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THE INVERSE SCATTERING TRANSFORM IN THE FORM OF A RIEMANN-HILBERT PROBLEM FOR THE DULLIN-GOTTWALD-HOLM EQUATION

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Abstract. The Cauchy problem for the Dullin-Gottwald-Holm (DGH) equation

$$u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3uu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx})$$

with zero boundary conditions (as $|x| \to \infty$) is treated by the Riemann-Hilbert approach to the inverse scattering transform method. The approach allows us to give a representation of the solution to the Cauchy problem, which can be efficiently used for further studying the properties of the solution, particularly, in studying its long-time behavior. Using the proposed formalism, smooth solitons as well as non-smooth cuspon solutions are presented.

Keywords: Dullin-Gottwald-Holm equation, Camassa-Holm equation, inverse scattering transform, Riemann-Hilbert problem.

Mathematics Subject Classification: 35Q53, 37K15, 35Q15, 35B40, 35Q51, 37K40.

1. INTRODUCTION

The Dullin-Gottwald-Holm (DGH) equation

$$u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3uu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}), \tag{1.1}$$

where ω, α and γ are real parameters, belongs to the class of unidirectional nonlinear wave equations, obtained via asymptotic expansions around simple wave motion of the Euler equations for shallow water in a particular Galilean frame. Actually, (1.1) was first derived, by using asymptotic expansions, in [12]. Before [12], integrable equations similar to (1.1) were derived in the context of the theory of hereditary symmetries [18]. In [17], equation (1.1) was re-derived by using asymptotic expansions directly in the Hamiltonian for the Euler equations in the shallow water regime and was proved to be correct to one order higher than for the Korteweg-de Vries (KdV) by using methods of asymptotic expansions and near-identity transformations.

Equation (1.1) is also called the CH-γ equation: it combines the linear dispersion of the Korteweg-de Vries (KdV) equation and the nonlinear/nonlocal dispersion of the Camassa-Holm (CH) equation [12]

$$u_t - u_{xxt} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.2}$$

Indeed, if $\gamma = 0$ and $\alpha = 1$, then (1.1) reduces to the CH equation whereas if $\alpha = 0$, (1.1) reduces to the KdV equation.

The DGH equation (1.1) is integrable in the sense that it possesses the Lax pair representation: (1.1) is the compatibility condition for the linear equations involving the spectral parameter η [17]:

$$\psi_{xx} = \frac{1}{4\alpha^2}\psi + \eta(m(x,t) + \Omega)\psi, \tag{1.3a}$$

$$\psi_t = \left(\frac{1}{2\alpha^2}\left(\frac{1}{\eta} + 2\gamma\right) - u(x,t)\right)\psi_x + \frac{1}{2}u_x(x,t)\psi, \tag{1.3b}$$

where m is the momentum variable: $m := u - \alpha^2 u_{xx}$, and $\Omega := \omega + \frac{\gamma}{2\alpha^2}$.

The Cauchy problem for (1.1) has been studied in [20, 21]. It has been shown that this equation is locally well-posed for initial data $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. The scattering problem for (1.1) is considered in [20] by examining the associated iso-spectral problem. The inverse scattering problem for the DGH equation is discussed in [1, 13, 20], where the reduction, by the Liouville transformation, of the iso-spectral problem to the the classical Sturm-Liouville problem is used. In [13], the Poisson brackets are computed and the action-angle variables are expressed in terms of the scattering data.

The change of variables

$$u(x,t) = v\left(\frac{x}{\alpha}, \frac{t}{\alpha}\right) + \frac{\gamma}{\alpha^2} \tag{1.4}$$

removes γ reducing (1.1) to the CH equation (1.2) (in variables $\tilde{x} = \frac{x}{\alpha}$, $\tilde{t} = \frac{x}{\alpha}$) with the linear dispersion parameter $\tilde{\omega} = \omega + \frac{3\gamma}{2\alpha^2}$ whereas the change of variables

$$u(x,t) = v\left(\frac{x}{\alpha} + \frac{\gamma}{\alpha^3}t, \frac{t}{\alpha}\right) \tag{1.5}$$

reduces (1.1) to (1.2) (in variables $\tilde{x} = \frac{x}{\alpha} + \frac{\gamma}{\alpha^3}t$, $\tilde{t} = \frac{x}{\alpha}$) with the linear dispersion parameter $\tilde{\omega} = \omega + \frac{\gamma}{2\alpha^2}$. Now notice that if one considers (1.1) on an infinite spatial domain (whole line or a half-line), then transformation (1.4) changes the boundary condition at infinity: if $u \to 0$ as $x \to \infty$ for (1.1), then $v \to \frac{\gamma}{\alpha^2}$ as $x \to \infty$ for (1.2). On the other hand, transformation (1.5), while keeping the boundary condition unchanged, uses a moving frame of reference, which would change the boundary conditions (from fixed to moving ones), if one wants to consider, say, the initial boundary value problem for the DGH equation (1.1) posed on the half-line $x > 0$ (with boundary values $u(0,t)$ and $u_x(0,t)$). These observations justify the necessity of the analysis of the DHG equation directly in the form (1.1).

The analysis of Camassa-Holm-type equations by using the inverse scattering approach was initiated in [14,15,19] for the Camassa-Holm equation (1.2) itself. A version of the inverse scattering method for the CH equation based on a Riemann-Hilbert (RH) factorization problem was proposed in [6,8] (another RH formulation of the inverse scattering transform is presented in [16]). The RH approach has proved its efficiency in the study of the long-time behavior of solutions of both initial value problems [4,5,7] and initial boundary value problems [9,10] for the CH equation.

In this paper we present the Riemann-Hilbert problem formalism for the inverse scattering approach to the initial value problem for the DGH equation:

$$u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3uu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}), \quad t > 0, -\infty < x < +\infty, \tag{1.6}$$

$$u(x,0) = u_0(x), \quad -\infty < x < +\infty. \tag{1.7}$$

We assume that $\omega + \frac{\gamma}{2\alpha^2} > 0$ and that $u_0(x)$ decays to 0 sufficiently fast:

$$u_0(x) \to 0, \quad x \to \pm\infty.$$

Moreover, $u_0(x)$ is assumed to satisfy the sign condition:

$$u_0(x) - \alpha^2 u_{0xx}(x) + \omega + \frac{\gamma}{2\alpha^2} > 0, \quad -\infty < x < +\infty.$$

The sign condition implies [20] the existence of a global solution $u(x,t)$ decaying to 0 for all $t > 0$:

$$u(x,t) \to 0, \quad x \to \pm\infty$$

and satisfying the positivity condition

$$u(x,t) - \alpha^2 u_{xx}(x,t) + \omega + \frac{\gamma}{2\alpha^2} > 0.$$

In Section 2 we present the appropriate Lax pairs associated with the DGH equation, whose dedicated solutions are used in Section 3 for formulating the matrix Riemann-Hilbert problem suitable for solving the Cauchy problem (1.6), (1.7). Then we give (Theorem 3.3) a representation of the solution $u(x,t)$ problem (1.6), (1.7) in terms of the solution of this RH problem evaluated at a distinguished point of the complex plane of the spectral parameter. In Section 4 we discuss the relationship between the matrix and vector formalism for the RH problem and show that the solution of the RH problem gives a solution to the nonlinear equation. The RH formalism is then used in Section 5 to present smooth as well as non-smooth (cusped) soliton solutions for (1.1).

