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HANKEL AND TOEPLITZ OPERATORS: CONTINUOUS AND DISCRETE REPRESENTATIONS

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Abstract. We find a relation guaranteeing that Hankel operators realized in the space of sequences $\ell^2(\mathbb{Z}_+)$ and in the space of functions $L^2(\mathbb{R}_+)$ are unitarily equivalent. This allows us to obtain exhaustive spectral results for two classes of unbounded Hankel operators in the space $\ell^2(\mathbb{Z}_+)$ generalizing in different directions the classical Hilbert matrix. We also discuss a link between representations of Toeplitz operators in the spaces $\ell^2(\mathbb{Z}_+)$ and $L^2(\mathbb{R}_+)$.

Keywords: unbounded Hankel and Toeplitz operators, various representations, moment problems, generalized Hilbert matrices.

Mathematics Subject Classification: 47B25, 47B35.

1. INTRODUCTION

1.1. This paper is based on the talk given by the author at the conference "Spectral Theory and Applications" held in May 2015 in Krakow. So, it is somewhat eclectic. Our aim is to discuss various properties Hankel and Toeplitz (known also as Wiener-Hopf) operators. We refer to the books [1, 5, 11, 12, 14] for basic information on these classes of operators.

Our main goal is to describe a relation between discrete and continuous representations of Hankel and Toeplitz operators in a sufficiently consistent way and to draw spectral consequences from this relation. We do not suppose that operators are bounded, and so we are naturally led to work with quadratic forms and distributional integral kernels. As is well known, the discrete (in the space $\ell^2(\mathbb{Z}_+)$) and continuous (in the space $L^2(\mathbb{R}_+)$) representations are linked by the Laguerre transform. For bounded operators, this yields the unitary equivalence of the corresponding operators in $\ell^2(\mathbb{Z}_+)$ and $L^2(\mathbb{R}_+)$. However in singular cases their equivalence may be lost because the natural domains of the quadratic forms in discrete and continuous representations are not linked by the Laguerre transform. As show simple examples, the continuous representation seems to be more general. Passing to the Fourier transforms, one

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can also realize discrete Hankel and Toeplitz operators in the Hardy space $\mathbb{H}^2_+(\mathbb{T})$ of functions analytic in the unit circle $\mathbb{T}$ and continuous operators in the Hardy space $\mathbb{H}^2_+(\mathbb{R})$ of functions analytic in the upper half-plane; see, e.g., the book [6], for the precise definition of these spaces.

Section 2 is of a preliminary nature. We first consider the discrete A and the continuous $\mathbf{A}$ convolution operators in the spaces $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{R};dx)$, respectively. Of course the Fourier transform allows one to reduce these operators to the multiplications B and $\mathbf{B}$ in the spaces $L^2(\mathbb{T})$ and $L^2(\mathbb{R};d\lambda)$. The operators B and $\mathbf{B}$ are obviously related by a change of variables. This yields a link between the operators A and $\mathbf{A}$ which is given by the Laguerre transform. Our main goal in this section is to discuss explicit formulas relating matrix elements of A and the integral kernel of $\mathbf{A}$. Then, we apply these results to Toeplitz operators realized in the spaces $\ell^2(\mathbb{Z}_+)$ and $L^2(\mathbb{R}_+)$. We are aiming at a systematic presentation of known results but insist upon the case of unbounded operators. Moreover, some formulas, for example, (2.20) and (2.21), are perhaps new.

1.2. In Section 3, we pass to the main subject of this paper, to Hankel operators. We recall that Hankel operators H are defined in the space $\ell^2(\mathbb{Z}_+)$ by the formula

$$(Hf)_n = \sum_{m\in\mathbb{Z}_+} a_{n+m} f_m, \quad f = \{f_n\}_{n\in\mathbb{Z}_+}. \tag{1.1}$$

Similarly, Hankel operators $\mathbf{H}$ act in the space $L^2(\mathbb{R}_+)$ by the formula

$$(\mathbf{H}\mathbf{f})(t) = \int_{\mathbb{R}_+} \mathbf{a}(t+s)\mathbf{f}(s)ds. \tag{1.2}$$

Note that spectral properties of the operators H are determined by the behavior of their matrix elements a_n as $n \to \infty$ while, as far as the operators $\mathbf{H}$ are concerned, both the behavior of integral kernels $\mathbf{a}(t)$ as $t \to \infty$ and $t \to 0$ as well as their local singularities at finite points $t \neq 0$ are essential.

Following the scheme of Section 2, we first find a link between the discrete (1.1) and continuous (1.2) realizations of Hankel operators. Then we consider the case where the matrix elements and the kernels of Hankel operators admit the integral representations

$$a_n = \int_{\operatorname{clos}\mathbb{D}} z^n dM(z), \quad n = 0, 1, \dots, \tag{1.3}$$

and

$$\mathbf{a}(t) = \int_{\operatorname{clos}\mathbb{C}^+} e^{-\zeta t} d\Sigma(\zeta), \quad t > 0, \tag{1.4}$$

with some complex measures $dM(z)$ and $d\Sigma(\zeta)$. Here $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc, $\mathbb{C}^+ = \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > 0\}$ is the right half-plane, and $\operatorname{clos}\mathbb{D}$, $\operatorname{clos}\mathbb{C}^+$ are the closures of these sets. Formulas (1.3) and (1.4) unify different types of integral representations of a_n and $\mathbf{a}(t)$, for example, representations in terms of Carleson measures or in terms of symbols of the corresponding bounded Hankel operators.

The central result of Section 3, Theorem 3.3, formally means that the “operators” H and $\mathbf{H}$ are unitarily equivalent provided the measures $dM(z)$ and $d\Sigma(\zeta)$ in (1.3) and (1.4) are linked by the equality

$$dM(z) = 2\alpha(\zeta+\alpha)^{-2} d\Sigma(\zeta), \quad z = \frac{\zeta-\alpha}{\zeta+\alpha}, \tag{1.5}$$

for some value of the parameter $\alpha > 0$. Although quite simple, Theorem 3.3 is very useful because it relates the discrete and continuous representations directly avoiding the general construction of Section 2. More important, it allows one to translate spectral results obtained for the operator $\mathbf{H}$ into the results for the operator H, and vice versa. Such examples are discussed in Section 4.

1.3. Section 4 is devoted to Hankel operators generalizing in different directions two classical examples: the Hilbert matrix and the Carleman operator. To put our results into the right context, let us briefly recall basic spectral properties of these operators. The Hilbert matrix is the Hankel operator H defined by formula (1.1) where $a_n = (n+1)^{-1}$ for all $n \geq 0$. As shown in the papers [8, 13], the spectrum of H is absolutely continuous, it is simple and coincides with the interval $[0, \pi]$. The Carleman operator is defined by formula (1.2) where $\mathbf{a}(t) = t^{-1}$. Using the Mellin transform, it is easy to show that the spectrum of the operator $\mathbf{H}$ is absolutely continuous, has multiplicity 2, and it also coincides with the interval $[0, \pi]$. So both these operators are bounded but not compact. It can be deduced from the results on the Hilbert matrix that Hankel operators H are bounded if $a_n = O(n^{-1})$ and they are compact if $a_n = o(n^{-1})$ as $n \to \infty$. Similarly, the results on the Carleman operator imply that Hankel operators $\mathbf{H}$ with integral kernels $\mathbf{a} \in L^\infty_{\rm loc}(\mathbb{R})$ are bounded if $\mathbf{a}(t) = O(t^{-1})$ and they are compact if $\mathbf{a}(t) = o(t^{-1})$ as $t \to 0$ and $t \to \infty$.

