big) \\
&= \Big(a + b\int_0^T \frac{1}{p(t)}|u_n'(t)|^{p(t)}\,dt\Big)\int_0^T |u_n'(t)|^{p(t)-2}u_n'(t)(u_n'(t) - u'(t))\,dt \\
&\quad - \sum_{j=1}^{l} I_j(u_n(t_j))(u_n(t_j) - u(t_j)) - \lambda\int_0^T h(t,u_n(t))(u_n(t) - u(t))\,dt \\
&\quad - \Big(a + b\int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\Big)\int_0^T |u'(t)|^{p(t)-2}u'(t)(u_n'(t) - u'(t))\,dt \\
&\quad + \sum_{j=1}^{l} I_j(u(t_j))(u_n(t_j) - u(t_j)) + \lambda\int_0^T h(t,u(t))(u_n(t) - u(t))\,dt \\
&= \Big(a + b\int_0^T \frac{1}{p(t)}|u_n'(t)|^{p(t)}\,dt\Big)\int_0^T |u_n'(t)|^{p(t)-2}u_n'(t)(u_n'(t) - u'(t))\,dt \\
&\quad - \Big(a + b\int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\Big)\int_0^T |u'(t)|^{p(t)-2}u'(t)(u_n'(t) - u'(t))\,dt \\
&\quad - \sum_{j=1}^{l}(I_j(u_n(t_j) - I_j(u(t_j))(u_n(t_j) - u(t_j)) \\
&\quad - \lambda\int_0^T (h(t,u_n(t)) - h(t,u(t)))(u_n(t) - u(t))\,dt.
\end{aligned}
$$

Since the embedding of X in $C([0,T])$ is compact, then (u_n) uniformly converges to u in $C([0,T])$, by using (h_1) and the Lebesgue Dominated Convergence Theorem, we have that

$$
\begin{cases}
\lambda\displaystyle\int_0^T \Big(h(t,u_n(t)) - h(t,u(t))\Big)\Big(u_n(t) - u(t)\Big)\,dt \to 0, \\
\displaystyle\sum_{j=1}^{l}\Big(I_j(u_n(t_j) - I_j(u(t_j))\Big)\Big(u_n(t_j) - u(t_j)\Big) \to 0, \qquad \text{as } n \to \infty.
\end{cases}
\tag{2.7}
$$

Since $\varphi'(u_n) \to 0$ and $u_n \rightharpoonup u$ in X, then we have $\big(\varphi'(u_n) - \varphi'(u)\big) \cdot \big(u_n - u\big) \to 0$ as $n \to \infty$.

Consequently, from (2.7) we have $S_n \to 0$ as $n \to \infty$, where

$$S_n = \Big(a + b\int_0^T \frac{1}{p(t)}|u_n'(t)|^{p(t)}\,dt\Big)\int_0^T |u_n'(t)|^{p(t)-2}u_n'(t)(u_n'(t)-u'(t))\,dt - \Big(a + b\int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\Big)\int_0^T |u'(t)|^{p(t)-2}u'(t)(u_n'(t)-u'(t))\,dt.$$

We can rewrite S_n as

$$\begin{aligned} S_n &= \Big(a + b\int_0^T \frac{1}{p(t)}|u_n'(t)|^{p(t)}\,dt\Big) \\ &\quad\times \int_0^T \big(|u_n'(t)|^{p(t)-2}u_n'(t) - |u'(t)|^{p(t)-2}u'(t)\big)(u_n'(t)-u'(t))\,dt \\ &\quad+ \Big(a + b\int_0^T \frac{1}{p(t)}|u_n'(t)|^{p(t)}\,dt\Big)\int_0^T |u'(t)|^{p(t)-2}u'(t)(u_n'(t)-u'(t))\,dt \\ &\quad- \Big(a + b\int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\Big)\int_0^T |u'(t)|^{p(t)-2}u'(t)(u_n'(t)-u'(t))\,dt. \end{aligned}$$

From the weak convergence of (u_n) in X, and since $|u'|^{p(t)-2}u' \in X' = W_0^{1,q(t)}(0,T)$ with $q(t) = \frac{p(t)}{p(t)-1}$, we deduce that

$$\int_0^T |u'(t)|^{p(t)-2}u'(t)(u_n'(t)-u'(t))\,dt \to 0 \quad \text{as } n \to \infty. \tag{2.8}$$

Hence, from (2.8) and since $S_n \to 0$, we get

$$\Big(a+b\int_0^T \frac{1}{p(t)}|u_n'(t)|^{p(t)}\,dt\Big)\int_0^T \big(|u_n'(t)|^{p(t)-2}u_n'(t)-|u'(t)|^{p(t)-2}u'(t)\big)(u_n'(t)-u'(t))\,dt \to 0$$

as $n \to \infty$. Since $a, b > 0$, then we have

$$\int_0^T \big(|u_n'(t)|^{p(t)-2}u_n'(t) - |u'(t)|^{p(t)-2}u'(t)\big)(u_n'(t)-u'(t))\,dt \to 0 \text{ as } n \to \infty. \tag{2.9}$$