2. LAX PAIRS AND EIGENFUNCTIONS

The Riemann-Hilbert formalism for integrable nonlinear equations is based on using appropriately defined eigenfunctions, i.e., solutions of the Lax pair equations, whose behavior as functions of the spectral parameter is well-controlled in the whole extended complex plane. For this purpose, it is convenient to work with matrix (2×2 in the case of the DGH equation) Lax pair equations, which are first order differential equations. Thus the first step in finding the appropriate form of the Lax pair is to introduce the vector

$$\Phi := \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}$$

and to rewrite the Lax pair (1.3) in the form

$$\Phi_x = U\Phi, \tag{2.1a}$$

$$\Phi_t = V\Phi, \tag{2.1b}$$

where $\Phi \equiv \Phi(x,t,\eta)$ and

$$U(x,t,\eta) = \begin{pmatrix} 0 & 1 \\ \frac{1}{4\alpha^2} + \eta(m+\Omega) & 0 \end{pmatrix}, \tag{2.1c}$$

$$V(x,t,\eta) = \begin{pmatrix} \frac{1}{2}u_x & \frac{1}{2\alpha^2}\left(\frac{1}{\eta}+2\gamma\right) - u \\ \left(\frac{1}{2\alpha^2}\left(\frac{1}{\eta}+2\gamma\right) - u\right)\left(\frac{1}{4\alpha^2}+\eta(m+\Omega)\right) + \frac{u-m}{2\alpha^2} & -\frac{1}{2}u_x \end{pmatrix}. \tag{2.1d}$$

Notice that the coefficient matrices U and V are traceless, which provides that the determinant of a matrix solution to (2.1) composed from two vector solutions is independent of x and t.

Also notice that U and V have singularities (in the extended complex η-plane) at $\eta = 0$ and at $\eta = \infty$. In order to control the behavior of solutions to (2.1) as functions of the spectral parameter η (which is crucial for the Riemann-Hilbert method), we follow a strategy similar to that adopted for the CH equation [6,8].

Namely, in order to control the large η behavior of solutions of (2.1), we will transform this Lax pair to the form (cf. [2,6,8]):

$$\hat\Phi_x + Q_x\hat\Phi = \hat U\hat\Phi, \tag{2.2a}$$

$$\hat\Phi_t + Q_t\hat\Phi = \hat V\hat\Phi, \tag{2.2b}$$

whose coefficients $Q(x,t,\eta)$, $\hat U(x,t,\eta)$, and $\hat V(x,t,\eta)$ have the following properties:

(i) Q is diagonal and is unbounded as $\eta\to\infty$;
(ii) $\hat U = \mathrm{O}(1)$ and $\hat V = \mathrm{O}(1)$ as $\eta\to\infty$;
(iii) the diagonal parts of $\hat U$ and $\hat V$ decay as $\eta\to\infty$;
(iv) $\hat U \to 0$ and $\hat V\to 0$ as $x\to\pm\infty$.

As in the case of the CH equation [6, 8], we perform this transformation in two steps:

(i) Transform (2.1) into a system where the leading (as $\eta \to \infty$) terms are represented as products of (x,t)-independent (matrix-valued) and (x,t)-dependent (scalar) factors.

(ii) Diagonalize the (x,t)-independent factors.

First, introducing the new spectral parameter k by

$$-k^2 = \eta + \frac{1}{4\alpha^2\Omega}$$

and the new vector function $\tilde{\Phi} \equiv \tilde{\Phi}(x,t,k)$ by

$$\tilde{\Phi} = D\Phi,$$

where

$$D(x,t) = \begin{pmatrix} (m+\Omega)^{\frac{1}{4}} & 0 \\ 0 & (m+\Omega)^{-\frac{1}{4}} \end{pmatrix},$$

transforms (2.1a) into

$$\tilde{\Phi}_x = \tilde{U}\tilde{\Phi}, \tag{2.3}$$

where

$$\tilde{U}(x,t;k) = \sqrt{(m+\Omega)}\begin{pmatrix} 0 & 1 \\ -k^2 & 0 \end{pmatrix} + \frac{m_x}{4(m+\Omega)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{m}{4\alpha^2\Omega\sqrt{(m+\Omega)}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

Second, introducing $\hat{\Phi} \equiv \hat{\Phi}(x,t,k)$ by $\hat{\Phi} = P\tilde{\Phi}$, where $P = \frac{1}{2}\begin{pmatrix} 1 & -\frac{1}{ik} \\ 1 & \frac{1}{ik} \end{pmatrix}$, diagonalizes the first term in (2.4) and transforms (2.3) into

$$\hat{\Phi}_x + ik\sqrt{m+\Omega}\sigma_3\hat{\Phi} = \hat{U}\hat{\Phi}, \tag{2.5}$$

where

$$\hat{U}(x,t,k) = \frac{m_x}{4(m+\Omega)}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{m}{8ik\alpha^2\Omega\sqrt{m+\Omega}}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \tag{2.6}$$

and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Accordingly, the t-equation (2.1b) of the Lax pair is transformed into

$$\hat{\Phi}_t + ik\left\{\frac{\sqrt{\Omega}}{2\alpha^2\eta(k)} - \left(u - \frac{\gamma}{\alpha^2}\right)\sqrt{m+\Omega}\right\}\sigma_3\hat{\Phi} = \hat{V}\hat{\Phi}, \tag{2.7}$$

where

$$\begin{aligned}\hat{V}(x,t,k) = &-\frac{m_x\left(u-\frac{\gamma}{\alpha^2}\right)}{4(m+\Omega)}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\sqrt{\Omega}-\sqrt{m+\Omega}}{2\alpha^2}\frac{ik}{\eta(k)}\sigma_3 \\ &+ \left\{\left(u - \frac{\gamma}{\alpha^2} - \frac{1}{2\alpha^2\eta(k)}\right)\frac{m}{8ik\alpha^2\Omega\sqrt{m+\Omega}} + \frac{u-m}{4ik\alpha^2\sqrt{m+\Omega}}\right\}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.\end{aligned} \tag{2.8}$$

Notice that the terms in $\{\dots\}$ in the l.h.s. of (2.7) were chosen in such a way that (2.7) be consistent with the desired form (2.2) of the Lax pair. Indeed, if one introduces

$$p(x,t,k) := \sqrt{\Omega}x - \int_x^\infty (\sqrt{m+\Omega} - \sqrt{\Omega})d\xi + \frac{1}{2\alpha^2}\left(\frac{1}{\eta(k)} + 2\gamma\right)\sqrt{\Omega}t \tag{2.9}$$

and

$$Q(x,t,k) := ikp(x,t,k)\sigma_3, \tag{2.10}$$

then it is clear that $p_x = \sqrt{m+\Omega}$. On the other hand, the fact that

$$p_t = \frac{\sqrt{\Omega}}{2\alpha^2\eta} - \left(u - \frac{\gamma}{\alpha^2}\right)\sqrt{m+\Omega}$$

follows from the "conservation law" form of the DHG equation (1.1):

$$(\sqrt{m+\Omega})_t = -\left(\left(u - \frac{\gamma}{\alpha^2}\right)\sqrt{m+\Omega}\right)_x. \tag{2.11}$$

The latter equation has also been used for obtaining $\hat{V}$ as it appears in (2.8), namely, for replacing the time derivative $\left(\sqrt{m+\Omega}\right)_t$ by the r.h.s. of (2.11).

Summarizing, equations (2.5) and (2.7) constitute the Lax pair of type (2.2) with $Q(x,t,k) = ikp(x,t,k)$, where p is given by (2.9).

In what follows we will determine solutions of (2.2) having well-controlled behavior as functions of the spectral parameter k for large values of k. For this purpose, introduce

$$\tilde{\tilde{\Phi}} = \hat{\Phi}e^Q \tag{2.12}$$

and think about $\tilde{\tilde{\Phi}}$ as a 2×2 matrix. Then (2.2) can be rewritten as

$$\begin{cases} \tilde{\tilde{\Phi}}_x + [Q_x, \tilde{\tilde{\Phi}}] = \hat{U}\tilde{\tilde{\Phi}}, \\ \tilde{\tilde{\Phi}}_t + [Q_t, \tilde{\tilde{\Phi}}] = \hat{V}\tilde{\tilde{\Phi}}, \end{cases} \tag{2.13}$$

where $[\cdot,\cdot]$ stands for the matrix commutator. Now determine the particular (Jost) solutions $\tilde{\tilde{\Phi}}_\pm(x,t)$ of (2.13) as the solutions of the associated Volterra integral equations:

$$\tilde{\tilde{\Phi}}_\pm(x,t,k) = I + \int_{\pm\infty}^x e^{Q(y,t,k)-Q(x,t,k)}\hat{U}(y,t,k)\tilde{\tilde{\Phi}}_\pm(y,t,k)e^{Q(x,t,k)-Q(y,t,k)}dy, \tag{2.14}$$

or, taking into account the definition of Q,

$$\tilde{\tilde{\Phi}}_+(x,t,k) = I - \int_x^\infty e^{ik\int_x^y \sqrt{m+\Omega}d\xi\sigma_3}\hat{U}(y,t,k)\tilde{\tilde{\Phi}}_+(y,t,k)e^{-ik\int_x^y \sqrt{m+\Omega}d\xi\sigma_3}dy, \tag{2.15}$$

$$\tilde{\tilde{\Phi}}_-(x,t,k) = I + \int_{-\infty}^x e^{-ik\int_y^x \sqrt{m+\Omega}d\xi\sigma_3}\hat{U}(y,t,k)\tilde{\tilde{\Phi}}_-(y,t,k)e^{ik\int_y^x \sqrt{m+\Omega}d\xi\sigma_3}dy \tag{2.16}$$

(I is the identity matrix).