In Section 4 we study Hankel operators H with matrix elements a_n such that $a_n n \to \infty$. These operators are unbounded. We exhibit two quite different cases where the spectral analysis of Hankel operators H can be carried out sufficiently explicitly. Our approach relies on Theorem 3.3 and the results on Hankel operators $\mathbf{H}$ with singular integral kernels $\mathbf{a}(t)$ obtained earlier in [19, 21].

We emphasize that some properties of Hankel operators are more transparent in the discrete representation while other properties – in the continuous representation. Such examples are given in Section 4. So when studying Hankel operators, it is very useful to keep in mind their various representations.

2. VARIOUS REPRESENTATIONS OF CONVOLUTIONS AND TOEPLITZ OPERATORS

2.1. First, we recall standard relations between various spaces we consider. Let us introduce the following diagrams:

$$\begin{array}{ccc} f=\{f_n\}_{n\in\mathbb{Z}} & \longrightarrow & \mathbf{f}(x)=(\Phi\mathbf{u})(x) \\ \downarrow & & \uparrow \\ u(\mu)=(\mathcal{F}^*f)(\mu) & \longrightarrow & \mathbf{u}(\lambda)=(\mathcal{U}u)(\lambda) \end{array} \qquad \begin{array}{ccc} \ell^2(\mathbb{Z}) & \xrightarrow{\ \mathcal{L}\ } & L^2(\mathbb{R};dx) \\ \downarrow \mathcal{F}^* & & \uparrow \Phi \\ L^2(\mathbb{T}) & \xrightarrow{\ \mathcal{U}\ } & L^2(\mathbb{R};d\lambda) \end{array} \tag{2.1}$$

Here the unitary mapping $\mathcal{F}: L^2(\mathbb{T})\to\ell^2(\mathbb{Z})$ corresponds to expanding a function in the Fourier series:

$$f_n=(\mathcal{F}u)_n:=\int_{\mathbb{T}} u(\mu)\mu^{-n}d\mathbf{m}_0(\mu),$$

where

$$d\mathbf{m}_0(\mu)=(2\pi i\mu)^{-1}d\mu$$

is the normalized Lebesgue measure on the unit circle $\mathbb{T}$. The adjoint operator $\mathcal{F}^*\colon \ell^2(\mathbb{Z})\to L^2(\mathbb{T})$ acts by the formula

$$u(\mu)=(\mathcal{F}^*f)(\mu)=\sum_{n\in\mathbb{Z}} f_n\mu^n. \tag{2.2}$$

Similarly, Φ is the Fourier transform,

$$\mathbf{f}(x)=(\Phi\mathbf{u})(x):=(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ix\lambda}\mathbf{u}(\lambda)d\lambda.$$

Of course the operator $\Phi: L^2(\mathbb{R};d\lambda)\to L^2(\mathbb{R};dx)$ is unitary. The unitary operator $\mathcal{U}=\mathcal{U}_\alpha: L^2(\mathbb{T})\to L^2(\mathbb{R};d\lambda)$ is defined by the equality

$$(\mathcal{U}u)(\lambda)=\sqrt{\tfrac{\alpha}{\pi}}(\lambda+i\alpha)^{-1}u\big(\tfrac{\lambda-i\alpha}{\lambda+i\alpha}\big), \tag{2.3}$$

where a positive parameter α can be fixed in an arbitrary way.

Let us set $\mathcal{L}=\Phi\mathcal{U}\mathcal{F}^*$. Then $\mathcal{L}\colon \ell^2(\mathbb{Z})\to\ell^2(\mathbb{R};dx)$ is the unitary operator, and it can be expressed in terms of the Laguerre polynomials. Recall that the Laguerre polynomials (see the book [2], Chapter 10.12) are defined by the formula

$$\mathsf{L}_n^p(t)=n!^{-1}e^t t^{-p}d^n(e^{-t}t^{n+p})/dt^n,\quad n=0,1,\dots,\quad t\ge 0,\quad p>-1.$$

Of course the polynomial $\mathsf{L}_n^p(t)$ has degree n; in particular, $\mathsf{L}_0^p(t)=1$. These polynomials are orthogonal with respect to the measure $t^pe^{-t}dt$ and

$$\int_0^\infty \mathsf{L}_n^p(t)\mathsf{L}_m^p(t)t^pe^{-t}dt=\frac{\Gamma(n+p+1)}{n!}\delta_{n,m}, \tag{2.4}$$

where $\Gamma(\cdot)$ is the gamma function and $\delta_{n,m}$ is the Kronecker symbol. The parameter $p > -1$ is arbitrary, but we need the cases $p = 0$ and $p = 1$ only.

Let us use the identity (see formula (10.12.32) in [2]) for $\mathsf{L}_n := \mathsf{L}_n^0$:

$$\int_0^\infty \mathsf{L}_n(t) e^{-(1/2+\zeta)t} dt = \frac{1}{\zeta + 1/2} \Big(\frac{2\zeta - 1}{2\zeta + 1}\Big)^n, \quad \operatorname{Re}\zeta > -1/2. \tag{2.5}$$

Putting here $\zeta = -i\lambda$ and making the inverse Fourier transform, we find that

$$\mathsf{L}_n(2\alpha x) e^{-\alpha x} \mathbb{1}_+(x) = i(2\pi)^{-1} \int_{-\infty}^\infty e^{-ix\lambda} (\lambda + i\alpha)^{-1} \big(\frac{\lambda - i\alpha}{\lambda + i\alpha}\big)^n d\lambda, \quad n = 0, 1, \dots, \tag{2.6}$$

where $\mathbb{1}_+(x)$ is the characteristic function of $\mathbb{R}_+$. Recall that the Hardy space $\mathbb{H}^2_+(\mathbb{T})$ (resp. $\mathbb{H}^2_-(\mathbb{T})$) consists of functions $u \in L^2(\mathbb{T})$ whose Fourier coefficients $(\mathcal{F}u)_n = 0$ for $n < 0$ (resp., for $n \geq 0$). Since the functions μ^n, $n = 0, 1, \dots$, form an orthonormal basis in the space $\mathbb{H}^2_+(\mathbb{T})$, it follows from relations (2.3) and (2.6) that the functions $-i(2\alpha)^{1/2} \mathsf{L}_n(2\alpha t) e^{-\alpha t}$ is an orthonormal basis in the space $L^2(\mathbb{R}_+)$. We also see that $i(2\alpha)^{1/2} \mathsf{L}_n(-2\alpha t) e^{\alpha t}$, $n = 0, 1, 2, \dots$, is an orthonormal basis in the space $L^2(\mathbb{R}_-)$. Moreover, relations (2.3) and (2.6) imply that the operator $\mathcal{L}\colon \ell^2(\mathbb{Z}) \to L^2(\mathbb{R}; dx)$ acts by the formula

$$\begin{aligned}(\mathcal{L}f)(x) = -i(2\alpha)^{1/2} &\sum_{n=0}^\infty f_n \mathsf{L}_n(2\alpha x) e^{-\alpha x} \mathbb{1}_+(x) \\ &+ i(2\alpha)^{1/2} \sum_{n=0}^\infty f_{-n-1} \mathsf{L}_n(-2\alpha x) e^{\alpha x} \mathbb{1}_+(-x), \quad f = \{f_n\}_{n\in\mathbb{Z}}.\end{aligned} \tag{2.7}$$

To be precise, the unitary operator $\mathcal{L}$ is first defined on the dense set $\mathcal{D} \subset \ell^2(\mathbb{Z})$ consisting of elements f with only a finite number of non-zero components f_n, and then it is extended by the continuity onto the whole space $\ell^2(\mathbb{Z})$.

