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ON LOCALLY HILBERT SPACES

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Abstract. This is an investigation of some basic properties of strictly inductive limits of Hilbert spaces, called locally Hilbert spaces, with respect to their topological properties, the geometry of their subspaces, linear functionals and dual spaces.

Keywords: locally Hilbert space, inductive limit, projective limit, orthocomplemented subspaces, linear functional, dual spaces.

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1. INTRODUCTION

The motivation to consider the topics of this article comes from our attempt to understand locally Hilbert C^*-modules from an operator theory point of view. More precisely, a generalisation of the concept of C^*-algebra that is called *locally* C^**-algebra*, cf. Inoue [7], triggered the investigations of locally Hilbert C^*-modules. Locally C^*-algebras have been called also LMC^**-algebras* [14], b^**-algebras* [1], and *pro* C^**-algebras* [15], [13]. Inoue also proved that, in order to have an operator model for locally C^*-algebra, a strictly inductive limit of Hilbert spaces, later called *locally Hilbert space*, is a natural concept to be used. Following this idea and employing techniques from dilation theory, in [4] we provided an operator model for locally Hilbert C^*-modules whose utility was first tested by obtaining a direct proof of existence of the exterior tensor products for locally Hilbert C^*-modules. Other applications are expected as well.

On the other hand, a locally Hilbert space bears an inductive limit topology, a pre-Hilbert topology, and a weak topology as well, and their relations require to be clarified. In this respect, some attempts performed in [8] turned out to be wrong, see our Examples 3.7 and 3.9.

This note is an investigation of some basic properties of locally Hilbert spaces with respect to their topological properties, the geometry of their subspaces, linear functionals and dual spaces, all these making the contents of Section 3. In this respect, using some classical duality theory, we first clarify for which subspaces of a locally

Hilbert space we expect to have orthocomplementarity and then we characterise the topological dual spaces of Hilbert spaces as projective limits of Hilbert spaces and some of their distinguished linear functionals, those norm continuous and those weakly continuous. Briefly, these are based on the observation that a strictly inductive system of Hilbert spaces gives naturally rise to a projective system of Hilbert space which characterises the topological dual of the underlying locally Hilbert space. This aspect is related to the concept of coherent transformation between inductive limits or between projective limits spaces. We also included a preliminary section that fixes the terminology and basic results on projective limits and inductive limits of locally convex spaces, cf. [5, 11, 12], that we use. In addition, in Subsection 2.4 we reviewed some consequences of the general duality theory for the geometry of subspaces of pre-Hilbert spaces.

2. SOME NOTATION AND PRELIMINARIES

2.1. PROJECTIVE LIMITS OF LOCALLY CONVEX SPACES.

A *projective system* of locally convex spaces is a pair $(\{\mathcal{V}_\alpha\}_{\alpha\in A};\{\varphi_{\alpha,\beta}\}_{\alpha\leq\beta})$ subject to the following properties:

(ps1) $(A;\leq)$ is a directed poset (partially ordered set);
(ps2) $\{\mathcal{V}_\alpha\}_{\alpha\in A}$ is a family of locally convex spaces;
(ps3) $\{\varphi_{\alpha,\beta}\mid \varphi_{\alpha,\beta}\colon \mathcal{V}_\beta\to\mathcal{V}_\alpha,\ \alpha,\beta\in A,\ \alpha\leq\beta\}$ is a family of continuous linear maps such that $\varphi_{\alpha,\alpha}$ is the identity map on $\mathcal{V}_\alpha$ for all $\alpha\in A$;
(ps4) the following transitivity condition holds

$$\varphi_{\alpha,\gamma}=\varphi_{\alpha,\beta}\circ\varphi_{\beta,\gamma},\ \text{for all } \alpha,\beta,\gamma\in A,\ \text{such that } \alpha\leq\beta\leq\gamma. \tag{2.1}$$

For such a projective system of locally convex spaces, consider the vector space

$$\prod_{\alpha\in A}\mathcal{V}_\alpha=\{(v_\alpha)_{\alpha\in A}\mid v_\alpha\in\mathcal{V}_\alpha,\ \alpha\in A\}, \tag{2.2}$$

with product topology, that is, the weakest topology which makes the canonical projections $\prod_{\alpha\in A}\mathcal{V}_\alpha\to\mathcal{V}_\beta$ continuous, for all $\beta\in A$. Then define $\mathcal{V}$ as the subspace of $\prod_{\alpha\in A}\mathcal{V}_\alpha$ consisting of all families of vectors $v=(v_\alpha)_{\alpha\in A}$ subject to the following *transitivity* condition

$$\varphi_{\alpha,\beta}(v_\beta)=v_\alpha,\ \text{for all } \alpha,\beta\in A,\ \text{such that } \alpha\leq\beta, \tag{2.3}$$

for which we use the notation

$$v=\varprojlim_{\alpha\in A} v_\alpha. \tag{2.4}$$

Further on, for each $\alpha\in A$, define $\varphi_\alpha\colon\mathcal{V}\to\mathcal{V}_\alpha$ as the linear map obtained by composing the canonical embedding of $\mathcal{V}$ in $\prod_{\alpha\in A}\mathcal{V}_\alpha$ with the canonical projection on $\mathcal{V}_\alpha$. Observe that $\mathcal{V}$ is a closed subspace of $\prod_{\alpha\in A}\mathcal{V}_\alpha$ and that the topology of $\mathcal{V}$ induced by the product topology from $\prod_{\alpha\in A}\mathcal{V}_\alpha$ can be seen as well as the weakest locally convex topology that makes the linear maps $\varphi_\alpha\colon\mathcal{V}\to\mathcal{V}_\alpha$ continuous, for all $\alpha\in A$. The pair $(\mathcal{V};\{\varphi_\alpha\}_{\alpha\in A})$ is called a *projective limit of locally convex spaces* induced by the projective system $(\{\mathcal{V}_\alpha\}_{\alpha\in A};\{\varphi_{\alpha,\beta}\}_{\alpha\leq\beta})$ and is denoted by

$$\mathcal{V}=\varprojlim_{\alpha\in A}\mathcal{V}_\alpha. \tag{2.5}$$

With notation as before, a locally convex space $\mathcal{W}$ and a family of continuous linear maps $\psi_\alpha\colon\mathcal{W}\to\mathcal{V}_\alpha$, $\alpha\in A$, are *compatible* with the projective system $(\{\mathcal{V}_\alpha\}_{\alpha\in A};\{\varphi_{\alpha,\beta}\}_{\alpha\leq\beta})$ if

$$\psi_\alpha=\varphi_{\alpha,\beta}\circ\psi_\beta,\ \text{for all } \alpha,\beta\in A \text{ with } \alpha\leq\beta. \tag{2.6}$$

For such a pair $(\mathcal{W};\{\psi_\alpha)\}_{\alpha\in A}$, there always exists a unique continuous linear map $\psi\colon\mathcal{W}\to\mathcal{V}=\varprojlim_{\alpha\in A}\mathcal{V}_\alpha$ such that

$$\psi_\alpha=\varphi_\alpha\circ\psi,\quad \alpha\in A. \tag{2.7}$$

Note that the projective limit $(\mathcal{V};\{\varphi_\alpha\}_{\alpha\in A})$ defined before is compatible with the projective system $(\{\mathcal{V}_\alpha\}_{\alpha\in A};\{\varphi_{\alpha,\beta}\}_{\alpha\leq\beta})$ and that, in this sense, the projective limit $(\mathcal{V}_\alpha;\{\varphi_\alpha\}_{\alpha\in A})$ is uniquely determined by the projective system $(\{\mathcal{V}_\alpha\}_{\alpha\in A};\{\varphi_{\alpha,\beta}\}_{\alpha\leq\beta})$.