Since equations (2.15) have the same form as in the case of the Camassa-Holm equation, see [6,8], the analytic properties of $\tilde{\tilde{\Phi}}_\pm(x,t)$ are also the same; they follow from the analysis of the Neumann series for the solutions of (2.15) and the symmetries

$$\overline{\tilde{\tilde{\Phi}}_\pm(\,\cdot\,,\,\cdot\,,\bar{k})} = \tilde{\tilde{\Phi}}_\pm(\,\cdot\,,\,\cdot\,,-k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{\tilde{\Phi}}_\pm(\,\cdot\,,\,\cdot\,,k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.17}$$

which are due to the symmetries of the coefficient matrix

$$\overline{\hat{U}(\,\cdot\,,\,\cdot\,,\bar{k})} = \hat{U}(\,\cdot\,,\,\cdot\,,-k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{U}(\,\cdot\,,\,\cdot\,,k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.18}$$

(overline means complex conjugation), and we just list them below. We denote $\mu^{(1)}$ and $\mu^{(2)}$ the columns of a 2×2 matrix $\mu = \begin{pmatrix} \mu^{(1)} & \mu^{(2)} \end{pmatrix}$. Then for all (x,t):

— $\det \tilde{\tilde{\Phi}}_\pm \equiv 1$;
— $\tilde{\tilde{\Phi}}_-^{(1)}$ and $\tilde{\tilde{\Phi}}_+^{(2)}$ are analytic in $\{k|\operatorname{Im} k > 0$ and continuous in $\{k|\operatorname{Im} k \geq 0, k \neq 0\}$;
— $\tilde{\tilde{\Phi}}_+^{(1)}$ and $\tilde{\tilde{\Phi}}_-^{(2)}$ are analytic in $\{k|\operatorname{Im} k < 0$ and continuous in $\{k|\operatorname{Im} k \leq 0, k \neq 0\}$;
— as $k \to \infty$ in $\{k|\operatorname{Im} k \geq 0\}$, $\begin{pmatrix} \tilde{\tilde{\Phi}}_-^{(1)} & \tilde{\tilde{\Phi}}_+^{(2)} \end{pmatrix} \to I$;
— as $k \to \infty$ in $\{k|\operatorname{Im} k \leq 0\}$, $\begin{pmatrix} \tilde{\tilde{\Phi}}_+^{(1)} & \tilde{\tilde{\Phi}}_-^{(2)} \end{pmatrix} \to I$;
— as $k \to 0$,

$$\tilde{\tilde{\Phi}}_\pm = \frac{\alpha_\pm(x,t)}{ik} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} \nu_1^\pm & \nu_2^\pm \\ \nu_2^\pm & \nu_1^\pm \end{pmatrix} + O(k) \quad \text{with } \alpha_\pm \in \mathbb{R} \text{ and } \nu_j^\pm \in \mathbb{R} \tag{2.19}$$

(notice that matrix $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$ involved in $\hat{U}$ is nilpotent: $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = 0$).

Again as in the case of the CH equation, one introduces the scattering matrix $s(k)$ (independent of (x,t)) by

$$\tilde{\tilde{\Phi}}_+(x,t,k) = \tilde{\tilde{\Phi}}_-(x,t,k) e^{-ikp(x,t,k)\sigma_3} s(k) e^{ikp(x,t,k)\sigma_3}, \quad k \in \mathbb{R},\ k \neq 0, \tag{2.20}$$

which, due to the symmetries (2.18), can be written in terms of two scalar spectral functions, $a(k)$ and $b(k)$:

$$s(k) = \begin{pmatrix} \overline{a(k)} & b(k) \\ \overline{b(k)} & a(k) \end{pmatrix}, \quad k \in \mathbb{R}, \tag{2.21}$$

such that $\overline{a(k)} = a(-k)$ and $\overline{b(k)} = b(-k)$. The spectral functions have the following properties [8]:

— $a(k)$ and $b(k)$ are determined by $u(x,0)$ through the solutions $\tilde{\tilde{\Phi}}_\pm(x,0)$ of equations (2.15);
— $a(k)$ is analytic in $\{k|\operatorname{Im} k > 0\}$ and continuous in $\{k|\operatorname{Im} k \geq 0, k \neq 0\}$; moreover, $a(k) \to 1$ as $k \to \infty$;

— $b(k)$ is continuous for $k \in \mathbb{R}, k \neq 0$ and $b(k) \to 0$ as $|k| \to \infty$;
— as $k \to 0$, $a(k) = \frac{\alpha_0}{ik} + O(1)$ and $b(k) = -\frac{\alpha_0}{ik} + O(1)$ with $\alpha_0 \in \mathbb{R}$;
— $|a(k)|^2 - |b(k)|^2 = 1$ for $k \in \mathbb{R}, k \neq 0$;
— let $\{k_j\}_1^N$ be the set of zeros of $a(k)$: $a(k_j) = 0$. Then $N < \infty$ and the zeros are simple with $\frac{\mathrm{d}a}{\mathrm{d}k}(k_j) \in i\mathbb{R}$; moreover, $k_j = i\nu_j$ with $0 < \nu_j < \frac{1}{2\alpha\sqrt{\Omega}}$ for all $1 \leq j \leq N$; and the eigenvectors are related by

$$\tilde{\tilde{\Phi}}_-^{(1)}(x,t,i\nu_j) = \varkappa_j e^{-2\nu_j p(x,t,i\nu_j)} \tilde{\tilde{\Phi}}_+^{(2)}(x,t,i\nu_j) \tag{2.22}$$

with $\varkappa_j \in \mathbb{R}$.

Notice that the case $\alpha_0 \neq 0$ is generic. On the other hand, in the non-generic case $\alpha_0 = 0$, i.e., when $\lim_{k\to 0} a(k) = a_0$ and $\lim_{k\to 0} b(k) = b_0$ are finite (then $a_0 \in \mathbb{R}$, $b_0 \in \mathbb{R}$, and $a_0^2 = 1 + b_0^2$), (2.20) implies that $\alpha_+(x,t)$ and $\alpha_-(x,t)$ in (2.19) are related by

$$\alpha_-(x,t) = (a_0 - b_0)\alpha_+(x,t). \tag{2.23}$$

3. THE RIEMANN-HILBERT PROBLEM

The analytic properties of $\tilde{\tilde{\Phi}}_\pm$ stated above allow rewriting the scattering relation (2.20) as a jump relation for a piece-wise meromorphic (w.r.t. k), 2×2-valued function (depending on x and t as parameters). Indeed, define $M(x,t,k)$ by

$$M(x,t,k) = \begin{cases} \begin{pmatrix} \frac{\tilde{\tilde{\Phi}}_-^{(1)}(x,t,k)}{a(k)} & \tilde{\tilde{\Phi}}_+^{(2)}(x,t,k) \end{pmatrix}, & \operatorname{Im} k > 0, \\ \begin{pmatrix} \tilde{\tilde{\Phi}}_+^{(1)}(x,t,k) & \frac{\tilde{\tilde{\Phi}}_-^{(2)}(x,t,k)}{\overline{a(\bar{k})}} \end{pmatrix}, & \operatorname{Im} k < 0. \end{cases} \tag{3.1}$$