2.2. Next, we discuss representations of the convolution/multiplication operators in all these spaces. We start with the space $\ell^2(\mathbb{Z})$ where the operator of the discrete convolution acts by the formula

$$(Af)_n = \sum_{m\in\mathbb{Z}} a_{n-m} f_m, \quad f = \{f_n\}_{n\in\mathbb{Z}}. \tag{2.8}$$

Without some assumptions on the sequence $a = \{a_n\}_{n\in\mathbb{Z}}$, in general $Af \notin \ell^2(\mathbb{Z})$ even for $f \in \mathcal{D}$. Therefore instead of the operator A, we consider its quadratic form

$$a[f, f] = \sum_{n,m\in\mathbb{Z}} a_{n-m} f_m \overline{f_n}, \quad f \in \mathcal{D}, \tag{2.9}$$

that consists of a finite number of terms for an arbitrary sequence a.

Similarly, the convolution operator $\mathbf{A}$ acts in the space $L^2(\mathbb{R}; dx)$ by the formula

$$(\mathbf{Af})(x) = \int_{\mathbb{R}} \mathbf{a}(x-y)\mathbf{f}(y)dy, \tag{2.10}$$

and its quadratic form is given by the equality

$$\mathbf{a}[\mathbf{f},\mathbf{f}] = \int_{\mathbb{R}} \mathbf{a}(x)\Big(\int_{\mathbb{R}} \mathbf{f}(y)\overline{\mathbf{f}(x+y)}dy\Big)dx \tag{2.11}$$

for test functions $\mathbf{f} \in C_0^\infty(\mathbb{R})$. Since, for such functions $\mathbf{f}$, the function

$$\mathbf{F}(x) = \int_{\mathbb{R}} \mathbf{f}(y)\overline{\mathbf{f}(x+y)}dy$$

also belongs to $C_0^\infty(\mathbb{R})$, the form (2.11) is correctly defined for a distribution $\mathbf{a}$ in the space $C_0^\infty(\mathbb{R})'$ dual to $C_0^\infty(\mathbb{R})$.

Of course the Fourier transform allows one to realize convolutions as multiplication operators. Let $\mathcal{P} = \mathcal{F}^*\mathcal{D}$ be the set of all quasi-polynomials (2.2). For a distribution $b \in \mathcal{P}'$ (the space dual to $\mathcal{P}$), we formally define the operator B in the space $L^2(\mathbb{T})$ by the equality

$$(Bu)(\mu) = b(\mu)u(\mu), \tag{2.12}$$

or, in precise terms, we introduce its quadratic form

$$b[u,u] = \int_{\mathbb{T}} b(\mu)|u(\mu)|^2 d\mathbf{m}_0(\mu), \quad u \in \mathcal{P}. \tag{2.13}$$

Then $B = \mathcal{F}^* A \mathcal{F}$ if

$$b(\mu) = (\mathcal{F}^* a)(\mu) = \sum_{n\in\mathbb{Z}} a_n \mu^n, \quad a = \{a_n\}_{n\in\mathbb{Z}}. \tag{2.14}$$

Strictly speaking, we have a relation between the quadratic forms

$$b[u,u] = a[f,f] \quad \text{if} \quad f = \mathcal{F}u.$$

Similarly, we put $\mathcal{Z} := \Phi^* C_0^\infty(\mathbb{R})$. Recall that the set $\mathcal{Z}$ consists of analytic functions satisfying a certain estimate at infinity (see, e.g., [4] for details). For a distribution $\mathbf{b} \in \mathcal{Z}'$ (the space dual to $\mathcal{Z}$), we formally define the multiplication operator $\mathbf{B}$ in the space $L^2(\mathbb{R}; d\lambda)$ by the equality

$$(\mathbf{Bu})(\lambda) = \mathbf{b}(\lambda)\mathbf{u}(\lambda), \tag{2.15}$$

or, in precise terms, we introduce its quadratic form

$$\mathbf{b}[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}} \mathbf{b}(\lambda)|\mathbf{u}(\lambda)|^2 d\lambda. \tag{2.16}$$

Then $\mathbf{B} = \Phi^* \mathbf{A} \Phi$ if $\mathbf{b} = (2\pi)^{1/2} \Phi^* \mathbf{a}$, where $\mathbf{a} \in C_0^\infty(\mathbb{R})'$. Strictly speaking, we have a relation between the quadratic forms

$$\mathbf{b}[\mathbf{u}, \mathbf{u}] = \mathbf{a}[\mathbf{f}, \mathbf{f}] \quad \text{if} \quad \mathbf{f} = \Phi \mathbf{u}.$$

Obviously, the operators A, $\mathbf{A}$, B and $\mathbf{B}$ are formally symmetric if $a_{-n} = \overline{a_n}$, $\mathbf{a}(-x) = \overline{\mathbf{a}(x)}$, $b(\mu) = \overline{b(\mu)}$ and $\mathbf{b}(\lambda) = \overline{\mathbf{b}(\lambda)}$. To be more precise, this means that the corresponding quadratic forms are real.

We emphasize that the bases $u_n(\mu) = \mu^n$, $n \in \mathbb{Z}$, and $\mathcal{F} u_n$ are the canonical bases in the spaces $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$, respectively. On the contrary, the bases $\mathbf{u}_n = \mathcal{U} u_n$ and, especially, $\Phi \mathbf{u}_n$ in the spaces $L^2(\mathbb{R}; d\lambda)$ and $L^2(\mathbb{R}; dx)$ do not apparently play any distinguished role. So, it seems more natural to consider the form defined by (2.11) on functions $\mathbf{f} \in C_0^\infty(\mathbb{R})$ for a distribution $\mathbf{a} \in C_0^\infty(\mathbb{R})'$. Similarly, we consider the form (2.16) for $\mathbf{u} \in \mathcal{Z}$ and $\mathbf{b} \in \mathcal{Z}'$.