The projective limit of a projective system of Hausdorff locally convex spaces is always Hausdorff and, if all locally convex spaces are complete, then the projective limit is complete.

2.2. INDUCTIVE LIMITS OF LOCALLY CONVEX SPACES

An *inductive system* of locally convex spaces is a pair $(\{\mathcal{X}_\alpha\}_{\alpha\in A};\{\chi_{\beta,\alpha}\}_{\alpha\leq\beta})$ subject to the following conditions:

(is1) $(A;\leq)$ is a directed poset;
(is2) $\{\mathcal{X}_\alpha\}_{\alpha\in A}$ is a net of locally convex spaces;
(is3) $\{\chi_{\beta,\alpha}\colon\mathcal{X}_\alpha\to\mathcal{X}_\beta\mid\alpha,\beta\in A,\ \alpha\leq\beta\}$ is a family of continuous linear maps such that $\chi_{\alpha,\alpha}$ is the identity map on $\mathcal{X}_\alpha$ for all $\alpha\in A$;
(is4) the following transitivity condition holds

$$\chi_{\delta,\alpha}=\chi_{\delta,\beta}\circ\chi_{\beta,\alpha},\ \text{for all } \alpha,\beta,\gamma\in A \text{ with } \alpha\leq\beta\leq\delta. \tag{2.8}$$

Recall that the *locally convex direct sum* $\bigoplus_{\alpha\in A}\mathcal{X}_\alpha$ is the algebraic direct sum, that is, the subspace of the direct product $\prod_{\alpha\in A}$ defined by all families $\{x_\alpha\}_{\alpha\in A}$ with finite support, endowed with the strongest locally convex topology that makes the canonical embedding $\mathcal{X}_\alpha\hookrightarrow\bigoplus_{\beta\in A}\mathcal{X}_\beta$ continuous, for all $\beta\in A$. In the following, we consider $\mathcal{X}_\alpha$ canonically identified with a subspace of $\bigoplus_{\beta\in A}\mathcal{X}_\beta$ and then, let the linear subspace $\mathcal{X}_0$ of $\bigoplus_{\alpha\in A}\mathcal{X}_\alpha$ be defined by

$$\mathcal{X}_0=\operatorname{Lin}\{x_\alpha-\chi_{\beta,\alpha}(x_\alpha)\mid\alpha,\beta\in A,\ \alpha\leq\beta,\ x_\alpha\in\mathcal{X}_\alpha\}. \tag{2.9}$$

The *inductive limit locally convex space* $(\mathcal{X};\{\chi_\alpha\}_{\alpha\in A})$ of the inductive system of locally convex spaces $(\{\mathcal{X}_\alpha\}_{\alpha\in A};\{\chi_{\beta,\alpha}\}_{\alpha\leq\beta})$ is defined as follows. Firstly,

$$\mathcal{X} = \varinjlim_{\alpha\in A} \mathcal{X}_\alpha = \big(\bigoplus_{\alpha\in A} \mathcal{X}_\alpha\big)/\mathcal{X}_0. \tag{2.10}$$

Then, for arbitrary $\alpha\in A$, the canonical linear map $\chi_\alpha\colon \mathcal{X}_\alpha\to\varinjlim_{\alpha\in A}\mathcal{X}_\alpha$ is defined as the composition of the canonical embedding $\mathcal{X}_\alpha\hookrightarrow\bigoplus_{\beta\in A}\mathcal{X}_\beta$ with the quotient map $\bigoplus_{\alpha\in A}\mathcal{X}_\beta\to\mathcal{X}$. The inductive limit topology of $\mathcal{X}=\varinjlim_{\alpha\in A}\mathcal{X}_\alpha$ is the strongest locally convex topology on $\mathcal{X}$ that makes the linear maps χ_α continuous, for all $\alpha\in A$.

An important distinction with respect to the projective limit is that, under the assumption that all locally convex spaces $\mathcal{X}_\alpha$, $\alpha\in A$, are Hausdorff, the inductive limit topology may not be Hausdorff, unless the subspace $\mathcal{X}_0$ is closed in $\bigoplus_{\alpha\in A}\mathcal{X}_\beta$, see [10] and [12]. Also, in general, the inductive limit of an inductive system of complete locally convex spaces is not complete.

With notation as before, a locally convex space $\mathcal{Y}$, together with a family of continuous linear maps $\kappa_\alpha\colon\mathcal{X}_\alpha\to\mathcal{Y}$, $\alpha\in A$, is *compatible* with the inductive system $(\{\mathcal{X}_\alpha\}_{\alpha\in A};\{\chi_{\beta,\alpha}\}_{\alpha\leq\beta})$ if

$$\kappa_\alpha = \kappa_\beta\circ\chi_{\beta,\alpha},\quad \alpha,\beta\in A,\ \alpha\leq\beta. \tag{2.11}$$

For such a pair $(\mathcal{Y};\{\kappa_\alpha)\}_{\alpha\in A}$, there always exists a unique continuous linear map $\kappa\colon\mathcal{Y}\to\mathcal{X}=\varinjlim_{\alpha\in A}\mathcal{X}_\alpha$ such that

$$\kappa_\alpha = \kappa\circ\chi_\alpha,\quad \alpha\in A. \tag{2.12}$$

Note that the inductive limit $(\mathcal{X};\{\chi_\alpha\}_{\alpha\in A})$ is compatible with $(\{\mathcal{X}_\alpha\}_{\alpha\in A};\{\chi_{\beta,\alpha}\}_{\alpha\leq\beta})$ and that, in this sense, the inductive limit $(\mathcal{X};\chi_\alpha\}_{\alpha\in A})$ is uniquely determined by the inductive system $(\{\mathcal{X}_\alpha\}_{\alpha\in A};\{\chi_{\beta,\alpha}\}_{\alpha\leq\beta})$.