Define $r(k) := b(k)/\overline{a(k)}$ for $k \in \mathbb{R}$. Then the limiting values $M_\pm(x,t,k)$, $k \in \mathbb{R}$ of M as k is approached from the domains $\pm \operatorname{Im} k > 0$ are related as follows:

$$M_-(x,t,k) = M_+(x,t,k) e^{-ikp(x,t,k)\sigma_3} J_0(k) e^{ikp(x,t,k)\sigma_3}, \quad k \in \mathbb{R}, \tag{3.2}$$

where

$$J_0(k) = \begin{pmatrix} 1 & -r(k) \\ \overline{r(k)} & 1 - |r(k)|^2 \end{pmatrix}. \tag{3.3}$$

Taking into account the properties of $\tilde{\tilde{\Phi}}_\pm$ and $s(k)$, as well as (2.23), $M(x,t,k)$ satisfies the following properties:
— $\det M \equiv 1$;
— $M \to I$ as $k \to \infty$;
— $M = \frac{\alpha_+(x,t)}{ik} \begin{pmatrix} -c & -1 \\ c & 1 \end{pmatrix} + O(1)$ as $k \to 0$ in $\operatorname{Im} k \geq 0$, where

$$c = \begin{cases} 0, & \text{if } \lim_{k\to 0} ka(k) \neq 0, \\ 1 - \frac{b_0}{a_0} \equiv 1 - r(0), & \text{if } a(k) = a_0 + O(k) \text{ as } k \to 0 \text{ (then } a_0 \neq 0) \end{cases}$$

(thus $c = 1 - r(0)$ in all cases);

—

$$\overline{M(\cdot,\cdot,\bar{k})} = M(\cdot,\cdot,-k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M(\cdot,\cdot,k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.4}$$

— $M^{(1)}$ has poles at the zeros $k_j = i\nu_j$ of $a(k)$ (in $\{k|\operatorname{Im} k > 0\}$) and $M^{(2)}$ has poles at at $\overline{k_j} = -i\nu_j$ (in $\{k|\operatorname{Im} k < 0\}$), $j = 1, 2, \dots, N$, where the following residue conditions are satisfied:

$$\begin{aligned} \operatorname{Res}_{k=i\nu_j} M^{(1)}(x,t,k) &= i\gamma_j e^{-2\nu_j p(x,t,i\nu_j)} M^{(2)}(x,t,i\nu_j), \\ \operatorname{Res}_{k=-i\nu_j} M^{(2)}(x,t,k) &= -i\gamma_j e^{-2\nu_j p(x,t,i\nu_j)} M^{(1)}(x,t,-i\nu_j) \end{aligned} \tag{3.5}$$

with $\gamma_j = -i\frac{\varkappa_j}{(\mathrm{d}a/\mathrm{d}k)(i\nu_j)} \in \mathbb{R}$.

The idea of the Riemann-Hilbert problem approach in the inverse scattering method consists in considering the jump relation (3.2) complemented by the normalization condition $M \to I$ as $k \to \infty$, by the residue conditions and by the structural condition at $k = 0$ as the factorization problem of finding $M(x,t,k)$ (and, consequently, $u(x,t)$) from the jump matrix in (3.2) and the other "defects of analyticity" of M mentioned above. As in the case of the CH equation, when realizing this idea, one faces two problems: (i) the determination of the jump matrix, which is $e^{-ikp}J_0(k)e^{ikp}$, involves not only the objects uniquely determined by the initial data $u(x,0)$ (the functions $a(k)$ and $b(k)$ involved in $J_0(k)$ and the constants involved in the residue conditions), but it also involves $p = p(x,t,k)$, which is obviously not determined by $u(x,0)$ (it involves $m(x,t)$ for $t \geq 0$); (ii) even if p in (3.2) were prescribed, the solution of the factorization problem described above would not be unique, since the condition at $k = 0$ is structural only: c is determined by the initial data and thus is prescribed in the framework of the Cauchy problem, but $\alpha_+(x,t)$ is not.

Similarly to the CH equation, item (i) can be resolved by introducing a new spatial variable dictated by the form of $p(x,t)$ (2.9):

$$y(x,t) = \sqrt{\Omega}x - \int_x^\infty (\sqrt{m(\xi,t)+\Omega} - \sqrt{\Omega})d\xi. \tag{3.6}$$

Then, in terms of the parameters y and t,

$$\hat{p}(y,t,k) = p(x(y,t),t,k) = y - \left(\frac{2\Omega^{\frac{3}{2}}}{1+4\alpha^2\Omega k^2} - \frac{\gamma\Omega^{\frac{1}{2}}}{\alpha^2}\right)t, \tag{3.7}$$

and the jump matrix becomes explicit in variables y and t:

$$J(y,t,k) = e^{-ik\hat{p}(y,t,k)}J_0(k)e^{ik\hat{p}(y,t,k)}. \tag{3.8}$$

Accordingly, the residue conditions (3.5) become also explicit in this scale. Namely, introducing $\hat{M}(y,t;k) := M(x(y,t),t;k)$, the residue conditions have the form

$$\begin{aligned} \operatorname{Res}_{k=i\nu_j} \hat{M}^{(1)}(y,t,k) &= i\gamma_j e^{-2\nu_j \hat{p}(y,t,i\nu_j)} \hat{M}^{(2)}(y,t,i\nu_j), \\ \operatorname{Res}_{k=-i\nu_j} \hat{M}^{(2)}(y,t,k) &= -i\gamma_j e^{-2\nu_j \hat{p}(y,t,i\nu_j)} \hat{M}^{(1)}(y,t,-i\nu_j) \end{aligned} \tag{3.9}$$

whereas the jump conditions become

$$\hat{M}_-(y,t,k) = \hat{M}_+(y,t,k)J(y,t,k), \quad k \in \mathbb{R}. \tag{3.10}$$

Recall that the jump and residue conditions for $\hat{M}(y,t;k)$ were obtained above assuming that there exists a solution $u(x,t)$ of the DGH equation decaying to 0 as $|x| \to \infty$ for any fixed $t > 0$. Now our goal is to show that $u(x,t)$ can be recovered in terms of $\hat{M}(y,t;k)$, which is considered as a unique solution of a factorization problem of the Riemann-Hilbert (RH) type, whose data are uniquely determined by $u(x,0)$. According to above, given $u(x,0)$, $-\infty < x < \infty$, we can determine the spectral data $\{r(k), k \in \mathbb{R}; \{\nu_j, \gamma_j\}_1^N\}$ via the solution of the integral equations (2.15) considered for $t = 0$. Then, the factorization problem is as follows: given $\{r(k), k \in \mathbb{R}; \{\nu_j, \gamma_j\}_1^N\}$, find a piece-wise meromorphic function $\hat{M}(y,t;k)$ satisfying (i) the jump conditions (3.10); (ii) the residue conditions (3.9); (iii) the normalization condition $\hat{M}(y,t,k) \to I$ as $k \to \infty$; (iv) the structural condition

$$\hat{M}(y,t,k) = \frac{\hat{\alpha}_+(y,t)}{ik}\begin{pmatrix} -c & -1 \\ c & 1 \end{pmatrix} + O(1), \quad k \to 0, \ \operatorname{Im} k \geq 0, \tag{3.11}$$

where $\alpha_+(y,t)$ is not specified.

As we have already mentioned above, since α_+ in condition (iv) is not specified, the solution of this RH problem is not unique. In order to have uniqueness, we add another structural condition, which, at the same time, will give simple means to recover $u(x,t)$ from $\hat{M}(y,t;k)$ evaluated at a particular value of k. Similarly to the case of the CH equation, this condition comes from the fact that for $\eta(k) = 0$, equation (2.1a) becomes independent of $m(x,t)$.