2.3. Let us find a link between the discrete and continuous representations. It is formally quite simple for the operators B and $\mathbf{B}$. By definitions (2.3) and (2.12), the operator $\mathbf{B} = \mathcal{U} B \mathcal{U}^*$ acts in the space $L^2(\mathbb{R}; d\lambda)$ as the multiplication by the function (distribution)

$$\mathbf{b}(\lambda) = b\big(\tfrac{\lambda - i\alpha}{\lambda + i\alpha}\big). \tag{2.17}$$

Its quadratic form is given by the formula (2.16), where

$$\mathbf{u}(\lambda) = (\mathcal{U} u)(\lambda) = \sqrt{\tfrac{\alpha}{\pi}} (\lambda + i\alpha)^{-1} \sum_{n \in \mathbb{Z}} f_n \big(\tfrac{\lambda - i\alpha}{\lambda + i\alpha}\big)^n$$

belongs to the set $\boldsymbol{\mathcal{P}} = \mathcal{U} \mathcal{P}$ and $\mathbf{b}$ belongs to the dual space $\boldsymbol{\mathcal{P}}'$. We emphasize however that this link is only formal because the sets $\mathcal{Z}$ and $\boldsymbol{\mathcal{P}}$ of test functions $\mathbf{u}(\lambda)$ are different.

It remains to directly link the representations of convolution operators in the spaces $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{R}; dx)$. Recall that the unitary operator $\mathcal{L} = \Phi \mathcal{U} \mathcal{F}^* \colon \ell^2(\mathbb{Z}) \to L^2(\mathbb{R}; dx)$ is given by equality (2.7). Let the operator A be defined in the space $\ell^2(\mathbb{Z})$ by formula (2.8) and $\mathbf{B} = \mathcal{U} \mathcal{F}^* A \mathcal{F} \mathcal{U}^*$. Then $\mathbf{A} = \Phi \mathbf{B} \Phi^* = \mathcal{L} A \mathcal{L}^*$ is the convolution in the space $L^2(\mathbb{R}; dx)$ acting by the formula (2.10) where $\mathbf{a} = (2\pi)^{-1/2} \Phi \mathbf{b}$ and $\mathbf{b}$ is defined by (2.14), (2.17). The quadratic form of the operator $\mathbf{A}$ is defined by formula (2.11), but we have the same problem as for the multiplication operators: the domains $C_0^\infty(\mathbb{R})$ and $\mathcal{L} \mathcal{D} = \Phi \boldsymbol{\mathcal{P}}$ of quadratic forms of the operators $\mathbf{A}$ and $\mathcal{L} A \mathcal{L}^*$ are different.

Our goal is to find an expression for $\mathbf{a}(x)$, $x \in \mathbb{R}$, in terms of $a = \{a_n\}_{n \in \mathbb{Z}}$. Recall the relation (see formula (10.12.15) in the book [2])

$$\frac{d}{dt} \mathsf{L}_n(t) = -\mathsf{L}_{n-1}^1(t), \quad t > 0, n \geq 1, \tag{2.18}$$

for the Laguerre polynomials. Let $\mathcal{S}'$ be the space dual to the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R})$ of rapidly decaying C^∞ functions. It follows from (2.6) that

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-ix\lambda}\Big(\frac{\lambda - i\alpha}{\lambda + i\alpha}\Big)^n d\lambda &= i\Big(\frac{d}{dx} + \alpha\Big) \int_{-\infty}^{\infty} e^{-ix\lambda}(\lambda + i\alpha)^{-1}\Big(\frac{\lambda - i\alpha}{\lambda + i\alpha}\Big)^n d\lambda \\ &= 2\pi(\frac{d}{dx} + \alpha)\big(\mathsf{L}_n(2\alpha x)e^{-\alpha x}\mathbb{1}_+(x)\big) \\ &= -4\pi\alpha \mathsf{L}^1_{n-1}(2\alpha x)e^{-\alpha x}\mathbb{1}_+(x) + 2\pi\delta(x),\end{aligned} \tag{2.19}$$

where $\delta(x)$ is the Dirac delta-function and the Fourier transform is understood in the sense of $\mathcal{S}'$. Passing here to the complex conjugation and making the change of the variables $x \mapsto -x$, we also see that

$$\int_{-\infty}^{\infty} e^{-ix\lambda}\Big(\frac{\lambda - i\alpha}{\lambda + i\alpha}\Big)^{-n} d\lambda = -4\pi\alpha \mathsf{L}^1_{n-1}(-2\alpha x)e^{\alpha x}\mathbb{1}_+(-x) + 2\pi\delta(x).$$

Therefore, it formally follows from equalities (2.14) and (2.17) that the distribution $\mathbf{a} = (2\pi)^{-1/2}\Phi\mathbf{b}$ satisfies the relation

$$\mathbf{a}(x) = \sum_{n\in\mathbb{Z}} a_n\delta(x) - 2\alpha\sum_{n=1}^{\infty} \mathsf{L}^1_{n-1}(2\alpha|x|)e^{-\alpha|x|}\big(a_n\mathbb{1}_+(x) + a_{-n}\mathbb{1}_+(-x)\big). \tag{2.20}$$

An expression for this distribution can also be given in a somewhat different form. Let $\varphi \in C_0^\infty(\mathbb{R})$ (or $\varphi \in \mathcal{S}$). Then using (2.18) and integrating by parts, we see that

$$\begin{aligned}\int_{-\infty}^{\infty} \mathbf{a}(x)\varphi(x)dx &= a_0\varphi(0) \\ &+ \sum_{n=1}^{\infty}\int_0^{\infty} \mathsf{L}_n(2\alpha x)e^{-\alpha x}\big(a_n(\alpha\varphi(x) - \varphi'(x)) + a_{-n}(\alpha\varphi(-x) + \varphi'(-x))\big)dx.\end{aligned} \tag{2.21}$$

Equation (2.20) formally shows that

$$\mathbf{a}(x) = \kappa\delta(x) + \mathbf{k}(x),$$

where

$$\sum_{n\in\mathbb{Z}} a_n = \kappa \tag{2.22}$$

and $\mathbf{k}(x)$ is the second term in the right-hand side of (2.20). Using the identities (2.4), we can solve equation (2.20) for the coefficients a_n:

$$a_{\pm n} = -2\alpha n^{-1}\int_0^{\infty} \mathbf{k}(\pm x)\mathsf{L}^1_{n-1}(2\alpha x)e^{-\alpha x}x dx, \quad n = 1, 2, \dots;$$

then (2.22) yields a_0.

The relations between various representations of convolution/multiplication operators can be summarized by the following diagrams complementing (2.1):

$$\begin{array}{ccc} a_n & \longrightarrow & \mathbf{a}(x) \\ \downarrow & & \uparrow \\ b(\mu) & \longrightarrow & \mathbf{b}(\lambda) \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & \mathbf{A} = \Phi\mathbf{B}\Phi^* \\ \downarrow & & \uparrow \\ B = \mathcal{F}^* A \mathcal{F} & \longrightarrow & \mathbf{B} = \mathcal{U} B \mathcal{U}^* \end{array} \tag{2.23}$$

2.4. Let us now consider Wiener-Hopf (Toeplitz) operators. We start with the space $\ell^2(\mathbb{Z}_+)$, where Wiener-Hopf operators W act again by formula (2.8) but now $n, m \in \mathbb{Z}_+$:

$$(Wf)_n = \sum_{m \in \mathbb{Z}_+} a_{n-m} f_m, \quad f = \{f_n\}_{n \in \mathbb{Z}_+}. \tag{2.24}$$

Quite similarly to (2.9), their quadratic forms are defined by the formula

$$w[f, f] = \sum_{n,m \in \mathbb{Z}_+} a_{n-m} f_m \overline{f_n}, \tag{2.25}$$

where $f \in \mathcal{D}_+ := \mathcal{D} \cap \ell^2(\mathbb{Z}_+)$ and the sequence $a = \{a_n\}_{n \in \mathbb{Z}}$ is again arbitrary.