2.3. COHERENT MAPS.

Let $(\mathcal{X};\{\chi_\alpha\}_{\alpha\in A})$, $\mathcal{X}=\varinjlim_{\alpha\in A}\mathcal{X}_\alpha$, and $(\mathcal{Y};\{\kappa_\alpha\}_{\alpha\in A})$, $\mathcal{Y}=\varinjlim_{\alpha\in A}\mathcal{Y}_\alpha$, be two inductive limits of locally convex spaces indexed by the same directed poset A. A linear map $g\colon\mathcal{X}\to\mathcal{Y}$ is called *coherent* if

(cim) There exists $\{g_\alpha\}_{\alpha\in A}$ a net of linear maps $g_\alpha\colon\mathcal{X}_\alpha\to\mathcal{Y}_\alpha$, $\alpha\in A$, such that $g\circ\chi_\alpha=\kappa_\alpha\circ g_\alpha$ for all $\alpha\in A$.

In terms of the underlying inductive systems $(\{\mathcal{X}_\alpha\}_{\alpha\in A};\{\chi_{\beta,\alpha}\}_{\alpha\leq\beta})$ and $(\{\mathcal{Y}_\alpha\}_{\alpha\in A};\{\kappa_{\beta,\alpha}\}_{\alpha\leq\beta})$, (cim) is equivalent with

(cim)′ There exists $\{g_\alpha\}_{\alpha\in A}$ a net of linear maps $g_\alpha\colon\mathcal{X}_\alpha\to\mathcal{Y}_\alpha$, $\alpha\in A$, such that $\kappa_{\beta,\alpha}\circ g_\alpha=g_\beta\circ\chi_{\beta,\alpha}$, for all $\alpha,\beta\in A$ with $\alpha\leq\beta$.

There is an one-to-one correspondence between the class of all coherent linear maps $g\colon\mathcal{X}\to\mathcal{Y}$ and the class of all nets $\{g_\alpha\}_{\alpha\in A}$ as in (cim) or, equivalently, as in (cim)′.

Let $(\mathcal{V};\{\varphi_\alpha\}_{\alpha\in A})$, $\mathcal{V}=\varprojlim_{\alpha\in A}\mathcal{V}_\alpha$, and $(\mathcal{W};\{\psi_\alpha\}_{\alpha\in A})$, $\mathcal{W}=\varprojlim_{\alpha\in A}\mathcal{W}_\alpha$, be two projective limits of locally convex spaces indexed by the same directed poset A. A linear map $f\colon\mathcal{V}\to\mathcal{W}$ is called *coherent* if (cpm) There exists $\{f_\alpha\}_{\alpha\in A}$ a net of linear maps $f_\alpha\colon \mathcal{V}_\alpha \to \mathcal{W}_\alpha$, $\alpha\in A$, such that $\psi_\alpha \circ f = f_\alpha \circ \varphi_\alpha$ for all $\alpha \in A$.

In terms of the underlying projective systems $(\{\mathcal{V}_\alpha\}_{\alpha\in A};\{\varphi_{\alpha,\beta}\}_{\alpha\leq\beta})$ and $(\{\mathcal{W}_\alpha\}_{\alpha\in A};\{\psi_{\alpha,\beta}\}_{\alpha\leq\beta})$, (cpm) is equivalent with

(cpm)′ There exists $\{f_\alpha\}_{\alpha\in A}$ a net of linear maps $f_\alpha\colon \mathcal{V}_\alpha \to \mathcal{W}_\alpha$, $\alpha\in A$, such that $\psi_{\alpha,\beta} \circ f_\beta = f_\alpha \circ \varphi_{\alpha,\beta}$, for all $\alpha,\beta \in A$ with $\alpha\leq\beta$.

There is an one-to-one correspondence between the class of all coherent linear maps $f\colon \mathcal{V}\to\mathcal{W}$ and the class of all nets $\{f_\alpha\}_{\alpha\in A}$ as in (cpm) or, equivalently, as in (cpm)′.

2.4. ORTHOCOMPLEMENTED SUBSPACES IN PRE-HILBERT SPACES.

In the following we consider an inner product space $\mathcal{H}$ and let $\widetilde{\mathcal{H}}$ denote its completion to a Hilbert space. We will denote by $\langle\cdot,\cdot\rangle$ the inner product on both $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, when there is no danger of confusion. The ambiental space is $\mathcal{H}$ and the weak topology on $\mathcal{H}$ is determined by the set of linear functionals $\mathcal{H}\ni h\mapsto\langle h,k\rangle$, for $k\in\mathcal{H}$. On the other hand, there is a weak topology on the Hilbert space $\widetilde{\mathcal{H}}$, determined by all linear functionals $\widetilde{\mathcal{H}}\ni h\mapsto\langle h,k\rangle$, for $k\in\widetilde{\mathcal{H}}$, and this induces a topology on $\mathcal{H}$, determined by all linear functionals $\mathcal{H}\ni h\mapsto\langle h,k\rangle$, for $k\in\widetilde{\mathcal{H}}$, different than the weak topology on $\mathcal{H}$; in general, the weak topology of $\mathcal{H}$ is weaker than the topology induced by the weak topology of $\widetilde{\mathcal{H}}$ on $\mathcal{H}$.

The following Proposition is a special case of a well-known result in duality theory, e.g. see Theorem 1.3.1 in [9].

Proposition 2.1. *If the linear functional φ on the inner product space $\mathcal{H}$ is weakly continuous then there exists a vector $h_0\in\mathcal{H}$ such that*

$$\varphi(h)=\langle h,h_0\rangle,\quad h\in\mathcal{H}.$$

For an arbitrary nonempty subset $\mathcal{S}$ of $\mathcal{H}$ we denote, as usually, the *orthogonal companion* of $\mathcal{S}$ by $\mathcal{S}^\perp=\{k\in\mathcal{H}\mid\langle h,k\rangle=0 \text{ for all } h\in\mathcal{H}\}$. Clearly, $\mathcal{S}^\perp$ is always weakly closed. We first show that, as in the Hilbert space case, in any pre-Hilbert space the weak topology provides a characterisation of those linear manifolds $\mathcal{L}$ in $\mathcal{H}$ such that $\mathcal{L}=\mathcal{L}^{\perp\perp}$. The next two results are also known, even under more general assumptions, e.g. see [3] for the case of indefinite inner product spaces, but we present short proofs for the reader's convenience.

Lemma 2.2. *Let $\mathcal{L}$ be a linear manifold of $\mathcal{H}$ and denote by $\overline{\mathcal{L}}$ its weak closure. Then $\mathcal{L}^\perp$ is weakly closed and $\mathcal{L}^\perp=\overline{\mathcal{L}}^\perp$.*

Proof. If $h_0\notin\mathcal{L}^\perp$ then there exists $k\in\mathcal{L}$ such that $\langle h_0,k\rangle\neq 0$. Since the inner product is weakly continuous in the first variable there exists a neighbourhood V of h_0, with respect to the weak topology, such that $[h,k]\neq 0$ for all $h\in V\cap\mathcal{L}^\perp$. Hence $\mathcal{L}^\perp$ is weakly closed.