To complete the proof of the Palais-Smale condition we follow the argument in the proof of Theorem 3.1 in [3]. We divide $I = (0,T)$ into two parts

$$I_1 = \{t \in (0,T) : 1 < p(t) < 2\}, \quad I_2 = \{t \in (0,T) : p(t) \geq 2\}.$$

We will let $|\cdot|_{p(t),I_i}$, $i = 1, 2$, to denote the norm in $L^{p(t)}(I_i)$.

On I_1, from Holder's inequality (See Proposition 1.2) and using the inequality (1.5) we get

$$\begin{aligned}
&\int\limits_{I_1} |u_n'(t) - u'(t)|^{p(t)}\,dt \\
&\leq \frac{\sqrt{2}}{p^- - 1}\int\limits_{I_1}\Big(\big(|u_n'(t)|^{p(t)-2}u_n'(t) - |u'(t)|^{p(t)-2}u'(t)\big)(u_n'(t) - u'(t))\Big)^{\frac{p(t)}{2}} \\
&\quad\times\Big(|u_n'(t)|^{p(t)} + |u'(t)|^{p(t)}\Big)^{\frac{2-p(t)}{2}}\,dt \\
&\leq c\Big|\Big(\big(|u_n'(t)|^{p(t)-2}u_n'(t) - |u'(t)|^{p(t)-2}u'(t)\big)(u_n'(t) - u'(t))\Big)^{\frac{p(t)}{2}}\Big|_{\frac{2}{p(t)},I_1} \\
&\quad\times\Big|\Big(|u_n'(t)|^{p(t)} + |u'(t)|^{p(t)}\Big)^{\frac{2-p(t)}{2}}\Big|_{\frac{2}{2-p(t)},I_1},
\end{aligned}$$

where $c = \frac{\sqrt{2}(2+p^+-p^-)}{2(p^- -1)}$. From (2.9) we get

$$\Big|\Big(\big(|u_n'(t)|^{p(t)-2}u_n'(t) - |u'(t)|^{p(t)-2}u'(t)\big)(u_n'(t) - u'(t))\Big)^{\frac{p(t)}{2}}\Big|_{\frac{2}{p(t)},I_1} \to 0 \quad \text{as} \quad n\to\infty.$$

Now using $\int_{I_1}\big(|u_n'(t)|^{p(t)} + |u'(t)|^{p(t)}\big)\,dt$ is bounded we get

$$\int\limits_{I_1} |u_n'(t) - u'(t)|^{p(t)}\,dt \to 0 \quad \text{as} \quad n\to\infty. \tag{2.10}$$

On I_2, from the inequality (1.4) we have

$$\int\limits_{I_2} |u_n'(t)-u'(t)|^{p(t)}\,dt \leq 2^{p^+}\int\limits_{I_2}\big(|u_n'(t)|^{p(t)-2}u_n'(t)-|u'(t)|^{p(t)-2}u'(t)\big)(u_n'(t)-u'(t))\,dt,$$

so

$$\int\limits_{I_2} |u_n'(t) - u'(t)|^{p(t)}\,dt \to 0 \quad \text{as} \quad n\to\infty. \tag{2.11}$$

From (2.10) and (2.11) and by Proposition 1.4, we conclude that $\|u_n - u\| \to 0$ as $n\to\infty$, so φ satisfies the Palais-Smale condition.

Notice that $W_0^{1,p^+}(0,T) \subset W_0^{1,p(t)}(0,T)$. Consider $\{e_1, e_2, \ldots\}$, a Schauder basis of the space $W_0^{1,p^+}(0,T)$ (see [18]), and for each $k\in\mathbb{N}$, consider X_k, the subspace of $W_0^{1,p^+}(0,T)$ generated by the k vectors $\{e_1, e_2, \ldots, e_k\}$. Clearly X_k is a subspace of $W_0^{1,p(t)}(0,T)$.