A Wiener-Hopf operator $\mathbf{W}$ acts in the space $L^2(\mathbb{R}_+)$ by the formula

$$(\mathbf{W}\mathbf{f})(x) = \int_{\mathbb{R}_+} \mathbf{a}(x - y)\mathbf{f}(y) dy,$$

and its quadratic form is again given by the equality

$$\mathbf{w}[\mathbf{f}, \mathbf{f}] = \int_{\mathbb{R}} \mathbf{a}(x) \Big(\int_{\mathbb{R}} \mathbf{f}(y) \overline{\mathbf{f}(x + y)} dy \Big) dx, \tag{2.26}$$

where $\mathbf{a} \in C_0^\infty(\mathbb{R})'$ but now $\mathbf{f} \in C_0^\infty(\mathbb{R}_+)$.

Next, we pass to the representation of Toeplitz operators in the Hardy spaces. Denote by P_+ the orthogonal projection in $L^2(\mathbb{T})$ onto the Hardy space $\mathbb{H}^2_+(\mathbb{T})$, and let, as before, the operator B be formally defined by equality (2.12), where the distribution $b \in \mathcal{P}'$. Then the Toeplitz operator $T : \mathbb{H}^2_+(\mathbb{T}) \to \mathbb{H}^2_+(\mathbb{T})$ is defined by the relation

$$Tu = P_+ Bu$$

on elements $u \in \mathcal{P}_+ := \mathcal{F}^* \mathcal{D}_+$; obviously, the set $\mathcal{P}_+$ consists of all polynomials $u(\mu) = \sum_{n \in \mathbb{Z}_+} f_n \mu^n$. The quadratic form $t[u, u]$ of the operator T is given by the right-hand side of (2.13), where $u \in \mathcal{P}_+$ and $b \in \mathcal{P}'$ are arbitrary.

Finally, we discuss the representation in the Hardy space $\mathbb{H}^2_+(\mathbb{R})$ consisting of functions $\mathbf{u} \in L^2(\mathbb{R})$ whose Fourier transforms $(\Phi\mathbf{u})(x) = 0$ for $x < 0$. Let $\mathbf{P}_+$ be the orthogonal projection in $L^2(\mathbb{R})$ onto $\mathbb{H}^2_+(\mathbb{R})$, and let $\mathbf{B}$ be the operator (2.15). Then the Toeplitz operator $\mathbf{T} : \mathbb{H}^2_+(\mathbb{R}) \to \mathbb{H}^2_+(\mathbb{T})$ is defined by the relation

$$\mathbf{T}\mathbf{u} = \mathbf{P}_+ \mathbf{B}\mathbf{u}.$$

Its quadratic form $\mathbf{t}[\mathbf{u},\mathbf{u}]$ is given by the the right-hand side of (2.16), where $\mathbf{u} \in \Phi^* C_0^\infty(\mathbb{R}_+) =: \mathcal{Z}_+$ and $\mathbf{b} \in \mathcal{Z}'$. The relation (2.17) between the functions (distributions) $b(\mu)$ and $\mathbf{b}(\lambda)$ remains of course true.

Note that the Laguerre operator $\mathcal{L}\colon \ell^2(\mathbb{Z}_\pm) \to L^2(\mathbb{R}_\pm)$ (here $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{Z}_+$) and

$$(\mathcal{L}f)(t) = -i(2\alpha)^{1/2} \sum_{n=0}^{\infty} f_n \mathsf{L}_n(2\alpha t) e^{-\alpha t}, \quad f = \{f_n\}_{n\in\mathbb{Z}_+}, t > 0, \tag{2.27}$$

but formula (2.20) for kernel of the operator $\mathbf{W}$ remains true.

For Toeplitz operators, instead of (2.1), (2.23) we have the diagrams

$$\begin{array}{ccc} \ell^2(\mathbb{Z}_+) & \xrightarrow{\mathcal{L}} & L^2(\mathbb{R}_+) \\ \big\downarrow \mathcal{F}^* & & \big\uparrow \Phi \\ \mathbb{H}^2_+(\mathbb{T}) & \xrightarrow{\mathcal{U}} & \mathbb{H}^2_+(\mathbb{R}) \end{array} \qquad \begin{array}{ccc} W & \longrightarrow & \mathbf{W} = \Phi \mathbf{T} \Phi^* \\ \big\downarrow & & \big\uparrow \\ T = \mathcal{F}^* W \mathcal{F} & \longrightarrow & \mathbf{T} = \mathcal{U} T \mathcal{U}^* \end{array} \tag{2.28}$$

Of course all the remarks above concerning a certain difference between the discrete and continuous representations of convolution operators apply also to Wiener-Hopf operators. The case of semibounded Wiener-Hopf operators is specially discussed in [23] (the discrete representation) and in [24] (the continuous representation).

2.5. Let us say a few words about bounded operators. For Wiener-Hopf operators W defined via the quadratic form (2.25), it is the classical Toeplitz result that the operator W is bounded, that is,

$$|w[f,f]| \le C \|f\|^2$$

for some $C > 0$, if and only if $a = \mathcal{F} b$ where $b \in L^\infty(\mathbb{T})$. The corresponding result for integral Wiener-Hopf operators $\mathbf{W}$ is stated explicitly in [24]. In the assertion below the Fourier transform is understood in the sense of the Schwartz space $\mathcal{S}'$.

Proposition 2.1 ([24, Theorem 1.1])**.** *Let the form* $\mathbf{w}[\mathbf{f},\mathbf{f}]$ *be defined by the relation* (2.26) *where* $\mathbf{f} \in C_0^\infty(\mathbb{R}_+)$ *and the distribution* $\mathbf{a} \in C_0^\infty(\mathbb{R})'$. *Then*

$$|\mathbf{w}[\mathbf{f},\mathbf{f}]| \le C \|\mathbf{f}\|^2,$$

(*so that the corresponding operator* $\mathbf{W}$ *is bounded*) *if and only if* $\mathbf{a} = (2\pi)^{-1/2} \Phi \mathbf{b}$ *where* $\mathbf{b} \in L^\infty(\mathbb{R})$. *Moreover,* $\|\mathbf{W}\| = \|\mathbf{b}\|_{\mathbf{L}^\infty(\mathbb{R})}$.

Observe that Proposition 2.1 is not a direct consequence of the Toeplitz criterion for the boundedness of the operators W in the space $\ell^2(\mathbb{Z}_+)$. The difference is that the domains $\mathcal{D}$ and $C_0^\infty(\mathbb{R}_+)$ of the corresponding quadratic forms are not linked by the Laguerre transform $\mathcal{L}$.