Since $\mathcal{L}\subseteq\overline{\mathcal{L}}$ we obtain $\mathcal{L}^\perp\supseteq\overline{\mathcal{L}}^\perp$. Conversely, if $h\notin\overline{\mathcal{L}}^\perp$ there exists $k_0\in\overline{\mathcal{L}}$ such that $\langle h,k_0\rangle\neq 0$. Then $\langle h,k\rangle\neq 0$ for all k in a neighbourhood U of k_0. Since $U\cap\mathcal{L}\neq\emptyset$ it follows $h\notin\mathcal{L}^\perp$. □

Proposition 2.3. *A linear manifold $\mathcal{L}$ of the inner product space $\mathcal{H}$ is weakly closed if and only if $\mathcal{L} = \mathcal{L}^{\perp\perp}$.*

Proof. From Lemma 2.2 we obtain $\overline{\mathcal{L}} \subseteq \mathcal{L}^{\perp\perp}$. Conversely, let $h_0 \notin \overline{\mathcal{L}}$. Then there exists $\epsilon > 0$ and $\{k_1, \ldots, k_n\} \subset \mathcal{H}$ such that

$$\{h \mid |\langle h - h_0, k_j\rangle| < \epsilon,\ 1 \leq j \leq n\} \cap \mathcal{L} = \emptyset.$$

Let us consider the seminorms on $\mathcal{H}$

$$p(h) = \max_{j=1}^{n} |\langle h, k_j\rangle|, \quad h \in \mathcal{H},$$
$$q(h) = \inf_{l\in\mathcal{L}} p(h - l), \quad l \in \mathcal{H}.$$

Then $q(h_0) \geq \epsilon$. By the complex version of the Hahn–Banach Theorem we obtain a linear functional φ on $\mathcal{H}$ such that $\varphi(h_0) = \epsilon$ and

$$|\varphi(h)| \leq q(h), \quad h \in \mathcal{H}. \tag{2.13}$$

By definition, q is weakly continuous hence φ is weakly continuous. Thus, by Proposition 2.1, there exists $k_0 \in \mathcal{H}$ such that

$$\varphi(h) = \langle h, k_0\rangle, \quad h \in \mathcal{H}.$$

From (2.13) and the definition of q, it follows $k_0 \in \mathcal{L}^\perp$ while $\langle h_0, k_0\rangle = \epsilon > 0$, hence $h_0 \notin \mathcal{L}^{\perp\perp}$. □

For two linear subspaces $\mathcal{S}$ and $\mathcal{L}$ of $\mathcal{H}$, that are mutually orthogonal, we denote by $\mathcal{S} \oplus \mathcal{L}$ their algebraic sum. Also, a linear operator $T\colon \mathcal{H} \to \mathcal{H}$ is called *projection* if $T^2 = T$ and *Hermitian* if $\langle Th, k\rangle = \langle h, Tk\rangle$ for all $h, k \in \mathcal{H}$. It is easy to see that any Hermitian projection T is *positive* in the sense $\langle Th, h\rangle \geq 0$ for all $h \in \mathcal{H}$ and that T is a Hermitian projection if and only if $I - T$ is the same.

The following result characterises the linear subspaces $\mathcal{S}$ of $\mathcal{H}$ that are *orthocomplemented*, that is, $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$. From the previous result it is clear that an orthocomplemented subspace $\mathcal{S}$ has the property that $\mathcal{S} = \mathcal{S}^{\perp\perp}$ and hence, it is necessarily weakly closed, but the general picture is a bit more involved than that.

Proposition 2.4. *Let $\mathcal{S}$ be a linear subspace of $\mathcal{H}$. The following assertions are equivalent:*

(i) *The weak topology of $\mathcal{S}$ coincides with the topology induced on $\mathcal{S}$ by the weak topology of $\mathcal{H}$.*
(ii) *For each $h \in \mathcal{H}$ the functional $\mathcal{S} \ni y \mapsto \langle y, h\rangle$ is continuous with respect to the weak topology of $\mathcal{S}$.*
(iii) $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$.
(iv) *There exists a Hermitian projection $P\colon \mathcal{H} \to \mathcal{H}$ such that* $\operatorname{Ran}(P) = \mathcal{S}$.

Proof. (i)⇒(ii). Let $h \in \mathcal{H}$ and observe that the linear functional $\mathcal{S} \ni y \mapsto \langle y, h\rangle$ is continuous with respect to the topology induced by the weak topology of $\mathcal{H}$ on $\mathcal{S}$. Since, by assumption, these two topologies coincide, it follows that this linear functional is weakly continuous on $\mathcal{S}$.

(ii)⇒(iii). Let $h \in \mathcal{H}$ be an arbitrary vector and consider the linear functional $\mathcal{S} \ni y \mapsto \langle y, h\rangle$ which, by assumption, is weakly continuous on $\mathcal{S}$ hence, by Proposition 2.1, it follows that there exists $h_0 \in \mathcal{S}$ such that $\langle y, h\rangle = \langle y, h_0\rangle$ for all $y \in \mathcal{S}$. This implies that $h_1 := h - h_0 \in \mathcal{S}^\perp$ hence $h \in \mathcal{S} \oplus \mathcal{S}^\perp$. This proves that $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$.

(iii)⇒(iv). The assumption means that for any $h \in \mathcal{H}$ there exist unique $h_0 \in \mathcal{S}$ and $h_1 \in \mathcal{S}^\perp$ such that $h = h_0$, so one can define $Ph = h_0$. It is easy to show that P is a Hermitian projection on $\mathcal{H}$ and that $\mathrm{Ran}(P) = \mathcal{S}$.

(iv)⇒(i). The topology induced by the weak topology of $\mathcal{H}$ on $\mathcal{S}$ is determined by the linear functionals $\mathcal{S} \ni y \mapsto \langle y, h\rangle$, when h runs in $\mathcal{H}$. Since, for any $h \in \mathcal{H}$ and any $y \in \mathcal{S}$ we have $\langle y, h\rangle = \langle Py, h\rangle = \langle y, Ph\rangle$, it follows that any of these linear functionals can be represented, for some $h_0 = Ph \in \mathcal{S}$, as a linear functional $\mathcal{S} \ni y \mapsto \langle y, h_0\rangle$, hence the two topologies coincide. □

3. MAIN RESULTS

3.1. LOCALLY HILBERT SPACES.

A *locally Hilbert space* is an *inductive limit*

$$\mathcal{H} = \varinjlim_{\lambda \in \Lambda} \mathcal{H}_\lambda = \bigcup_{\lambda \in \Lambda} \mathcal{H}_\lambda, \tag{3.1}$$

of a *strictly inductive system of Hilbert spaces* $\{\mathcal{H}_\lambda\}_{\lambda \in \Lambda}$, that is,