For $r > 0$, consider

$$K_k(r) = \Big\{u \in X_k : \|u\|^2 = \sum_{i=1}^{k}\xi_i^2 = r^2\Big\}.$$

For any $r > 0$. We consider the odd homeomorphism $\chi : K_k(r) \to S^{k-1}$ defined by $\chi(u) = (\xi_1, \xi_2, \dots, \xi_k)$, where S^{k-1} is the sphere in $\mathbb{R}^k$. From Theorem 1.7 and Proposition 1.8, we conclude that $\gamma(K_k(r)) = k$. Let $0 < r < \frac{1}{c_0}$ where c_0 is the best constant of the embedding of X in $C([0,T])$, so $\|u\|_\infty \le c_0\|u\| < 1$. It follows from hypothesis (h_2) that $\int_0^T H(t,u(t))\,dt > 0$ for any $u \in K_k(r)$. Then

$$\mu_k = \inf_{u \in K_k(r)} \int_0^T H(t,u(t))\,dt$$

is strictly positive (note the compactness of $K_k(r)$). If we set

$$\nu_k = \inf_{u \in K_k(r)} \sum_{j=1}^{l} \int_0^{u(t_j)} I_j(s)\,ds,$$

we see that $\nu_k \le 0$. Let

$$\lambda_k = \frac{1}{\mu_k}\Big(\frac{a}{p^-} r^{p^-} + \frac{b}{2(p^-)^2} r^{2p^-} - \nu_k\Big),$$

and note $\lambda_k > 0$. Then when $\lambda > \lambda_k$, we take $0 < r \le 1$, and then for any $u \in K_k(r)$ we have $\|u\| \le 1$ and

$$\begin{aligned}
\varphi(u) &\le \frac{a}{p^-} \int_0^T |u'(t)|^{p(t)}\,dt + \frac{b}{2(p^-)^2}\Big(\int_0^T |u'(t)|^{p(t)}\,dt\Big)^2 - \nu_k - \lambda\mu_k \\
&\le \frac{a}{p^-}\|u\|^{p^-} + \frac{b}{2(p^-)^2}\|u\|^{2p^-} - \nu_k - \lambda\mu_k \\
&< \frac{a}{p^-} r^{p^-} + \frac{b}{2(p^-)^2} r^{2p^-} - \nu_k - \lambda_k\mu_k = 0.
\end{aligned}$$

Theorem 1.10 guarantees that the functional φ has at least k pairs of different critical points. Hence, Problem (1.1) has at least k distinct pairs of nontrivial solutions. □

Corollary 2.2. *Assume that (h_2) and (h_3) hold, and*

(h_4) $h(t,u)$ is bounded.

Then for any $k \in \mathbb{N}$, there exists a λ_k such that for any $\lambda > \lambda_k$, Problem (1.1) has at least k distinct pairs of nontrivial solutions.

Theorem 2.3. *Assume that (h_1) holds, and assume the following are satisfied.*

(h_5) There exist $\alpha_j, \beta_j > 0$ and γ_j with $0 < \gamma_j < 2p^- - 1$ $(j = 1, 2, \dots, l)$ such that

$$|I_j(u)| \le \alpha_j + \beta_j |u|^{\gamma_j} \quad \textit{for any } u \in \mathbb{R} \ (j = 1, \dots, l).$$

(h_6) *$h(t,u)$ and $I_j(u)$ $(j = 1, 2, \ldots, l)$ are odd with respect to u and $H(t,u) > 0$ for every $(t,u) \in [0,T] \times \mathbb{R} \setminus \{0\}$.*

Then for any $k \in \mathbb{N}$, there exists a λ_k such that for any $\lambda > \lambda_k$, Problem (1.1) *has at least k distinct pairs of nontrivial solutions.*

Proof. From assumptions (h_1) and (h_6), we see that $\varphi \in C^1(X,\mathbb{R})$ is an even functional and $\varphi(0) = 0$. We now show that φ is bounded from below. Let $\alpha_0 = \max\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, $\beta_0 = \max\{\beta_1, \beta_2, \ldots, \beta_l\}$. We have for any $u \in X$ with $\|u\| > 1$ that

$$\begin{aligned} \varphi(u) = a \int_0^T \frac{1}{p(t)} |u'(t)|^{p(t)}\, dt + \frac{b}{2}\Big(\int_0^T \frac{1}{p(t)} |u'(t)|^{p(t)}\, dt\Big)^2 \\ - \lambda \int_0^T H(t,u(t))\, dt - \sum_{j=1}^{l} \int_0^{u(t_j)} I_j(s)\, ds \\ \geq \frac{a}{p^+}\|u\|^{p^-} + \frac{b}{2(p^+)^2}\|u\|^{2p^-} - \int_0^T (\alpha(t)|u(t)| + \beta(t)|u(t)|)^{\gamma(t)+1} \\ - \sum_{j=1}^{l} (\alpha_j |u(t_j)| + \beta_j |u(t_j)|^{\gamma_j + 1}) \\ \geq \frac{a}{p^+}\|u\|^{p^-} + \frac{b}{2(p^+)^2}\|u\|^{2p^-} - |\alpha|_{L^1} c_0 \|u\| - |\beta|_{L^1} c \|u\|^{\gamma^+ + 1} \\ - \alpha_0 l c_0 \|u\| - \beta_0 c' \sum_{j=1}^{l} \|u\|^{\gamma_j + 1}. \end{aligned} \tag{2.12}$$