In order to have a good control of solutions of the Lax pair equations at $k = \pm\frac{i}{2\alpha\sqrt{\Omega}}$ (which corresponds to $\eta = 0$), we introduce another transformation of the original Lax pair (2.1). Introduce $\widetilde{\Phi}_0 = P_0\Phi$, where $P_0 = \frac{1}{2}\begin{pmatrix} 1 & -\frac{1}{ik\sqrt{\Omega}} \\ 1 & \frac{1}{ik\sqrt{\Omega}} \end{pmatrix}$. Then (2.1) reduces to

$$\begin{aligned} &\widetilde{\Phi}_{0x} + ik\sqrt{\Omega}\sigma_3\widetilde{\Phi}_0 = U_0\widetilde{\Phi}_0, \\ &\widetilde{\Phi}_{0t} + \frac{ik\sqrt{\Omega}}{2\alpha^2}\left(\frac{1}{\eta(k)} + 2\gamma\right)\sigma_3\widetilde{\Phi}^0 = V_0\widetilde{\Phi}_0, \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} U_0 &= \frac{\eta(k)m}{2ik\sqrt{\Omega}}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \\ V_0 &= \frac{u_x}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + uik\sqrt{\Omega}\sigma_3 + \frac{1}{2}\left(\frac{1}{2\alpha^2}\left(\frac{1}{\eta} + 2\gamma\right) - u\right) \times \\ &\quad \times \left(ik\sqrt{\Omega} - \frac{1}{4ik\alpha^2\sqrt{\Omega}} - \frac{\eta(m+\Omega)}{ik\sqrt{\Omega}}\right)\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

(notice that $U_0 \to 0$ and $V_0 \to 0$ as $|x| \to \infty$). Consequently, introducing $\widetilde{\widetilde{\Phi}}_0$ by

$$\widetilde{\widetilde{\Phi}}_0 = \widetilde{\Phi}_0 e^{\left(ik\sqrt{\Omega}x + \frac{ik\sqrt{\Omega}}{2\alpha^2}\left(\frac{1}{\eta} + 2\gamma\right)t\right)\sigma_3} \tag{3.13}$$

reduces (3.12) to the Lax pair in the commutator form:

$$\begin{gathered}\tilde{\tilde{\Phi}}_{0x} + ik\sqrt{\Omega}[\sigma_3, \tilde{\tilde{\Phi}}_0] = U_0\tilde{\tilde{\Phi}}_0, \\ \tilde{\tilde{\Phi}}_{0t} + \frac{ik\sqrt{\Omega}}{2\alpha^2}\left(\frac{1}{\eta} + 2\gamma\right)[\sigma_3, \tilde{\tilde{\Phi}}_0] = V_0\tilde{\tilde{\Phi}}_0.\end{gathered} \tag{3.14}$$

The Jost solutions $\tilde{\tilde{\Phi}}_{0\pm}$ to (3.14) are determined as the solutions of the integral equations

$$\tilde{\tilde{\Phi}}_{0\pm}(x,t,k) = I + \int\limits_{\pm\infty}^{x} e^{-ik\sqrt{\Omega}(x-y)\sigma_3} U_0 \tilde{\tilde{\Phi}}_{0\pm}(x,t,k) e^{ik\sqrt{\Omega}(x-y)\sigma_3} dy. \tag{3.15}$$

Now, since $U_0 \equiv 0$ at $k = \pm\frac{i}{2\alpha\sqrt{\Omega}}$ (at $\eta = 0$), we have an important property:

$$\tilde{\tilde{\Phi}}_{0\pm}\left(x, t, \pm\frac{i}{2\alpha\sqrt{\Omega}}\right) \equiv I \tag{3.16}$$

for all x and t.

Further, we notice that $\tilde{\tilde{\Phi}}_{\pm}$ and $\tilde{\tilde{\Phi}}_{0\pm}$, being related to the same system of equations (2.1), are related as

$$\begin{aligned}&\tilde{\tilde{\Phi}}_{\pm}(x,t,k) \\ &= P(k)D(x,t)(P_0)^{-1}(k)\tilde{\tilde{\Phi}}_{0\pm}(x,t,k)e^{-\left(ik\sqrt{\Omega}x + \frac{ik\sqrt{\Omega}}{2\alpha^2}\left(\frac{1}{\eta}+2\gamma\right)t\right)\sigma_3} C_{\pm}(k)e^{Q(x,t,k)},\end{aligned} \tag{3.17}$$

where $C_{\pm}(k)$ are some matrices independent of x and t. Passing to the limits $x \to \pm\infty$ determines $C_{\pm}(k)$:

$$C_+(k) = \frac{1}{\sqrt[4]{\Omega}} I, \quad C_-(k) = \frac{1}{\sqrt[4]{\Omega}} e^{ik\int\limits_{-\infty}^{\infty}(\sqrt{m+\Omega}-\sqrt{\Omega})d\xi\sigma_3}.$$

Calculate $F := P(k)D(x,t)(P_0)^{-1}(k)$:

$$F = \frac{1}{2}\begin{pmatrix} q + \sqrt{\Omega}q^{-1} & q - \sqrt{\Omega}q^{-1} \\ q - \sqrt{\Omega}q^{-1} & q + \sqrt{\Omega}q^{-1}\end{pmatrix}, \tag{3.18}$$

where $q := (m+\Omega)^{\frac{1}{4}}$. Evaluating (3.17) at $k = \frac{i}{2\alpha\sqrt{\Omega}}$ and using (3.16) we have

$$\begin{aligned}\tilde{\tilde{\Phi}}_{-}^{(1)}\left(x,t,\frac{i}{2\alpha\sqrt{\Omega}}\right) &= \frac{1}{\sqrt[4]{\Omega}} F^{(1)} e^{-\frac{1}{2\alpha\sqrt{\Omega}}\int\limits_{-\infty}^{x}(\sqrt{m+\Omega}-\sqrt{\Omega})d\xi}, \\ \tilde{\tilde{\Phi}}_{+}^{(2)}\left(x,t,\frac{i}{2\alpha\sqrt{\Omega}}\right) &= \frac{1}{\sqrt[4]{\Omega}} F^{(2)} e^{\frac{1}{2\alpha\sqrt{\Omega}}\int\limits_{x}^{\infty}(\sqrt{m+\Omega}-\sqrt{\Omega})d\xi}.\end{aligned} \tag{3.19}$$

Then, calculating

$$a\left(\frac{i}{2\alpha\sqrt{\Omega}}\right) = \det\left(\tilde{\tilde{\Phi}}_-^{(1)}\left(x,t,\tfrac{i}{2\alpha\sqrt{\Omega}}\right) \quad \tilde{\tilde{\Phi}}_+^{(2)}\left(x,t,\tfrac{i}{2\alpha\sqrt{\Omega}}\right)\right) = e^{-\frac{1}{2\alpha\sqrt{\Omega}}\int_{-\infty}^{\infty}(\sqrt{m+\Omega}-\sqrt{\Omega})d\xi} \tag{3.20}$$

and substituting this and (3.19) into (3.1) evaluated at $k = \frac{i}{2\alpha\sqrt{\Omega}}$, we have

$$M\left(x,t,\frac{i}{2\alpha\sqrt{\Omega}}\right) = \frac{1}{\sqrt[4]{\Omega}}F(x,t)\begin{pmatrix} e^{\frac{1}{2\alpha\sqrt{\Omega}}\int_x^{\infty}(\sqrt{m+\Omega}-\sqrt{\Omega})d\xi} & 0 \\ 0 & e^{-\frac{1}{2\alpha\sqrt{\Omega}}\int_x^{\infty}(\sqrt{m+\Omega}-\sqrt{\Omega})d\xi}\end{pmatrix}. \tag{3.21}$$

Relation (3.21) is important from two points of view: first, it provides a structural condition for the Riemann-Hilbert problem needed to guarantee the uniqueness of its solution; second, it gives means to express, parametrically, the solution $u(x,t)$ of the Cauchy problem (1.6), (1.7) in terms of the solution of the RH problem.

Indeed, in view of (3.18), (3.21) suggests the structural condition for $\hat{M}(y,t,\frac{i}{2\alpha\sqrt{\Omega}})$ in the form:

$$\hat{M}\left(y,t,\frac{i}{2\alpha\sqrt{\Omega}}\right) = \frac{1}{2}\begin{pmatrix}\tilde{q}+\tilde{q}^{-1} & \tilde{q}-\tilde{q}^{-1} \\ \tilde{q}-\tilde{q}^{-1} & \tilde{q}+\tilde{q}^{-1}\end{pmatrix}\begin{pmatrix} f & 0 \\ 0 & f^{-1}\end{pmatrix}, \tag{3.22}$$

where $\tilde{q}(y,t) > 0$ and $f(y,t) > 0$ are not specified (in the framework of the RH problem for $\hat{M}$).