(lhs1) $(\Lambda; \leq)$ is a directed poset;
(lhs2) $\{\mathcal{H}_\lambda; \langle \cdot, \cdot \rangle_{\mathcal{H}_\lambda}\}_{\lambda \in \Lambda}$ is a net of Hilbert spaces;
(lhs3) for each $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$ we have $\mathcal{H}_\lambda \subseteq \mathcal{H}_\mu$;
(lhs4) for each $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$ the inclusion map $J_{\mu,\lambda} \colon \mathcal{H}_\lambda \to \mathcal{H}_\mu$ is isometric, that is,

$$\langle x, y\rangle_{\mathcal{H}_\lambda} = \langle x, y\rangle_{\mathcal{H}_\mu}, \text{ for all } x, y \in \mathcal{H}_\lambda. \tag{3.2}$$

As in Subsection 2.3, for each $\lambda \in \Lambda$, letting $J_\lambda \colon \mathcal{H}_\lambda \to \mathcal{H}$ be the inclusion of $\mathcal{H}_\lambda$ in $\bigcup_{\lambda \in \Lambda} \mathcal{H}_\lambda$, the *inductive limit topology* on $\mathcal{H}$ is the strongest that makes the linear maps J_λ continuous for all $\lambda \in \Lambda$. Also, it is clear that a locally Hilbert space is uniquely determined by the strictly inductive system of Hilbert spaces as in (lhs1)–(lhs4).

On $\mathcal{H}$ a canonical inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ can be defined as follows:

$$\langle h, k\rangle_{\mathcal{H}} = \langle h, k\rangle_{\mathcal{H}_\lambda}, \quad h, k \in \mathcal{H}, \tag{3.3}$$

where $\lambda \in \Lambda$ is any index for which $h, k \in \mathcal{H}_\lambda$. It follows that this definition of the inner product is correct and, for each $\lambda \in \Lambda$, the inclusion map $J_\lambda \colon (\mathcal{H}_\lambda; \langle \cdot, \cdot \rangle_{\mathcal{H}_\lambda}) \to$

$(\mathcal{H};\langle\cdot,\cdot\rangle_{\mathcal{H}})$ is isometric. This implies that, letting $\|\cdot\|_{\mathcal{H}}$ denote the norm induced by the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathcal{H}$, the *norm topology* on $\mathcal{H}$ is weaker than the inductive limit topology of $\mathcal{H}$. Since the norm topology is Hausdorff, it follows that the inductive limit topology on $\mathcal{H}$ is Hausdorff as well. In addition, on $\mathcal{H}$ we consider the *weak topology* as well, that is, the locally convex topology induced by the family of seminorms $\mathcal{H}\ni h\to|\langle h,k\rangle|$, indexed by $k\in\mathcal{H}$. Of course, the weak topology on any locally Hilbert space is Hausdorff separated as well.

Note that, a locally Hilbert space is a rather special type of locally convex space and, in general, not a Hilbert space, although it bears a canonical structure of a pre-Hilbert space. In particular, the results presented in Subsection 2.4 with respect to orthogonal companions and orthocomplentarity apply. For $\lambda\leq\mu$ we denote by $\mathcal{H}_\mu\ominus\mathcal{H}_\lambda$ the orthogonal companion of $\mathcal{H}_\lambda$ in $\mathcal{H}_\mu$.

Lemma 3.1. *For each $\lambda\in\Lambda$ we have $\mathcal{H}=\mathcal{H}_\lambda\oplus\mathcal{H}_\lambda^\perp$, in particular there exists a unique Hermitian projection $P_\lambda\colon\mathcal{H}\to\mathcal{H}$ such that* $\operatorname{Ran}(P_\lambda)=\mathcal{H}_\lambda$.

Proof. Clearly, $\mathcal{H}\supseteq\mathcal{H}_\lambda\oplus\mathcal{H}_\lambda^\perp$. Conversely, let $h\in\mathcal{H}$ arbitrary. Then there exists $\mu\in\Lambda$ such that $\lambda\leq\mu$, and hence $\mathcal{H}_\lambda\subseteq\mathcal{H}_\mu$, with $h\in\mathcal{H}_\mu=\mathcal{H}_\lambda\oplus(\mathcal{H}_\mu\ominus\mathcal{H}_\lambda)$. Since $\mathcal{H}_\mu\ominus\mathcal{H}_\lambda\subseteq\mathcal{H}_\lambda^\perp$ it follows that $h\in\mathcal{H}_\lambda\oplus\mathcal{H}_\lambda^\perp$.

The existence (and uniqueness) of the Hermitian projection $P_\lambda\colon\mathcal{H}\to\mathcal{H}$ with $\operatorname{Ran}(P_\lambda)=\mathcal{H}_\lambda$ follows as in the proof of Proposition 2.4 (iii)⇒(iv). □

With respect to the decomposition provided by Lemma 3.1, the underlying locally Hilbert space structure of $\mathcal{H}_\lambda^\perp$ can be explicitly described.

Proposition 3.2. *Fix $\lambda\in\Lambda$ arbitrary and denote by $\Lambda_\lambda=\{\mu\in\Lambda\mid\lambda\leq\mu\}$ the branch of Λ defined by λ. Then, with respect to the induced order relation $\leq$, Λ_λ is a directed poset, $\{\mathcal{H}_\mu\ominus\mathcal{H}_\lambda\mid\mu\in\Lambda_\lambda\}$ is a strictly inductive system of Hilbert spaces, and*

$$\mathcal{H}_\lambda^\perp=\varinjlim_{\mu\in\Lambda_\lambda}(\mathcal{H}_\mu\ominus\mathcal{H}_\lambda). \tag{3.4}$$

Proof. The fact that for each $\lambda\in\Lambda$ the branch Λ_λ is a directed poset is clear. Clear is also the fact that $\{\mathcal{H}_\mu\ominus\mathcal{H}_\lambda^\perp\}_{\mu\in\Lambda_\lambda}$ is a strictly inductive system of Hilbert spaces. In order to finish the proof we only have to prove that

$$\mathcal{H}_\lambda^\perp=\bigcup_{\mu\in\Lambda_\lambda}(\mathcal{H}_\mu\ominus\mathcal{H}_\lambda). \tag{3.5}$$

One inclusion of (3.5) is clear. In order to prove the converse inclusion, let $h\in\mathcal{H}_\lambda^\perp$ be an arbitrary vector. Then $h\in\mathcal{H}=\bigcup_{\nu\in\Lambda}\mathcal{H}_\nu$ and hence there exists $\nu\in\Lambda$ such that $h\in\mathcal{H}_\nu$. Since Λ is directed, without loss of generality we can assume that $\lambda\leq\nu$, hence $\mathcal{H}_\lambda\subseteq\mathcal{H}_\nu$. Then $h\in\mathcal{H}_\nu\ominus\mathcal{H}_\lambda$, hence (3.5) holds. □

The following remark shows that any pre-Hilbert space can be viewed as a locally Hilbert space.