Now, we show that φ satisfies the Palais-Smale condition. For any $u \in X$ with $\|u\| \leq 1$, we have

$$\begin{aligned} \varphi(u) \geq \frac{a}{p^+}\|u\|^{p^+} + \frac{b}{2(p^+)^2}\|u\|^{2p^+} - |\alpha|_{L^1} c_0 \|u\| - |\beta|_{L^1} c \|u\|^{\gamma^- + 1} \\ - \alpha_0 l c \|u\| - \beta_0 c' \sum_{j=1}^{l} \|u\|^{\gamma_j + 1}. \end{aligned} \tag{2.13}$$

Let $(u_n) \subset X$ be a sequence such that $(\varphi(u_n))$ is a bounded sequence and $\varphi'(u_n) \to 0$ in X'. From (2.12), (2.13), and since $\gamma^+, \gamma_j < 2p^- - 1$, in all cases we deduce that (u_n) is bounded in X. The rest of the proof of the Palais-Smale condition is similar to that in Theorem 2.1.

Consider $K_k(r)$ as in Theorem 2.1. For any $r > 0$, there exits an odd homeomorphism $\chi : K_k(r) \to S^{k-1}$. From assumption (h_6) we have

$$\mu_k = \inf_{u \in K_k(r)} \int_0^T H(t,u(t))\, dt > 0.$$

Let

$$\nu_k = \inf_{u\in K_k(r)} \sum_{j=1}^{l} \int_0^{u(t_j)} I_j(s)\,ds \quad \text{and} \quad \lambda_k = \max\left\{0, \frac{1}{\mu_k}\left(\frac{a}{p^-}r^{p^-} + \frac{b}{2(p^-)^2}r^{2p^-} - \nu_k\right)\right\}.$$

Then when $\lambda > \lambda_k$, we take $0 < r \leq 1$, and then for any $u \in K_k(r)$ we have $\|u\| \leq 1$ and

$$\begin{aligned}\varphi(u) &\leq \frac{a}{p^-}r^{p^-} + \frac{b}{2(p^-)^2}r^{2p^-} - \nu_k - \lambda\mu_k \\ &< \frac{a}{p^-}r^{p^-} + \frac{b}{2(p^-)^2}r^{2p^-} - \nu_k - \lambda_k\mu_k \leq 0.\end{aligned}$$

From Theorem 1.10, φ has at least k pairs of different critical points. Consequently, Problem (1.1) has at least k distinct pairs of nontrivial solutions. □

Corollary 2.4. *Assume that the assumptions* (h_4), (h_6) *hold, and*

(h_7) $I_j(u)$ $(j = 1, 2, \ldots, l)$ *are bounded.*

Then for any $k \in \mathbb{N}$, *there exists a* λ_k *such that for any* $\lambda > \lambda_k$, *Problem* (1.1) *has at least* k *distinct pairs of nontrivial solutions.*

Theorem 2.5. *Assume that* (h_3) *holds, and*

- (h_8) *There exists a constant* $\sigma > 0$ *such that* $h(t,\sigma) = 0, h(t,u) > 0$ *for every* $u \in (0,\sigma)$,
- (h_9) $h(t,u)$ *is odd with respect to* u.

Then for any $k \in \mathbb{N}$, *there exists a* λ_k *such that when* $\lambda > \lambda_k$, *Problem* (1.1) *has at least* k *distinct pairs of nontrivial solutions.*

Proof. Define the bounded function

$$f(t,u) = \begin{cases} 0, & \text{if } |u| > \sigma, \\ h(t,u), & \text{if } |u| \leq \sigma. \end{cases}$$

Consider

$$\begin{cases} -\left(a + b\int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\right)\left(|u'(t)|^{p(t)-2}\cdot u'(t)\right)' = \lambda f(t,u(t)), & t \neq t_j,\ t \in [0,T], \\ -\Delta u'(t_j) = I_j(u(t_j)), \quad j = 1, 2, \ldots, l, \\ u(0) = u(T) = 0, \end{cases} \tag{2.14}$$

and we now show that solutions of Problem (2.14) are also solutions of Problem (1.1). Let u_0 be a solution of Problem (2.14). We now prove that $-\sigma \leq u_0(t) \leq \sigma$. Suppose that $\max_{0\le t\le T} u_0(t) > \sigma$, then there exists an interval $[d_1, d_2] \subset [0, T]$ such that $u_0(d_1) = u_0(d_2) = \sigma$ and for any $t \in (d_1, d_2)$ we have $u_0(t) > \sigma$, and so

$$-\Big(a + b\int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\Big)\Big(|u'(t)|^{p(t)-2}\cdot u'(t)\Big)' = \lambda f(t, u(t)) = 0, \quad t \in (d_1, d_2).$$