Proposition 3.1. *Consider the RH problem: find a piece-wise meromorphic,* 2×2 *function* $\hat{M}(y,t,k)$ *satisfying: the jump condition* (3.10) (*where the jump* $J(y,t,k)$ *as in* (3.8) *with* (3.7) *and* (3.3) *with given* $r(k)$), *the residue conditions* (3.9) (*with given* $\{\nu_j,\gamma_j\}$), *the normalization condition* $\hat{M}\to I$ *as* $k\to\infty$, *the symmetry condition* (3.4)*, the structural condition at* $k=0$ (3.11) (*where* $\hat{\alpha}(y,t)$ *is not specified*), *and the structural condition at* $k=\frac{i}{2\alpha\sqrt{\Omega}}$ (3.22) (*where* $\tilde{q}(y,t)>0$ *and* $f(y,t)>0$ *are not specified*). *Then the solution of this RH problem, if exists, is unique.*

Proof. First, notice that if $\hat{M}$ is a solution of the RH problem formulated in Proposition 3.1, then $\det\hat{M}\equiv 1$. Indeed, the conditions on $\hat{M}$ imply that $\det\hat{M}$ has no jump across $\mathbb{R}$, has no singularities at $\{i\nu_j\}$, and approaches 1 as $k\to\infty$; then, (3.11) and the Liouville theorem imply that $\det\hat{M} = 1+\frac{\beta}{ik}$ with some β; finally, the symmetry (3.4) yields $\beta\equiv 0$ and thus $\det\hat{M}\equiv 1$.

Moreover, from $\det\hat{M}\equiv 1$ it follows that the matrix $n=\{n_{ij}\}_{i,j=1,2}$ in

$$\hat{M}(y,t,k) = \frac{\hat{\alpha}_+(y,t)}{ik}\begin{pmatrix}-c & -1 \\ c & 1\end{pmatrix} + \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22}\end{pmatrix} + O(k), \quad k\to 0,\ \operatorname{Im} k\geq 0, \tag{3.23}$$

satisfies the condition

$$n_{11}+n_{21} = c(n_{12}+n_{22}). \tag{3.24}$$

Now assume that there are two solutions to the RH problem, $\hat{M}_1$ and $\hat{M}_2$. Define $N := \hat{M}_1(\hat{M}_2)^{-1}$. Then the conditions on $\hat{M}$ imply that N has no jump and can have the only singularity at $k = 0$, which, due to (3.24), has the form

$$N = \frac{\phi}{ik}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + O(1) \tag{3.25}$$

with some scalar function $\phi(y, t)$. Then, by the Liouville theorem,

$$N = I + \frac{\phi}{ik}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

and thus

$$\hat{M}_1(y, t, k) = \left(I + \frac{\phi(y, t)}{ik}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\right)\hat{M}_2(y, t, k).$$

Now the observation that the multiplication from the left by N preserves the structure (3.22) only if $\phi \equiv 0$ completes the proof.

Remark 3.2. In the formulation of the RH problem, the global symmetry condition (3.4) and the structural condition (3.11) can be replaced by the local symmetry condition (at $k = 0$):

$$\hat{M}(k) = \frac{\hat{\alpha}_+(y, t)}{ik}\begin{pmatrix} -c & -1 \\ c & 1 \end{pmatrix} + \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} + O(k), \quad k \to 0, \ \operatorname{Im} k > 0,$$

$$\hat{M}(k) = \frac{\hat{\alpha}_+(y, t)}{ik}\begin{pmatrix} -1 & -c \\ 1 & c \end{pmatrix} + \begin{pmatrix} n_{22} & n_{21} \\ n_{12} & n_{11} \end{pmatrix} + O(k), \quad k \to 0, \ \operatorname{Im} k < 0.$$

Then the global symmetry condition will follow taking into account the symmetry of type (2.18) of the jump matrix J.

Assuming the existence of the solution $u(x, t)$ of the Cauchy problem (1.6), (1.7), the existence of a solution to the RH problem in Proposition 3.1 follows by construction. On the other hand, assuming that the solution of the RH $\hat{M}(y, t, k)$ is found and evaluated at $k = \frac{i}{2\alpha\sqrt{\Omega}}$, relations (3.21) and (3.22) imply that the solution $u(x, t)$ of the Cauchy problem (1.6), (1.7) can be expressed, in a parametric form, in terms of $\hat{M}$.

Theorem 3.3. *Let $u_0(x)$ satisfy assumptions made for the Cauchy problem* (1.6), (1.7) *for the DGH equation. Let $\{r(k), k \in \mathbb{R}; c; \{\nu_j, \gamma_j\}_1^N\}$ be the spectral data determined by $u_0(x)$, and let $\hat{M}(y, t; k)$ be the solution of the associated RH problem form Proposition 3.1. Then, by evaluating $\hat{M}$ at $k = \frac{i}{2\alpha\sqrt{\Omega}}$ we get the parametric representation for the solution $u(x, t)$ of the Cauchy problem* (1.6), (1.7)*:*

$$u(x, t) = \hat{u}(y(x, t), t)$$

with

$$x(y, t) = 2\alpha \log f(y, t) + \frac{y}{\sqrt{\Omega}}, \tag{3.26}$$

$$\hat{u}(y, t) = 2\alpha\frac{\partial}{\partial t}\log f(y, t) - \frac{2\gamma\sqrt{\Omega}}{\alpha}\frac{\partial}{\partial y}\log f(y, t), \tag{3.27}$$

where f *comes from* (3.22)*:*

$$f^2(y,t) := \frac{\hat{\mu}_1(y,t,\frac{i}{2\alpha\sqrt{\Omega}})}{\hat{\mu}_2(y,t,\frac{i}{2\alpha\sqrt{\Omega}})}, \tag{3.28}$$

$$\hat{\mu}_1(y,t) := \hat{M}_{11}(y,t,\frac{i}{2\alpha\sqrt{\Omega}}) + \hat{M}_{21}(y,t,\frac{i}{2\alpha\sqrt{\Omega}}), \tag{3.29}$$

$$\hat{\mu}_2(y,t) := \hat{M}_{12}(y,t,\frac{i}{2\alpha\sqrt{\Omega}}) + \hat{M}_{22}(y,t,\frac{i}{2\alpha\sqrt{\Omega}}). \tag{3.30}$$

Indeed, comparing (3.21) and (3.22) it follows that

$$f(y(x,t),t) = e^{\frac{1}{2\alpha\sqrt{\Omega}}\int\limits_x^\infty (\sqrt{m+\Omega}-\sqrt{\Omega})d\xi},$$

which, taking into account (3.6), implies (3.26). Then (3.27) follows from (2.11).

Remark 3.4. Eigenfunctions associated with the Lax pair equations (2.13) and (3.14) via integral Fredholm equations of type

$$\Psi(x,t,k) = I + \int\limits_{(x^*,t^*)}^{(x,t)} e^{Q(\xi,t;z)-Q(x,\tau,k)} \left(\hat{U}\Psi(\xi,\tau,k)\mathrm{d}\xi + \hat{V}\Psi(\xi,\tau,k)\mathrm{d}\tau\right) e^{Q(x,t,k)-Q(\xi,\tau,k)} \tag{3.31}$$

with an appropriate choice of (x^*,t^*) (as $(0,0)$, $(0,\infty)$, and $(\infty,0)$) allows formulating a RH problem suitable for analyzing initial boundary value (or half-line) problems following the procedure presented in [9] in the case of the CH equation.