Remark 3.3. Let $\mathcal{H}$ be an arbitrary pre-Hilbert space. Consider $\mathcal{F}(\mathcal{H})$ the collection of all finite dimensional subspaces of $\mathcal{H}$ and note that the inclusion makes $\mathcal{F}(\mathcal{H})$

a directed ordered set. Then observe that $\mathcal{F}(\mathcal{H})$ can be viewed as a strictly inductive system of Hilbert spaces in a canonical way and that

$$\mathcal{H} = \varinjlim_{\mathcal{L}\in\mathcal{F}(\mathcal{H})} \mathcal{L} = \bigcup_{\mathcal{L}\in\mathcal{F}(\mathcal{H})} \mathcal{L}.$$

3.2. LINEAR FUNCTIONALS ON LOCALLY HILBERT SPACES.

Let $\mathcal{H} = \varinjlim_{\lambda\in\Lambda} \mathcal{H}_\lambda$ be a locally Hilbert space. Let $\mathcal{H}^\sharp$ be the linear space consisting of all linear functionals $f\colon \mathcal{H} \to \mathbb{C}$ that are continuous with respect to the inductive limit topology of $\mathcal{H}$. For each $\lambda \in \Lambda$ we consider the canonical projection $\mathcal{H}^\sharp \ni f \mapsto f_\lambda = f|\mathcal{H}_\lambda \in \mathcal{H}_\lambda^\sharp$, where $\mathcal{H}_\lambda^\sharp$ denotes the topological dual space of $\mathcal{H}_\lambda$, viewed as a Banach space with the functional norm, and then, for each $\lambda, \mu \in \Lambda$ such that $\lambda \leq \mu$, there is a canonical projection $\mathcal{H}_\mu^\sharp \ni f \mapsto f|\mathcal{H}_\lambda \in \mathcal{H}_\lambda^\sharp$. It is easy to see that, in this way, we obtain a projective system of Banach spaces $\{\mathcal{H}_\lambda^\sharp\}_{\lambda\in\Lambda}$ and that $\mathcal{H}^\sharp$ is canonically identified with its projective limit

$$\mathcal{H}^\sharp = \varprojlim_{\lambda\in\Lambda} \mathcal{H}_\lambda^\sharp, \tag{3.6}$$

such that, for each $f \in \mathcal{H}^\sharp$, letting $f_\lambda = f|\mathcal{H}_\lambda \in \mathcal{H}_\lambda^\sharp$, we identify f with $\varprojlim_{\lambda\in\Lambda} f_\lambda$. We consider on $\mathcal{H}^\sharp$ the projective limit topology induced by (3.6).

Lemma 3.4. *For every $f \in \mathcal{H}^\sharp$ there exists a unique net $(\hat{f}_\lambda)_{\lambda\in\Lambda}$ subject to the following properties:*

(i) *$\hat{f}_\lambda \in \mathcal{H}_\lambda$ for each $\lambda \in \Lambda$.*
(ii) *For every $\lambda \leq \mu$ we have $P_{\lambda,\mu}\hat{f}_\mu = \hat{f}_\lambda$, where $P_{\lambda,\mu}$ is the orthogonal projection of $\mathcal{H}_\mu$ onto $\mathcal{H}_\lambda$.*
(iii) *For every $\lambda \in \Lambda$, $f(h) = \langle h, \hat{f}_\lambda\rangle_{\mathcal{H}_\lambda}$, for all $h \in \mathcal{H}_\lambda$.*

Proof. For each $\lambda \in \Lambda$ consider the linear map $\Phi_\lambda\colon \mathcal{H}_\lambda^\sharp \to \mathcal{H}_\lambda$ defined by $\Phi_\lambda(\varphi) = \hat{\varphi}$, for all $\varphi \in \mathcal{H}_\lambda^\sharp$, where $\hat{\varphi} \in \mathcal{H}_\lambda$ is, via the Riesz Representation Theorem, the unique vector such that $\varphi(h) = \langle h, \hat{\varphi}\rangle$ for all $h \in \mathcal{H}_\lambda$.

If $f \in \mathcal{H}^\sharp$ and $\lambda \leq \mu$ then, considering an arbitrary vector $h \in \mathcal{H}_\lambda \subseteq \mathcal{H}_\mu$, we have

$$\begin{aligned} f(h) &= \langle h, \hat{f}_\mu\rangle_{\mathcal{H}_\mu} = \langle P_{\lambda,\mu}h, \hat{f}_\mu\rangle_{\mathcal{H}_\mu} \\ &= \langle h, P_{\lambda,\mu}\hat{f}_\mu\rangle_{\mathcal{H}_\mu} = \langle h, P_{\lambda,\mu}\hat{f}_\mu\rangle_{\mathcal{H}_\lambda}, \end{aligned}$$

hence, taking into account the uniqueness of the vector $\hat{f}_\lambda$, it follows that $P_{\lambda,\mu}\hat{f}_\mu = \hat{f}_\lambda$. □

We observe that $(\{\mathcal{H}_\lambda\}_{\lambda\in\Lambda}; \{P_{\lambda,\mu}\}_{\lambda\leq\mu})$, where $P_{\lambda,\mu}$ is the orthogonal projection of $\mathcal{H}_\mu$ onto $\mathcal{H}_\lambda$ for every $\lambda \leq \mu$, is a projective system of Hilbert spaces, with respect to which there is a unique projective limit of Hilbert spaces $\varprojlim_{\lambda\in\Lambda} \mathcal{H}_\lambda$. Then observe that Lemma 3.4 implies that there exists a canonical map

$$\mathcal{H}^\sharp \ni f \mapsto \hat{f} = \varprojlim_{\lambda\in\Lambda} \hat{f}_\lambda \in \varprojlim_{\lambda\in\Lambda} \mathcal{H}_\lambda. \tag{3.7}$$

Proposition 3.5. *The transformation defined at* (3.7) *is a coherent conjugate linear isomorphism between the two projective limit locally convex spaces* $\mathcal{H}^\sharp$*, defined as in* (3.6)*, and* $\varprojlim_{\lambda\in\Lambda}\mathcal{H}_\lambda$.