We deduce that there is a constant c such that $|u_0'(t)|^{p(t)-2}\cdot u_0'(t) = c$ for any $t \in [d_1, d_2]$, and since $u_0'(d_1) \ge 0$ and $u_0'(d_2) \le 0$, then we have

$$|u_0'(d_1)|^{p(d_1)-2}\cdot u_0'(d_1) = c \ge 0,$$
$$|u_0'(d_2)|^{p(d_2)-2}\cdot u_0'(d_2) = c \le 0,$$

so, $u_0'(t) = 0$ for any $t \in [d_1, d_2]$, i.e. $u_0(t) = \sigma$ for any $t \in [d_1, d_2]$, which is a contradiction. From a similar argument we see that $\min_{0\le t\le T} u_0(t) \ge -\sigma$.

The functional $\varphi_1 : X \to \mathbb{R}$ defined by

$$\begin{aligned}\varphi_1(u) = a\int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt + \frac{b}{2}\Big(\int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\Big)^2 \\ -\sum_{j=1}^{l}\int_0^{u(t_j)} I_j(s)\,ds - \lambda\int_0^T F(t, u(t))\,dt\end{aligned} \tag{2.15}$$

is continuously Fréchet differentiable at any $u \in X$, where $F(t, u) = \int_0^u f(t, s)\,ds$. We have

$$\begin{aligned}\varphi_1'(u)\cdot v = \Big(a + b\int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\Big)\int_0^T |u'|^{p(t)-2}u'(t)v'(t)\,dt \\ -\sum_{j=1}^{l} I_j(u(t_j))v(t_j) - \lambda\int_0^T f(t, u(t))v(t)\,dt\end{aligned} \tag{2.16}$$

for all $v \in X$. It is clear that φ_1 is an even functional, $\varphi_1(0) = 0$ and bounded from below. To see this note for $u \in X$ with $\|u\| \ge 1$ we have

$$\begin{aligned}\varphi_1(u) &\ge \frac{a}{p^+}\int_0^T |u'(t)|^{p(t)}\,dt + \frac{b}{2(p^+)^2}\Big(\int_0^T |u'(t)|^{p(t)}\,dt\Big)^2 - \lambda\int_0^T\int_0^{u(t)} f(t, s)\,dsdt \\ &\ge \frac{a}{p^+}\int_0^T |u'(t)|^{p(t)}\,dt + \frac{b}{2(p^+)^2}\Big(\int_0^T |u'(t)|^{p(t)}\,dt\Big)^2 - \lambda\underbrace{\int_0^T\int_0^{\sigma} f(t, s)\,dsdt}_{=\tau>0} \\ &\ge \frac{a}{p^+}\|u\|^{p^-} + \frac{b}{2(p^+)^2}\|u\|^{2p^-} - \lambda\tau,\end{aligned}$$

so it follows that φ_1 is bounded from below. Let (u_n) be a Palais-Smale sequence. It is easy to see that (u_n) is bounded in X and the rest of the proof of the Palais-Smale condition is similar to that in the proof in Theorem 2.1.

Consider $K_k(r) = \{u \in X_k : \|u\| = r\}$. For any $r > 0$ the odd homeomorphism $\chi : K_k(r) \to S^{k-1}$ gives $\gamma(K_k(r)) = k$. Let $0 < r < \min\{1, \frac{\sigma}{c_0}\}$, where c_0 is the best constant of the embedding of X in $C([0,T])$, so $\|u\|_\infty \le c_0\|u\| < \sigma$ for any $u \in K_k(r)$. Using assumptions (h_8) and (h_9), we have $F(t, u(t)) > 0$ as $u(t) \neq 0$. Then $\int_0^T F(t, u(t))\, dt > 0$ for any $u \in K_k(r)$. If we set

$$\mu_k = \inf_{u \in K_k(r)} \int\limits_0^T F(t, u(t))\, dt \quad \text{and} \quad \nu_k = \inf_{u \in K_k(r)} \sum_{j=1}^{l} \int\limits_0^{u(t_j)} I_j(s)\, ds,$$

then $\mu_k > 0$ and $\nu_k \le 0$. Let

$$\lambda_k = \frac{1}{\mu_k}\left(\frac{a}{p^-} r^{p^-} + \frac{b}{2(p^-)^2} r^{2p^-} - \nu_k\right),$$

so $\lambda_k > 0$, and for any $\lambda > \lambda_k$ and any $u \in K_k(r)$ with $\|u\| < 1$ we have

$$\begin{aligned}\varphi(u) &\le \frac{a}{p^-} r^{p^-} + \frac{b}{2(p^-)^2} r^{2p^-} - \nu_k - \lambda\mu_k \\ &< \frac{a}{p^-} r^{p^-} + \frac{b}{2(p^-)^2} r^{2p^-} - \nu_k - \lambda_k\mu_k = 0.\end{aligned}$$

From Theorem 1.10, φ_1 has at least k pairs of different critical points. Then, Problem (2.14) has at least k distinct pairs of nontrivial solutions. Consequently, Problem (1.1) has at least k distinct pairs of nontrivial solutions. □

A similar argument to that in Theorem 2.5, yields the following result.