4. MATRIX VERSUS VECTOR RH PROBLEM. DEDUCING NONLINEAR EQUATIONS FROM THE MATRIX RH PROBLEM

In the inverse scattering formalism for the Camassa-Holm equation, similarly to the Korteweg-de Vries equation, vector RH problems are more habitual. In the case of the DGH equation, the row vector RH problem is as follows: given $\{r(k), k\in\mathbb{R}; \{\nu_j,\gamma_j\}_1^N\}$ find a 2-vector function $\hat{\mu}(y,t,k) = (\hat{\mu}_1\ \hat{\mu}_2)$ satisfying the following conditions:
— jump condition (3.10) in the form

$$\hat{\mu}_-(y,t,k) = \hat{\mu}_+(y,t,k)J(y,t,k), \quad k\in\mathbb{R}, \tag{4.1}$$

— residue conditions (3.9),
— normalization condition $\hat{\mu} \to (1\ 1)$ as $k\to\infty$,
— symmetry condition

$$\hat{\mu}(-k) = \hat{\mu}(k)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{4.2}$$

Since the RH problem above has the same structure as in the case of the KdV equation, its unique solvability follows from the "vanishing lemma" associated with the one-dimensional Schrödinger operator (see, e.g., [3]) stating that the associated homogeneous RH problem (with the normalization condition $\hat{\mu} \to (0\ 0)$ as $k \to \infty$) has the trivial solution only.

Provided the solution $\hat{\mu}(y,t,k)$ to the vector RH problem satisfies the condition

$$\hat{\mu}_j\left(y,t,\frac{i}{2\alpha\sqrt{\Omega}}\right) > 0, \quad j = 1,2, \tag{4.3}$$

the solution of the DGH equation can be represented by (3.26), (3.27), where f is given by (3.28).

Notice that given the solution $\hat{M}$ of the matrix RH problem, the solution of the vector RH problem is obviously $\hat{\mu} = (1\ 1)\hat{M}$. Conversely, the following proposition holds true.

Proposition 4.1. *Given the solution $\hat{\mu}$ of the vector RH problem satisfying* (4.3), *the solution $\hat{M}$ of the matrix RH problem formulated in Proposition 3.1 can be constructed as follows.*

— *Evaluate $\hat{\mu}$ at $k = \frac{i}{2\alpha\sqrt{\Omega}}$ and determine $\tilde{q}(y,t)$ and $f(y,t)$ by*

$$f(y,t) = \left(\frac{\hat{\mu}_1\left(y,t,\frac{i}{2\alpha\sqrt{\Omega}}\right)}{\hat{\mu}_2\left(y,t,\frac{i}{2\alpha\sqrt{\Omega}}\right)}\right)^{\frac{1}{2}}, \ \tilde{q}(y,t) = \left(\hat{\mu}_1\left(y,t,\frac{i}{2\alpha\sqrt{\Omega}}\right)\hat{\mu}_2\left(y,t,\frac{i}{2\alpha\sqrt{\Omega}}\right)\right)^{\frac{1}{2}};$$

— *Determine $\hat{M}$ by*

$$\hat{M}(k) = \begin{pmatrix} \hat{\mu}_1 - \frac{1}{2ik}\tilde{q}\left(\frac{\hat{\mu}_1}{\tilde{q}}\right)_y & -\frac{1}{2ik}\tilde{q}\left(\frac{\hat{\mu}_2}{\tilde{q}}\right)_y \\ \frac{1}{2ik}\tilde{q}\left(\frac{\hat{\mu}_1}{\tilde{q}}\right)_y & \hat{\mu}_2 + \frac{1}{2ik}\tilde{q}\left(\frac{\hat{\mu}_2}{\tilde{q}}\right)_y \end{pmatrix} \tag{4.4}$$

(*subscript y denotes the derivative*).

Indeed, the jump, residue, symmetry, and normalization conditions for $\hat{M}$ obviously follow from the respective conditions for $\hat{\mu}$. Then, as in the proof of Proposition 3.1, we have $\det \hat{M} \equiv 1$. On the other hand, evaluating $\det \hat{M}(k)$ using (4.4) at $k = \frac{i}{2\alpha\sqrt{\Omega}}$ and equating the result to 1 gives

$$\tilde{q}^2\left(1 + 2\alpha\sqrt{\Omega}\frac{f_y}{f}\right) = 1. \tag{4.5}$$

Using (4.5) in evaluating (4.4) at $k = \frac{i}{2\alpha\sqrt{\Omega}}$ we arrive at the structure (3.22). Finally, from (4.1) at $k = 0$ and (4.2) it follows that $\hat{\mu}_1(0) = (1 - \bar{r}(0))\hat{\mu}_2(0)$ and thus the structural condition (3.11) holds with $c = 1 - \bar{r}(0) = 1 - r(0)$.

An advantage of the vector version of the RH problem is that it does not involve a structural condition at the singular (in the matrix case) point $k=0$: indeed, due to a specific matrix structure of this singularity, the multiplication by (1 1) from the left cancels the singularity. On the other hand, the matrix formulation has the advantage that it allows establishing that a function constructed from a solution of the respective matrix RH problem, where the spectral data are not apriori assumed to be generated by a solution of the associated nonlinear equation, indeed satisfy this nonlinear equation. This can be done by constructing $\hat{U}$ and $\hat{V}$ involved in the Lax pair (zero curvature) representation

$$\Psi_y = \hat{U}\Psi, \quad \Psi_t = \hat{V}\Psi.$$

Then the compatibility relation $\hat{U}_t - \hat{V}_y + [\hat{U},\hat{V}] = 0$ reduces to a system of equations (relating the values of $\hat{M}$ at the dedicated points), that can be shown to be equivalent to the nonlinear equation in question.

For the sake of simplicity, let us illustrate this scheme in the case of the CH equation (1.2) with $\omega = 1$ (to which the DGH equation can be reduced; see Introduction). Assuming that $\hat{M}$ is a solution of the RH problem with $\hat{p}(y,t,k) = y - \frac{2}{4k^2+1}t$, define $\Psi = \hat{M}e^{ikp\sigma_3}$ and calculate $\hat{U} := \Psi_y\Psi^{-1}$ and $\hat{V} := \Psi_y\Psi^{-1}$ by calculating the main terms as $k\to\infty$, $k\to 0$ and $k\to\frac{i}{2}$ (their structure is dictated by the structural conditions for $\hat{M}$ at these points) and using the Liouville theorem; this gives

$$\hat{U} = -ik\sigma_3 + u_\infty\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{i\beta_1}{k}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix},$$
$$\hat{V} = \frac{ik}{4k^2+1}\begin{pmatrix}\tilde{q}^2 + \dfrac{1}{\tilde{q}^2} & -\tilde{q}^2 + \dfrac{1}{\tilde{q}^2}\\[2ex] \tilde{q}^2 - \dfrac{1}{\tilde{q}^2} & -\tilde{q}^2 - \dfrac{1}{\tilde{q}^2}\end{pmatrix} + \frac{i\beta_2}{k}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}.$$

Here $\tilde{q}$ comes from (3.22), $u_\infty = 2i\hat{M}^{(1)}_{12}$, where $\hat{M}^{(1)}$ comes from the development

$$\hat{M} = I + \frac{\hat{M}^{(1)}}{k} + O(k^{-2}) \quad \text{as } k\to\infty,$$

and β_j can be expressed in terms of the expansion of $\hat{M}$ at $k=0$. Then the compatibility condition $\hat{U}_t - \hat{V}_y + [\hat{U},\hat{V}] = 0$ gives a set of equations relating the coefficients $u_\infty, \beta_1, \beta_2, \tilde{q}$:

$$\begin{aligned}
u_{\infty t} + 2\beta_2 - \frac{p_2}{2} &= 0,\\
\beta_{1t} - \beta_{2y} - 2\beta_2 u_\infty &= 0,\\
p_2 + 4\beta_1(p_1+p_2) &= 0,\\
p_{1y} - 2u_\infty p_2 &= 0,\\
p_{2y} - 2u_\infty p_1 &= 0,
\end{aligned}$$

where $p_1 := \tilde{q}^2 + \frac{1}{\tilde{q}^2}$ and $p_2 := \tilde{q}^2 - \frac{1}{\tilde{q}^2}$. From this system one deduces