3. HANKEL OPERATORS

3.1. Let us pass to Hankel operators. We start with the space $\ell^2(\mathbb{Z}_+)$ where Hankel operators H act according to the formula (cf. (2.24))

$$(Hf)_n = \sum_{m\in\mathbb{Z}_+} a_{n+m} f_m, \quad f = \{f_n\}_{n\in\mathbb{Z}_+},$$

and their quadratic forms (cf. (2.25)) are defined by the formula

$$h[f,f] = \sum_{n,m\in\mathbb{Z}_+} a_{n+m} f_m \overline{f_n}, \tag{3.1}$$

where $f \in \mathcal{D}_+ = \mathcal{D} \cap \ell^2(\mathbb{Z}_+)$ and the sequence $a = \{a_n\}_{n\in\mathbb{Z}_+}$ is arbitrary.

A Hankel operator $\mathbf{H}$ acts in the space $L^2(\mathbb{R}_+)$ by the formula

$$(\mathbf{Hf})(t) = \int_{\mathbb{R}_+} \mathbf{a}(t+s)\mathbf{f}(s)ds,$$

and its quadratic form is given by the equality

$$\mathbf{h}[\mathbf{f},\mathbf{f}] = \int_{\mathbb{R}_+} \mathbf{a}(t)\Big(\int_0^t \mathbf{f}(s)\overline{\mathbf{f}(t-s)}ds\Big)dt. \tag{3.2}$$

Since, for all test functions $\mathbf{f} \in C_0^\infty(\mathbb{R}_+)$, the function

$$\mathbf{F}(t) = \int_0^t \mathbf{f}(s)\overline{\mathbf{f}(t-s)}ds \tag{3.3}$$

also belongs to $C_0^\infty(\mathbb{R}_+)$, the form (3.2) is correctly defined for all distributions $\mathbf{a} \in C_0^\infty(\mathbb{R}_+)'$ in the space dual to $C_0^\infty(\mathbb{R}_+)$.

Let us now pass to the representation of Hankel operators in the Hardy spaces $\mathbb{H}^2_+(\mathbb{T})$ and $\mathbb{H}^2_+(\mathbb{R})$. A Hankel operator G in the space $\mathbb{H}^2_+(\mathbb{T})$ is formally defined by the relation

$$Gu = P_+ BJu,$$

where $(Ju)(\mu) = \bar{\mu}u(\bar{\mu})$ so that the involution $J\colon \mathbb{H}^2_\pm(\mathbb{T}) \to \mathbb{H}^2_\mp(\mathbb{T})$. As before B is the multiplication operator (2.12) by the function $b(\mu)$ in the space $L^2(\mathbb{T})$. To be precise, we define G via its quadratic form

$$g[u,u] = \int_{\mathbb{T}} \omega(\mu)u(\bar{\mu})\overline{u(\mu)}d\mathbf{m}_0(\mu), \tag{3.4}$$

where $\omega(\mu) = \bar{\mu}b(\mu)$, $u \in \mathcal{P}_+ = \mathcal{F}^*\mathcal{D}_+$ and $\omega \in \mathcal{P}'_+$. Obviously, we have

$$g[u,u] = h[\mathcal{F}u, \mathcal{F}u] \quad \text{if} \quad \omega = \mathcal{F}^* a. \tag{3.5}$$

Hankel operators in the Hardy space $\mathbb{H}^2_+(\mathbb{R}) \subset L^2(\mathbb{R})$ of functions analytic in the upper half-plane are defined quite similarly. Let, as before, $\mathbf{P}_+$ be the orthogonal projection in $L^2(\mathbb{R})$ onto $\mathbb{H}^2_+(\mathbb{R})$, and let $\mathbf{B}$ be the operator (2.15). A Hankel operator $\mathbf{G}$ is formally defined in the space $\mathbb{H}^2_+(\mathbb{R})$ by the relation

$$\mathbf{G}\mathbf{u} = \mathbf{P}_+\mathbf{B}\mathbf{J}\mathbf{u},$$

where $(\mathbf{J}\mathbf{u})(\lambda) = \mathbf{u}(-\lambda)$. To be precise, we have to pass to the Fourier transform in (3.2) which yields the representation

$$\mathbf{g}[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}} \Omega(\lambda)\mathbf{u}(-\lambda)\overline{\mathbf{u}(\lambda)}d\lambda, \quad \mathbf{u} \in \mathcal{Z}_+, \tag{3.6}$$

for the quadratic form of the operator $\mathbf{G}$. Obviously, we have

$$\mathbf{g}[\mathbf{u},\mathbf{u}] = \mathbf{h}[\Phi\mathbf{u},\Phi\mathbf{u}] \quad \text{if} \quad \Omega = (2\pi)^{1/2}\Phi^*\mathbf{a}. \tag{3.7}$$

The functions $\omega(\mu)$ and $\Omega(\lambda)$ are known as symbols of the discrete and continuous Hankel operators. Making the change of the variables $\mu = (\lambda - i\alpha)(\lambda + i\alpha)^{-1}$ in (3.4) we see that

$$\mathbf{g}[\mathcal{U}u,\mathcal{U}u] = g[u,u]$$

if

$$\Omega(\lambda) = -\big(\tfrac{\lambda-i\alpha}{\lambda+i\alpha}\big)\omega\big(\tfrac{\lambda-i\alpha}{\lambda+i\alpha}\big). \tag{3.8}$$

For Hankel operators G and $\mathbf{G}$, this equality plays the role of (2.17). Since $\mathbf{a} = (2\pi)^{-1/2}\Phi\Omega$ and $\omega = \mathcal{F}^*a$, relation (2.19) where $x > 0$ yields the representation (cf. (2.20))

$$\mathbf{a}(t) = 2\alpha\sum_{n=0}^{\infty} a_n \mathsf{L}_n^1(2\alpha t)e^{-\alpha t}, \quad t > 0. \tag{3.9}$$

Let us now use that $(n+1)^{-1/2}2\alpha t^{1/2}\mathsf{L}_n^1(2\alpha t)e^{-\alpha t}$, $n = 0, 1, \ldots$, is the orthonormal basis in the space $L^2(\mathbb{R}_+)$. Therefore, it follows from (3.9) that

$$a_n = 2\alpha(n+1)^{-1}\int_{\mathbb{R}_+} \mathbf{a}(t)\mathsf{L}_n^1(2\alpha t)e^{-\alpha t}tdt. \tag{3.10}$$

Similarly to Toeplitz operators (cf. (2.28)), the relations between various representations of Hankel operators can be summarized by the following diagram:

$$\begin{array}{ccc} H & \longrightarrow & \mathbf{H} = \Phi\mathbf{H}\Phi^* \\ \Big\downarrow & & \Big\uparrow \\ G = \mathcal{F}^*H\mathcal{F} & \longrightarrow & \mathbf{G} = \mathcal{U}G\mathcal{U}^* \end{array}$$

It is easy to see that Hankel operators H, $\mathbf{H}$, G and $\mathbf{G}$ are formally symmetric if

$$a_n = \bar{a}_n, \quad \mathbf{a}(t) = \overline{\mathbf{a}(t)}, \quad \omega(\bar{\mu}) = \overline{\omega(\mu)} \qquad \text{and} \quad \Omega(-\lambda) = \overline{\Omega(\lambda)}.$$

To be more precise, this means that the corresponding quadratic forms are real.