Theorem 2.6. *Let conditions (h_5), (h_8) hold and*

(h_{10}) *$h(t,u)$ and $I_j(u)$ $(j = 1, 2, \ldots, l)$ are odd with respect to u.*

Then for any $k \in \mathbb{N}$, there exists a λ_k such that for any $\lambda > \lambda_k$, Problem (1.1) has at least k distinct pairs of nontrivial solutions.

Theorem 2.7. *Suppose that (h_5) holds, and*

(h_{11}) *There exist a constant $\sigma_1 > 0$ such that $h(t, \sigma_1) \le 0$,*

(h_{12}) *$h(t,u)$ and $I_j(u)$ $(j = 1, 2, \ldots, l)$ are odd with respect to u and $\lim_{u\to 0} \frac{h(t,u)}{u} = 1$ uniformly for $t \in [0,T]$.*

Then for any $k \in \mathbb{N}$, there exists a λ_k such that for any $\lambda > \lambda_k$, Problem (1.1) has at least k distinct pairs of nontrivial solutions.

Proof. Define the bounded function

$$g(t,u)=\begin{cases} h(t,\sigma_1), & \text{if } u>\sigma_1,\\ h(t,u), & \text{if } |u|\leq\sigma_1,\\ h(t,-\sigma_1), & \text{if } u<-\sigma_1.\end{cases}$$

We will verify that the solutions of the problem

$$\begin{cases} -\Big(a+b\int\limits_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\Big)\big(|u'(t)|^{p(t)-2}\cdot u'(t)\big)'+\lambda g(t,u(t))=0, t\neq t_j,\ t\in[0,T],\\ -\Delta u'(t_j)=I_j(u(t_j)),\quad j=1,2,\dots,l,\\ u(0)=u(T)=0,\end{cases}\tag{2.17}$$

are solution of Problem (1.1). Let u_0 be a solution of Problem (2.17). We prove that $-\sigma_1\leq u_0(t)\leq\sigma_1$ for any $t\in[0,T]$. Suppose that $\max_{0\leq t\leq T}u_0(t)>\sigma_1$, then there exists an interval $[d_1,d_2]\subset[0,T]$ such that $u_0(d_1)=u_0(d_2)=\sigma_1$ and for any $t\in(d_1,d_2)$ we have $u_0(t)>\sigma_1$, and then when $t\in(d_1,d_2)$ we obtain

$$\Big(a+b\int\limits_0^T\frac{1}{p(t)}|u_0'(t)|^{p(t)}\,dt\Big)\Big(|u_0'(t)|^{p(t)-2}\cdot u_0'(t)\Big)'=-\lambda g(t,u_0(t))=-\lambda h(t,\sigma_1)\geq 0.$$

Therefore, we deduce that

$$\Big(|u_0'(t)|^{p(t)-2}\cdot u_0'(t)\Big)'\geq 0,\quad t\in(d_1,d_2),$$

thus $t\mapsto|u_0'(t)|^{p(t)-2}\cdot u_0'(t)$ is nondecreasing in (d_1,d_2), so then

$$0\leq|u_0'(d_1)|^{p(d_1)-2}u_0'(d_1)\leq|u_0'(t)|^{p(t)-2}u_0'(t)\leq|u_0'(d_2)|^{p(d_2)-2}u_0'(d_2)\leq 0,$$

for every $t\in[d_1,d_2]$. Hence $u_0'=0$ on $[d_1,d_2]$, so, since $u_0(d_1)=u_0(d_2)=\sigma_1$, then $u(t)=\sigma_1$ for every $t\in[d_1,d_2]$, which is a contradiction. From a similar argument we see that $\min_{0\leq t\leq T}u_0(t)\geq-\sigma_1$, i.e. u_0 is a solution of Problem (1.1).