$$u_\infty = \frac{1}{2}\frac{(\tilde{q}^2)_y}{\tilde{q}^2}; \quad \beta_1 = -\frac{1}{8}\frac{\tilde{q}^4 - 1}{\tilde{q}^4}$$

and, if one introduces $\hat{u}_y := \left(\frac{1}{\tilde{q}^2}\right)_t$,

$$\beta_2 = \frac{W}{4\tilde{q}^2}, \quad \hat{u}_y = W_y,$$

where

$$W := \tilde{q}^4 - 1 + \left(\hat{u}_y \tilde{q}^2\right)_y \tilde{q}^2.$$

The system

$$\begin{aligned} \left(\frac{1}{\tilde{q}^2}\right)_t &= u_y, \\ \left(\hat{u} - q^4 + 1 - \left(\hat{u}_y q^2\right)_y q^2\right)_y &= 0, \end{aligned}$$

after the change of variable $(y,t) \mapsto (x,t)$: $x_y(y,t) = \tilde{q}^{-2}$ and the introduction of

$$m(x,t) := \tilde{q}^4(y(x,t),t) - 1, \quad u(x,t) := \hat{u}(y(x,t),t)$$

reduces to the CH equation in the "conservative law" form:

$$\begin{aligned} \left(\sqrt{m+1}\right)_t &= \left(-u\sqrt{m+1}\right)_x, \\ (u - m - u_{xx})_x &= 0. \end{aligned}$$

5. SOLITONS

In the Riemann-Hilbert variant of the inverse scattering transform method, the pure soliton solutions comes from the solutions of the RH problem with trivial jump conditions ($J \equiv I$) and thus can be obtained by solving the system of linear algebraic equations generated by the residue conditions.

For instance, the one-soliton solution of the DGH equation, similarly to the case of the CH equation, can be obtained from the solution $(\hat{\mu}_1\ \hat{\mu}_2)$ of the vector RH problem corresponding to $J \equiv I$, $N = 1$, $k_1 = i\nu$ with $0 < \nu < \frac{1}{2\alpha\sqrt{\Omega}}$, and $\gamma_1 > 0$. From the RH conditions on $\hat{\mu}$ it follows that

$$(\hat{\mu}_1(y,t,k)\ \hat{\mu}_2(y,t,k)) = \begin{pmatrix} \frac{k - B(y,t)}{k - i\nu} & \frac{k + B(y,t)}{k + i\nu} \end{pmatrix} \tag{5.1}$$

with some B. Substituting this into the residue conditions (3.9) gives

$$B(y,t) = i\nu\frac{1-g}{1+g}, \tag{5.2}$$

where

$$g(y,t) = \exp\left\{-2\nu\left(y - \left(\frac{2\Omega^{\frac{3}{2}}}{1-4\alpha^2\Omega\nu^2} - \frac{\gamma\Omega^{\frac{1}{2}}}{\alpha^2}\right)t - y_0\right)\right\} \tag{5.3}$$

with $y_0 = \frac{1}{2\nu}\log\frac{\gamma_1}{2\nu}$.

Now, substituting (5.1) with (5.3) and (5.2) into the formulas (3.26)–(3.28) for the solution of the DGH equation gives the parametric representation for the one-soliton solution: $u(x,t) = \hat{u}(y(x,t),t)$, where

$$\hat{u}(y,t) = \frac{32\alpha^2\Omega^2\nu^2}{(1-4\alpha^2\Omega\nu^2)^2}\frac{g}{1+2\frac{1+4\alpha^2\Omega\nu^2}{1-4\alpha^2\Omega\nu^2}g+g^2}, \tag{5.4}$$

$$x(y,t) = \alpha\log\frac{1+g\frac{1+2\alpha\sqrt{\Omega}\nu}{1-2\alpha\sqrt{\Omega}\nu}}{1+g\frac{1-2\alpha\sqrt{\Omega}\nu}{1+2\alpha\sqrt{\Omega}\nu}} + \frac{y}{\sqrt{\Omega}}. \tag{5.5}$$

Introducing the soliton speed (in the scale (y,t))

$$\hat{v}_s = \frac{2\Omega^{3/2}}{1-4\alpha^2\Omega\nu^2} - \frac{\gamma\sqrt{\Omega}}{\alpha^2} \tag{5.6}$$

and rewriting g as $g = e^{\phi}$ with $\phi = -2\nu(y - \hat{v}_s t - y_0)$, the formula for $\hat{u}$ takes the form

$$\hat{u}(y,t) = \frac{16\alpha^2\Omega^2\nu^2}{1-4\alpha^2\Omega\nu^2}\frac{1}{(1-4\alpha^2\Omega\nu^2)\cosh\{\phi(y,t)\}+1+4\alpha^2\Omega\nu^2}. \tag{5.7}$$

According to (5.5), the soliton speed in the original scale (x,t) is $v_s = \frac{2\Omega}{1-4\alpha^2\Omega\nu^2} - \frac{\gamma}{\alpha^2}$; thus the solitons can propagate in the sector $\frac{x}{t} > 2\Omega - \frac{\gamma}{\alpha^2}$ of the (x,t)-plane.

Moreover, similarly to the CH equation, see [5,7], the Riemann-Hilbert formalism allows studying in details the long-time asymptotics of the solution of the Cauchy problem (1.6), (1.7). Particularly, one can show that the solitons dominate the asymptotics in the soliton sector $\frac{x}{t} > 2\Omega - \frac{\gamma}{\alpha^2}$: as $t \to \infty$, the solution approaches the sum of one-solitons, whose parameters are determined by the parameters involved in the residue conditions.

Notice that (5.5) implies that

$$\frac{\partial x}{\partial y}(y,t) = \frac{1}{\sqrt{\Omega}}\frac{(1+g)^2}{1+2\frac{1+4\alpha^2\Omega\nu^2}{1-4\alpha^2\Omega\nu^2}g+g^2} \tag{5.8}$$

and thus $\frac{\partial x}{\partial y}(y,t) > 0$ for all y and t provided that g is as in (5.3) (particularly, $g > 0$ for all y and t) and that $0 < \nu < \frac{1}{2\alpha\sqrt{\Omega}}$. Consequently, the variable change $y \mapsto x$ is smooth and bijective for any fixed t. Therefore, the smooth soliton solution (5.4) in the variables (y,t) remains smooth in the original variables (x,t) and thus is a classical, smooth solution of the DGH equation.

Now notice that the considerations in the previous section allows constructing (local) solutions of the DGH equation starting from a Riemann-Hilbert problem, whose data do not necessarily correspond to "good" (smooth, decaying) initial data. Provided the RH problem is solved, the construction will give a solution of the DGH equation, that needn't be a classical solution but solves the DGH equation locally, probably in a certain week sense (see, e.g., [11] for the case of the short wave limit of the CH equation).

For example, consider the RH problem with the trivial jump and with the residue conditions corresponding to $N = 1$ and $k = i\nu$ with $\nu > \frac{i}{2\alpha\sqrt{\Omega}}$. Assume also that the associated γ_1 is negative: $\gamma_1 < 0$. Then the formulas (5.4), (5.5) with g given by

$$g(y,t) = -\exp\left\{-2\nu\left(y - \left(\frac{2\Omega^{\frac{3}{2}}}{1-4\alpha^2\Omega\nu^2} - \frac{\gamma\Omega^{\frac{1}{2}}}{\alpha^2}\right)t - y_0\right)\right\} \tag{5.9}$$

where $y_0 = \frac{1}{2\nu}\log\frac{|\gamma_1|}{2\nu}$, still determine a bounded function that solves, locally, the DGH equation. Notice that $\hat{u}(y,t)$ in this case, see (5.4), is a smooth function for all y and t, but the variable change $y \mapsto x$, see (5.5), is singular at a single point corresponding to $g = -1$, where $\frac{\partial x}{\partial y} = 0$ (which, as it is seen from (5.4), corresponds to the single maximum of $\hat{u}(y,t)$). Therefore, in the original variables (x,t), $u(x,t)$ given by (5.4), (5.5) with g given by (5.9) has a cusp (with $\frac{\partial u}{\partial x} = \infty$) at his maximum and thus can be called the cuspon solution of the DGH equation.

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