Similarly to the case of convolutions and Toeplitz operators, the relations (3.9) and (3.10) are only formal. We also note that the domains $\mathcal{D}_+$ and $C_0^\infty(\mathbb{R}_+)$ of the forms (3.1) and (3.2) are not related by the Laguerre transform (2.27); likewise, the domains $\mathcal{P}_+$ and $\mathcal{Z}_+$ of the forms (3.4) and(3.6) are not related by the change of variables (2.3). We emphasize however that this discrepancy is essential in singular cases only.

3.2. As far as the conditions of boundedness are concerned, the results on Hankel operators are formally similar to those on Toeplitz operators. For Hankel operators H defined via the quadratic form (3.1), it is the classical Nehari result that the operator H is bounded if and only if there exists $\omega \in L^\infty(\mathbb{T})$ such that $a = \mathcal{F}\omega$. An important difference compared to Toeplitz operators is that now the equation $a = \mathcal{F}\omega$ does not determine the function ω uniquely and may also be satisfied with unbounded ω.

The corresponding result for integral Hankel operators $\mathbf{H}$ is stated explicitly in [18].

Proposition 3.1 ([18, Theorem 3.4])**.** *Let the form* $\mathbf{h}[\mathbf{f}, \mathbf{f}]$ *be defined by the relation* (3.2) *where the distribution* $\mathbf{a} \in C_0^\infty(\mathbb{R}_+)'$. *Then the corresponding operator* $\mathbf{H}$ *is bounded if and only if there exists a function* $\Omega \in L^\infty(\mathbb{R})$ *such that*

$$\mathbf{a} = (2\pi)^{-1/2}\Phi\Omega.$$

In this case $\|\mathbf{H}\| = \|\Omega\|_{L^\infty(\mathbb{R})}$.

By the same reasons as for Toeplitz operators (see the remark after Proposition 2.1), this result is not a direct consequence of the Nehari theorem.

Of course, the operators $\mathcal{F}$, Φ, $\mathcal{U}$ and $\mathcal{L}$ establishing the equivalence of various representations of Hankel operators are not unique. For example, the operator $\mathcal{U} = \mathcal{U}_\alpha$ defined by equality (2.3) depends on the parameter $\alpha > 0$, and there is no distinguished choice of this parameter. To state the problem precisely, for each of the spaces $\mathcal{H} = \ell^2(\mathbb{Z}_+), L^2(\mathbb{R}_+), \mathbb{H}^2_+(\mathbb{T}), \mathbb{H}^2_+(\mathbb{R})$, let us introduce the group $\mathbb{G}(\mathcal{H})$ of all unitary automorphisms of the set $\mathbb{A}(\mathcal{H})$ of bounded Hankel operators in the space $\mathcal{H}$. By definition, a unitary operator $\mathsf{U} \in \mathbb{G}(\mathcal{H})$ if and only if $\mathsf{U}\mathsf{H}\mathsf{U}^* \in \mathbb{A}(\mathcal{H})$ for all operators $\mathsf{H} \in \mathbb{A}(\mathcal{H})$. For different $\mathcal{H}$, the groups $\mathbb{G}(\mathcal{H})$ are related by the transformations $\mathcal{F}$, Φ, $\mathcal{U}$ and $\mathcal{L}$. So it suffices to describe this group for one of the choices of $\mathcal{H}$.

It turns out that the group $\mathbb{G}(\mathbb{H}^2_+(\mathbb{R}))$ admits a very explicit description. Let D_ρ,

$$(\mathsf{D}_\rho \mathbf{u})(\lambda) = \rho^{1/2}\mathbf{u}(\rho\lambda), \quad \rho > 0,$$

be the dilation operator in $\mathbb{H}^2_+(\mathbb{R})$, and let the involution J be defined in this space by the equation

$$(\mathsf{J}u)(\lambda) = i\lambda^{-1}u(-\lambda^{-1}).$$

Obviously, $\mathsf{D}_\rho \mathbf{G} \mathsf{D}_\rho^*$ and $\mathsf{J}\mathbf{G}\mathsf{J}^*$ are Hankel operators for all Hankel operators $\mathbf{G} \in \mathbb{A}(\mathbb{H}^2_+(\mathbb{R}))$. Surprisingly, the group $\mathbb{G}(\mathbb{H}^2_+(\mathbb{R}))$ is exhausted by these transformations. Let us state the precise result.

Theorem 3.2 ([20, Theorem A.1])**.** *A unitary operator* $\mathsf{U} \in \mathbb{G}(\mathbb{H}^2_+(\mathbb{R}))$ *if and only if it has one of the two forms:* $\mathsf{U} = \mu\mathsf{D}_\rho$ *or* $\mathsf{U} = \mu\mathsf{D}_\rho\mathsf{J}$ *for some* $\mu \in \mathbb{T}$ *and* $\rho > 0$.

3.3. Now we consider the case when the matrix elements a_n of a Hankel operator H and the integral kernel $\mathbf{a}(t)$ of a Hankel operator $\mathbf{H}$ are given by formulas (1.3) and (1.4), respectively. Here $dM(z)$ is a finite complex measure on $\operatorname{clos}\mathbb{D}$ and $d\Sigma(\zeta)$ is a locally finite complex measure on $\operatorname{clos}\mathbb{C}^+$. We suppose that the measures $dM(z)$ and $d\Sigma(\zeta)$ are linked by equation (1.5); in this case $M(\{1\}) = 0$. We denote by $d|M|(z)$ and $d|\Sigma|(\zeta)$ the variations of these measures and assume that

$$|M|(\operatorname{clos}\mathbb{D}) = 2\alpha \int\limits_{\operatorname{clos}\mathbb{C}^+} |\zeta + \alpha|^{-2} d|\Sigma|(\zeta) < \infty. \tag{3.11}$$

Relation (1.3) only implies that the sequence a_n is bounded as $n \to \infty$, and hence the operator H is not defined in $\ell^2(\mathbb{Z}_+)$ even on the set $\mathcal{D}$. Similarly, the operator $\mathbf{H}$ is not defined in $L^2(\mathbb{R}_+)$ even on the set $C_0^\infty(\mathbb{R}_+)$. So as usual, instead of operators we have to work with the corresponding quadratic forms (3.1) and (3.2).

Formula (1.4) defines $\mathbf{a}(t)$ as a distribution on a set of test functions, denoted $\mathcal{X}$, that can be chosen as follows. A function $\mathbf{F} \in \mathcal{X}$ if and only if $\mathbf{F} \in C^\infty(\mathbb{R}_+)$, there exist limits $\mathbf{F}^{(k)}(+0)$ for $k = 0, 1, 2$, $\mathbf{F}(+0) = 0$ and $\mathbf{F}^{(k)} \in L^1(\mathbb{R}_+)$ for $k = 0, 1, 2$. Integrating twice by parts, we find that, for such $\mathbf{F}$, the estimate

$$\Big| \int\limits_0^\infty e^{-\zeta t} \mathbf{F}(t) dt \Big| \leq C|\zeta + 1|^{-2}, \quad \zeta \in \operatorname{clos}\mathbb{C}^+,$$

holds. Therefore, the form

$$\langle \mathbf{a}, \mathbf{F} \rangle := \int\limits_{\operatorname{clos}\mathbb{C}^+} \Big(\int\limits_0^\infty e^{-\zeta t} \mathbf{F}(t) dt \Big) d\Sigma(\zeta) \tag{3.12}$$

is well defined for all $\mathbf{F} \in \mathcal{X}$.