We consider the functional $\varphi_2:X\to\mathbb{R}$ defined by

$$\begin{aligned}\varphi_2(u)=a\int\limits_0^T\frac{1}{p(t)}|u'(t)|^{p(t)}\,dt+\frac{b}{2}\Big(\int\limits_0^T\frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\Big)^2\\ -\sum_{j=1}^{l}\int\limits_0^{u(t_j)}I_j(s)\,ds-\lambda\int\limits_0^T G(t,u(t))\,dt,\end{aligned}\tag{2.18}$$

where $G(t,u)=\int\limits_0^u g(t,s)\,ds$. Obviously, φ_2 is continuously Fréchet differentiable at any $u\in X$ and

$$\begin{aligned}\varphi_2'(u)\cdot v=\Big(a+b\int\limits_0^T\frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\Big)\int\limits_0^T|u'(t)|^{p(t)-2}u'(t)v'(t)\,dt\\ -\sum_{j=1}^{l}I_j(u(t_j))v(t_j)-\lambda\int\limits_0^T g(t,u(t))v(t)\,dt,\end{aligned}\tag{2.19}$$

for all $v\in X$. It is clear that critical points of φ_2 are solutions of Problem (2.17). Now $\varphi_2\in C^1(X,\mathbb{R})$ is an even functional and $\varphi_2(0)=0$.

Let $\alpha_0=\max\{\alpha_1,\alpha_2,\dots,\alpha_l\}$, $\beta_0=\max\{\beta_1,\beta_2,\dots,\beta_l\}$, and we see that

$$\int\limits_0^T G(t,u(t))\,dt=\int\limits_0^T\int\limits_0^{u(t)}g(t,s)\,dsdt\le\int\limits_0^T\int\limits_0^{\sigma_1}g(t,s)\,dsdt=\eta.\tag{2.20}$$

Using assumption (h_5) and (2.20), we have for any $u\in X$ with $\|u\|>1$ that

$$\begin{aligned}\varphi_2(u)&=a\int\limits_0^T\frac{1}{p(t)}|u'(t)|^{p(t)}\,dt+\frac{b}{2}\Big(\int\limits_0^T\frac{1}{p(t)}|u'(t)|^{p(t)}\,dt\Big)^2\\&\quad-\sum_{j=1}^{l}\int\limits_0^{u(t_j)}I_j(s)\,ds-\lambda\int\limits_0^T G(t,u(t))\,dt\\&\ge\frac{a}{p^+}\|u\|^{p^-}+\frac{b}{2(p^+)^2}\|u\|^{2p^-}-\sum_{j=1}^{l}(\alpha_j|u(t_j)|+\beta_j|u(t_j)|^{\gamma_j+1})-\lambda\eta\\&\ge\frac{a}{p^+}\|u\|^{p^-}+\frac{b}{2(p^+)^2}\|u\|^{2p^-}-\alpha_0lc_0\|u\|-\beta_0c'\sum_{j=1}^{l}\|u\|^{\gamma_j+1}-\lambda\eta,\end{aligned}\tag{2.21}$$

so it follows that φ_2 is bounded from below.

Now we show that φ_2 satisfies the Palais-Smale condition. For any $u\in X$ with $\|u\|\le 1$, we have

$$\begin{aligned}\varphi(u)\ge\frac{a}{p^+}\|u\|^{p^+}+\frac{b}{2(p^+)^2}\|u\|^{2p^+}-\alpha_0lc_0\|u\|\\-\beta_0c'\sum_{j=1}^{l}\|u\|^{\gamma_j+1}-\lambda\eta.\end{aligned}\tag{2.22}$$

Let $(u_n)\subset X$ be a sequence such that $(\varphi(u_n))$ is a bounded sequence and $\varphi'(u_n)\to 0$ as $n\to+\infty$. From (2.21), (2.22), and since $\gamma,\gamma_j<2p^--1$, in all cases we deduce that (u_n) is bounded in X. The proof of the Palais-Smale condition is now similar to that in Theorem 2.1.

Consider $K_k(r)$ as in Theorem 2.1. From assumption (h_{12}), for any $\varepsilon > 0$, there exists $\delta > 0$, when $|u| \le \delta$, we have $h(t,u) \ge u - \varepsilon|u|$. Take $0 < r \le 1$ sufficiently small such that $\|u\|_\infty < \min\{\sigma_1, \delta\}$ for any $u \in K_k(r)$. Then, taking $0 < \varepsilon < 1$ we have

$$\int_0^T G(t,u(t))\,dt = \int_0^T \int_0^{u(t)} g(t,s)\,dsdt \ge \frac{1}{2}\int_0^T (1-\varepsilon)|u(t)|^2\,dt > 0,$$

for any $u \in K_k(r)$.