Proof. For each $\lambda\in\Lambda$ consider the linear map $\Phi_\lambda\colon \mathcal{H}_\lambda^\sharp\to\mathcal{H}_\lambda$ defined as in the proof of Lemma 3.4. Clearly, Φ_λ is a conjugate isometric isomorphism. Then, modulo Lemma 3.4, define $\Phi\colon\mathcal{H}^\sharp\to\varprojlim_{\lambda\in\Lambda}\mathcal{H}_\lambda$ by

$$\Phi(f)=(\hat{f}_\mu)_{\mu\in\Lambda},\quad f\in\mathcal{H}^\sharp.$$

Letting $R_\lambda\colon\mathcal{H}^\sharp\to\mathcal{H}_\lambda^\sharp$ denote the canonical projection, defined by restriction, and considering the canonical projection $P_\lambda\colon\varprojlim_{\mu\in\Lambda}\mathcal{H}_\mu\to\mathcal{H}_\lambda$, for all $\lambda\in\Lambda$, it follows that

$$P_\lambda\circ\Phi=\Phi_\lambda\circ R_\lambda,\quad \lambda\in\Lambda,$$

hence Φ is a coherent conjugate linear transformation. The continuity of Φ comes for free, taking into account that the maps Φ_λ are isometric, for all $\lambda\in\Lambda$, hence continuous, and the definition of the projective limit topologies.

In order to show that Φ is a coherent conjugate linear isomorphism of projective limit spaces, we explicitly determine its inverse. Let $h=\varprojlim_{\lambda\in\Lambda}h_\lambda\in\varprojlim_{\lambda\in\Lambda}\mathcal{H}_\lambda$ be an arbitrary vector, that is, $h_\lambda\in\mathcal{H}_\lambda$, for each λ, and $P_{\lambda,\mu}h_\mu=h_\lambda$ whenever $\lambda\le\mu$. For arbitrary $\lambda\in\Lambda$, let $f_\lambda\in\mathcal{H}_\lambda^\sharp$ be the linear functional on $\mathcal{H}_\lambda$ determined by h_λ, that is, $f_\lambda(k)=\langle k,h_\lambda\rangle$, for all $k\in\mathcal{H}_\lambda$. We show that $(f_\lambda)_{\lambda\in\Lambda}$ satisfies the transitivity condition, that is, $f_\mu|\mathcal{H}_\lambda=f_\lambda$ whenever $\lambda\le\mu$. Indeed, for each $k\in\mathcal{H}_\lambda$,

$$\begin{aligned}f_\mu(k)&=\langle k,h_\mu\rangle_{\mathcal{H}_\mu}=\langle P_{\lambda,\mu}k,h_\mu\rangle_{\mathcal{H}_\mu}\\&=\langle k,P_{\lambda,\mu}h_\mu\rangle_{\mathcal{H}_\lambda}=\langle k,h_\lambda\rangle_{\mathcal{H}_\lambda}=f_\lambda(k).\end{aligned}$$

Letting

$$\varprojlim_{\lambda\in\Lambda}\mathcal{H}_\lambda\ni h=(h_\lambda)_{\lambda\in\Lambda}\mapsto\Psi(h)=f=(f_\lambda)_{\lambda\in\Lambda}\in\varprojlim_{\lambda\in\Lambda}\mathcal{H}_\lambda^\sharp=\mathcal{H}^\sharp,$$

we obtain a coherent conjugate linear transformation Ψ which is the inverse of Φ. The continuity of Ψ follows since Ψ_λ are isometric, for all $\lambda\in\Lambda$, hence continuous, and the definition of the projective limit topologies. □

Since the inductive limit topology on $\mathcal{H}$ is stronger than the norm topology, it is clear that any norm continuous linear functional on $\mathcal{H}$ is continuous with respect to the inductive limit topology on $\mathcal{H}$. The converse implication does not hold, in general.

Proposition 3.6. *Let* $f\in\mathcal{H}^\sharp$ *and consider* $(\hat{f}_\lambda)_{\lambda\in\Lambda}\in\varprojlim_{\lambda\in\Lambda}\mathcal{H}_\lambda$ *as in* (3.7)*. The following assertions are equivalent:*

(i) f *is norm continuous.*
(ii) *There exists* $z_f\in\widetilde{\mathcal{H}}$*, the Hilbert space completion of* $\mathcal{H}$*, such that* $f(h)=\langle h,z_f\rangle_{\widetilde{\mathcal{H}}}$ *for all* $h\in\mathcal{H}$.

(iii) *The net* $(\hat{f}_\lambda)_{\lambda\in\Lambda}$ *is norm bounded, that is,* $\sup_{\lambda\in\Lambda}\|\hat{f}_\lambda\|_{\mathcal{H}_\lambda}<\infty$.

(iv) *The net* $(\hat{f}_\lambda)_{\lambda\in\Lambda}$ *is Cauchy with respect to the norm topology on* $\mathcal{H}$.

Proof. (i)⇒(ii). If f is norm continuous then it has a (unique) extension to a bounded linear functional $\tilde{f}$ on the Hilbert space $\widetilde{\mathcal{H}}$ hence, by the Riesz Representation Theorem there exists $z_f\in\widetilde{\mathcal{H}}$ such that $\tilde{f}(h)=\langle h,z_f\rangle_{\widetilde{\mathcal{H}}}$ for all $h\in\widetilde{\mathcal{H}}$, in particular, for all $h\in\mathcal{H}$.

(ii)⇒(iii). For each $\lambda\in\Lambda$, letting $\widetilde{P}_\lambda$ denote the orthogonal projection of $\widetilde{\mathcal{H}}$ onto its closed subspace $\mathcal{H}_\lambda$, we have $\widetilde{P}_\lambda z_f=\hat{f}_\lambda$. Indeed, for arbitrary $h\in\mathcal{H}_\lambda$,

$$f(h)=\langle h,z_f\rangle_{\widetilde{\mathcal{H}}}=\langle \widetilde{P}_\lambda h,z_f\rangle_{\widetilde{\mathcal{H}}}=\langle h,\widetilde{P}_\lambda z_f\rangle_{\widetilde{\mathcal{H}}}=\langle h,\widetilde{P}_\lambda z_f\rangle_{\mathcal{H}_\lambda},$$

and then apply the uniqueness of $\hat{f}_\lambda$. Then, $\|\hat{f}_\lambda\|_{\mathcal{H}_\lambda}=\|\widetilde{P}_\lambda z_f\|_{\mathcal{H}_\lambda}\le\|z_f\|_{\widetilde{\mathcal{H}}}$.

(iii)⇒(i). Let $M=\sup_{\lambda\in\Lambda}\|\hat{f}_\lambda\|_{\mathcal{H}_\lambda}$ and let $h\in\mathcal{H}$ be arbitrary. Then there exists $\lambda\in\Lambda$ such that $h\in\mathcal{H}_\lambda$ and hence

$$\begin{aligned}|f(h)|&=|f_\lambda(h)|=|\langle h,\hat{f}_\lambda\rangle_{\mathcal{H}_\lambda}|\\&\le\|h\|_{\mathcal{H}_\lambda}\|\hat{f}_\lambda\|_{\mathcal{H}_\lambda}\le M\|h\|_{\mathcal{H}_\lambda}=M\|h\|_{\mathcal{H}}.\end{aligned}$$

(ii)⇒(iv). As before, for all $\lambda\in\Lambda$, we have $\widetilde{P}_\lambda z_f=\hat{f}_\lambda$. Since the net of orthogonal projections $(\widetilde{P}_\lambda)_{\lambda\in\Lambda}$ converges to $I_{\widetilde{\mathcal{H}}}$, the identity operator on $\widetilde{\mathcal{H}}$, with respect to the strong operator topology, e.g. see Proposition 2.5.6 in [9], it follows that $\hat{f}_\lambda=\widetilde{P}_\lambda z_f\to z_f$ in norm, hence the net $(\hat{f}_\lambda)_{\lambda\in\Lambda}$ is Cauchy with respect to the norm topology.