Set

$$\mu_k = \inf_{u \in K_k(r)} \int_0^T G(t,u(t))\,dt \quad \text{and} \quad \nu_k = \inf_{u\in K_k(r)} \sum_{j=1}^{l} \int_0^{u(t_j)} I_j(s)\,ds.$$

Let $\lambda_k = \max\left\{0, \frac{1}{\mu_k}\left(\frac{a}{p^-}r^{p^-} + \frac{b}{2(p^-)^2}r^{2p^-} - \nu_k\right)\right\}$, then for all λ such that $\lambda > \lambda_k$ and every $u \in K_k(r)$, we have

$$\begin{aligned}\varphi_2(u) &\le \frac{a}{p^-}r^{p^-} + \frac{b}{2(p^-)^2}r^{2p^-} - \nu_k - \lambda\mu_k \\ &< \frac{a}{p^-}\|u\|^{p^-} + \frac{b}{2(p^-)^2}\|u\|^{2p^-} - \nu_k - \lambda_k\mu_k \le 0.\end{aligned}$$

Theorem 1.10 guarantees that φ_2 has at least k pairs of different critical points. That is, Problem (2.17) has at least k distinct pairs of nontrivial solutions. Therefore we have the same result for Problem (1.1). □

Theorem 2.8. *Assume that (h_{11}) and (h_{12}) hold, and*

(h_{13}) $\int_0^u I_j(s)\,ds \le 0$ *for any* $u \in \mathbb{R}$ $(j = 1,\dots,l)$.

Then for any $k \in \mathbb{N}$, there exists a λ_k such that for any $\lambda > \lambda_k$, Problem (1.1) has at least k distinct pairs of nontrivial solutions.

Proof. The argument is similar to that in Theorem 2.7. □

REFERENCES

[1] A. Ambrosetti, A. Malchiodi, *Nonlinear Analysis and Semilinear Elliptic Problems*, Cambridge University Press, Cambridge, 2007.

[2] A. Castro, *Metodos Variacionales y Analisis Funcional no Lineal*, X Coloquio Colombiano de Matematicas, 1980.

[3] J. Chabrowski, Y. Fu, *Existence of solutions for $p(t)$-Laplacian problems on a bounded domain*, J. Math. Anal. Appl. **306** (2005), 604–618.

[4] H. Chen, J. Li, *Variational approach to impulsive differential equation with Dirichlet boundary conditions*, Bound. Value Probl. **2010** (2010), 1–16.

[5] D.C. Clarke, *A variant of the Lusternik-Schnirelmann theory*, Indiana Univ. Math. J. **22** (1972), 65–74.

[6] F.J.S.A. Corrêa, G.M. Figueiredo, *On an elliptic equation of p-Kirchhoff-type via variational methods*, Bull. Aust. Math. Soc. **74** (2006), 263–277.

[7] F.J.S.A. Corrêa, G.M. Figueiredo, *On a p-Kirchhoff equation via Krasnoselskii's genus*, Appl. Math. Lett. **22** (2009), 819–822.

[8] G. Dai, J. Wei, *Infinitely many non-negative solutions for a $p(t)$-Kirchhoff-type problem with Dirichlet boundary condition*, Nonlinear Anal. **73** (2010), 3420–3430.

[9] L. Diening, P. Harjulehto, P. Hastö, M. Ružička, *Lebesgue and Sobolev spaces with variable exponents*, Lectures Notes in Mathematics, 2011.

[10] X.L. Fan, Q.H. Zhang, *Existence of solutions for $p(t)$-Laplacian Dirichlet problem*, Nonlinear Anal. **52** (2003), 1843–1852.

[11] X.L. Fan, D. Zhao, *On the spaces $L^{p(t)}$ and $W^{m,p(t)}$*, J. Math. Anal. Appl. **263** (2001), 424–446.

[12] M. Galewski, D. O'Regan, *Impulsive boundary value problems for $p(t)$-Laplacian's via critical point theory*, Czechoslovak Math. J. **62** (2012), 951–967.

[13] M. Galewski, D. O'Regan, *On well posed impulsive boundary value problems for $p(t)$-Laplacian's*, Math. Model. Anal. **18** (2013), 161–175.

[14] O. Kavian, *Introduction à la Théorie des Points Critiques et Applications aux Problèmes Elliptiques*, Springer-Verlag, 1993.

[15] M.A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, MacMillan, New York, 1964.

[16] I. Peral, *Multiplicity of solutions for the p-Laplacian*, Second School of Nonlinear Functional Analysis and Applications to Differential Equations, ICTP, Trieste, 1997.

[17] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, Conference board of the mathematical sciences, American Mathematical Society, Providence, Rhode Island, 1984.

[18] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.

A. Mokhtari
mokhtarimaths@yahoo.fr

Laboratory of Fixed Point Theory and Applications
Department of Mathematics
E.N.S. Kouba, Algiers, Algeria

T. Moussaoui
moussaoui@ens-kouba.dz

Laboratory of Fixed Point Theory and Applications
Department of Mathematics
E.N.S. Kouba, Algiers, Algeria

D. O'Regan
donal.oregan@nuigalway.ie

School of Mathematics, Statistics and Applied Mathematics
National University of Ireland
Galway, Ireland

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