(iv)⇒(iii). If the net $(\hat{f}_\lambda)_{\lambda\in\Lambda}$ is norm Cauchy then clearly it is norm bounded. □

For the next examples, recall that a linear functional on a topological vector space is continuous if and only if its null space is closed, e.g. see Corollary 1.2.5 in [9].

Example 3.7. Let $\mathbb{C}^{\mathbb{N}}$ be the vector space of all complex sequences and for each $n\in\mathbb{N}$ let $\mathcal{H}_n=\{x=(x^{(k)})_k\in\mathbb{C}^{\mathbb{N}}\mid x^{(k)}=0\text{ for all }k\ge n\}$. Then $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ is a strictly inductive system of Hilbert spaces and its inductive limit is the space $\mathcal{H}=\{x\in\mathbb{C}^{\mathbb{N}}\mid \operatorname{supp}(x)<\infty\}$ of all complex sequences with finite support. The inner product on $\mathcal{H}$ is that induced from $\ell^2_{\mathbb{C}}$, which is the Hilbert space completion $\widetilde{\mathcal{H}}$ of $\mathcal{H}$.

On $\mathcal{H}$ consider the linear functional

$$f(x)=\sum_{k=1}^{\infty}x^{(k)},\quad x=(x^{(k)})_{k\in\mathbb{N}}\in\mathcal{H}.$$

The sequence $(\hat{f}_n)_{n\in\mathbb{N}}$ associated to f as in Lemma 3.4 is $\hat{f}_n^{(k)}=1$ for $k\le n$ and $k=0$ for $k>n$. Observe that $\|\hat{f}_n\|=\sqrt{n}$, hence the sequence $(\hat{f}_n)_{n\in\mathbb{N}}$ is unbounded. This shows that f is continuous with respect to the inductive limit topology on $\mathcal{H}$ but it is not norm continuous. In particular, this shows that Theorem 2.7 and Corollary 2.8 in [8] are false.

In addition, letting $\mathcal{L}=\operatorname{Null}(f)=\{x\in\mathcal{H}\mid\sum_{k=1}^{\infty}x^{(k)}=0\}$, this is an example of a subspace of $\mathcal{H}$ that is closed with respect to the inductive limit topology but not closed with respect to the norm topology.

Clearly, any weakly continuous linear functional on $\mathcal{H}$ is norm continuous and hence continuous with respect to the inductive limit topology on $\mathcal{H}$. The converse implication does not hold, in general.

Proposition 3.8. *Let $f \in \mathcal{H}^\sharp$ and consider $(\hat{f}_\lambda)_{\lambda\in\Lambda} \in \varprojlim_{\lambda\in\Lambda} \mathcal{H}_\lambda$ as in* (3.7). *The following assertions are equivalent:*

(i) *f is weakly continuous.*
(ii) *There exists $z_f \in \mathcal{H}$ such that $f(h) = \langle h, z_f\rangle_{\mathcal{H}}$ for all $h \in \mathcal{H}$.*
(iii) *The net $(\hat{f}_\lambda)_{\lambda\in\Lambda}$ is eventually constant, that is, there exists $\lambda_0 \in \Lambda$ such that, for each $\lambda \in \Lambda$ with $\lambda_0 \leq \lambda$ we have $\hat{f}_\lambda = \hat{f}_{\lambda_0}$.*

Proof. (i)⇒(ii). This is a consequence of Proposition 2.1.

(ii)⇒(i). This is obvious.

(ii)⇒(iii). Since $z_f \in \mathcal{H}$ it follows that there exists $\lambda_0 \in \Lambda$ such that $z_f \in \mathcal{H}_{\lambda_0}$. Then, for any $\lambda \in \Lambda$ with $\lambda_0 \leq \lambda$ we have $z_f \in \mathcal{H}_\lambda$ and hence, for all $h \in \mathcal{H}_\lambda$ we have

$$\langle h, z_f\rangle_{\mathcal{H}_\lambda} = f(h) = \langle h, \hat{f}_\lambda\rangle_{\mathcal{H}_\lambda}.$$

By uniqueness, it follows that $\hat{f}_\lambda = z_f = \hat{f}_{\lambda_0}$.

(iii)⇒(ii). Let $z_f = \hat{f}_{\lambda_0} \in \mathcal{H}$. For any $h \in \mathcal{H}$ there exists $\lambda \in \Lambda$, with $\lambda_0 \leq \lambda$ and $h \in \mathcal{H}_\lambda$, hence,

$$f(h) = \langle h, \hat{f}_\lambda\rangle_{\mathcal{H}_\lambda} = \langle h, \hat{f}_{\lambda_0}\rangle_{\mathcal{H}_\lambda} = \langle h, z_f\rangle_{\mathcal{H}}.$$

□

Example 3.9. With notation as in Example 3.7, the subspace $\mathcal{M} = \{x \in \mathcal{H} \mid \sum_{k=1}^\infty \frac{x^{(k)}}{k} = 0\}$ is norm closed, hence closed with respect to the inductive limit topology of $\mathcal{H}$, but not weakly closed. Indeed, consider the linear functional $g\colon \mathcal{H} \to \mathbb{C}$ defined by

$$g(x) = \sum_{k=1}^\infty \frac{x^{(k)}}{k}, \quad x = (x^{(k)})_k \in \mathcal{H},$$

and observe that $\mathcal{M} = \mathrm{Null}(g)$. The functional g is norm continuous, since letting $z_g = (\frac{1}{k})_k \in \ell^2 = \widetilde{\mathcal{H}}$, we have $g(x) = \langle x, z_g\rangle$ for all $x \in \mathcal{H}$, but it is not weakly continuous, since $z_g \notin \mathcal{H}$. Therefore, by Proposition 3.6 $\mathcal{M} = \mathrm{Null}(g)$ is a norm closed subspace of $\mathcal{H}$ but, by Proposition 3.8 it is not weakly closed,.

Also, since $\mathcal{M} \neq \mathcal{H}$ and $\mathcal{M}^\perp = \{0\}$ it follows that $\mathcal{H} \neq \mathcal{M} \oplus \mathcal{M}^\perp$, see [6]. In particular, this shows that Theorem 2.6 in [8] is false.

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