Let us also introduce a set of test functions $\mathbf{f} \in C^\infty(\mathbb{R}_+)$, denoted $\mathcal{Y}$, such that there exist limits $\mathbf{f}^{(k)}(+0)$ and $\mathbf{f}^{(k)} \in L^1(\mathbb{R}_+)$ for $k = 0, 1, 2$. By definition (3.3), we have $\mathbf{F} \in \mathcal{X}$ if $\mathbf{f} \in \mathcal{Y}$. Therefore if $\mathbf{h} \in \mathcal{X}'$, then the quadratic form $\mathbf{h}[\mathbf{f}, \mathbf{f}]$ is well defined for all $\mathbf{f} \in \mathcal{Y}$. Note that $\mathbf{D}_+ := \mathcal{L}\mathcal{D}_+ \subset \mathcal{Y}$.

Our main result in this subsection formally means that the "operators" H and $\mathbf{H}$ are unitarily equivalent provided the corresponding measures are linked by the equality (1.5). Let us state this result precisely.

Theorem 3.3. *Let the matrix elements a_n and the integral kernel $\mathbf{a}(t)$ be given by equalities* (1.3) *and* (1.4)*, and let $h[f, f]$ and $\mathbf{h}[\mathbf{f}, \mathbf{f}]$ be the corresponding quadratic forms* (3.1) *and* (3.2) *defined for $f \in \mathcal{D}_+$ and $\mathbf{f} \in \mathcal{Y}$. Suppose that the measures $dM(z)$ and $d\Sigma(\zeta)$ are linked by the relation* (1.5) *for some $\alpha > 0$ and satisfy the condition* (3.11)*. Then for all $f \in \mathcal{D}_+$ the identity*

$$h[f, f] = \mathbf{h}[\mathcal{L}f, \mathcal{L}f] \tag{3.13}$$

holds.

Proof. Let $f \in \mathcal{D}_+$ and $\mathbf{f} = \mathcal{L}f \in \mathcal{Y}$. It follows from formulas (3.3) and (3.12) that in this case

$$\mathbf{h}[\mathbf{f},\mathbf{f}] = \int\limits_{\operatorname{clos}\mathbb{C}^+} d\Sigma(\zeta)\Big(\int\limits_0^\infty e^{-\zeta t}\overline{\mathbf{f}(t)}dt\Big)\Big(\int\limits_0^\infty e^{-\zeta t}\mathbf{f}(s)ds\Big). \tag{3.14}$$

Moreover, in view of the identity (2.5) and definition (2.27), we have

$$\int\limits_0^\infty e^{-\zeta t}\mathbf{f}(t)dt = \frac{-i\sqrt{2\alpha}}{\zeta+\alpha}\sum_{n=0}^\infty f_n\Big(\frac{\zeta-\alpha}{\zeta+\alpha}\Big)^n$$

where the sum consists of a finite number of terms. Substituting this expression into relation (3.14), we find that

$$\mathbf{h}[\mathbf{f},\mathbf{f}] = 2\alpha\sum_{n,m=0}^\infty f_m\overline{f_n}\int\limits_{\operatorname{clos}\mathbb{C}^+}\Big(\frac{\zeta-\alpha}{\zeta+\alpha}\Big)^{n+m}\frac{1}{(\zeta+\alpha)^2}d\Sigma(\zeta).$$

After the change of the variables $z = \frac{\zeta-\alpha}{\zeta+\alpha}$, we see that this expression equals

$$\mathbf{h}[\mathbf{f},\mathbf{f}] = \sum_{n,m=0}^\infty f_m\overline{f_n}\int\limits_{\operatorname{clos}\mathbb{D}} z^{n+m}dM(z)$$

where the measure $dM(z)$ is defined by relation (1.5). According to (1.3) this expression coincides with (3.1). This concludes the proof of the identity (3.13). ☐

Corollary 3.4. *Suppose that the expression* (3.13) *is estimated by* $C\|f\|^2$ *with some* $C>0$. *Then there exist bounded operators* H *and* $\mathbf{H}$ *corresponding to these quadratic forms, and they are unitarily equivalent:* $\mathbf{H}\mathcal{L} = \mathcal{L}H$.

Obviously, coefficients (1.3) and function (1.4) are real if the measures $dM(z)$ and $d\Sigma(\zeta)$ are invariant with respect to the complex conjugation:

$$M(\overline{X}) = \overline{M(X)} \quad \text{and} \quad \Sigma(\overline{Y}) = \overline{\Sigma(Y)} \tag{3.15}$$

for all $X \subset \operatorname{clos}\mathbb{D}$ and $Y \subset \operatorname{clos}\mathbb{C}^+$.

Corollary 3.5. *Under the assumptions of Theorem 3.3, let condition* (3.15) *be satisfied. Suppose that an operator* H *is defined on* $\mathcal{D}_+$ *and* $(Hf,f) = h[f,f]$ *for* $f \in \mathcal{D}_+$, *or equivalently an operator* $\mathbf{H}$ *is defined on* $\mathbf{D}_+$ *and* $(\mathbf{H}\mathbf{f},\mathbf{f}) = \mathbf{h}[\mathbf{f},\mathbf{f}]$ *for* $\mathbf{f} \in \mathbf{D}_+$. *If* H *is essentially self-adjoint on* $\mathcal{D}_+$ (*or equivalently* $\mathbf{H}$ *is essentially self-adjoint on* $\mathbf{D}_+$), *then their closures are unitarily equivalent:* $(\operatorname{clos}\mathbf{H})\,\mathcal{L} = \mathcal{L}(\operatorname{clos}H)$.

In particular, the measures $dM(z)$ and $d\Sigma(\zeta)$ may be supported by the intervals $[-1,1]$ and $[0,\infty)$, respectively; as before we suppose also that $M(\{1\}) = 0$. Then Theorem 3.3 reads as follows.

Theorem 3.6. *Let the matrix elements a_n and the kernel $\mathbf{a}(t)$ be given by the equalities*

$$a_n = \int_{-1}^{1} \nu^n dM(\nu), \quad n = 0, 1, \ldots, \tag{3.16}$$

and

$$\mathbf{a}(t) = \int_0^\infty e^{-\lambda t} d\Sigma(\lambda), \quad t > 0. \tag{3.17}$$

Suppose that the measures $dM(\nu)$ and $d\Sigma(\lambda)$ are linked by the relation

$$dM(\nu) = 2\alpha(\lambda+\alpha)^{-2} d\Sigma(\lambda), \quad \nu = \frac{\lambda-\alpha}{\lambda+\alpha}, \tag{3.18}$$

for some $\alpha > 0$ and satisfy the condition

$$|M|([-1,1)) = 2\alpha \int_0^\infty |\lambda+\alpha|^{-2} d|\Sigma|(\lambda) < \infty.$$

Then for all $f \in \mathcal{D}_+$ the identity (3.13) *holds.*

The following result is a combination of Theorem 3.6, of Theorems 1.2 and 3.4 in [22] on the discrete case and of the preceding results (Theorem 3.10) of [19] on the continuous case.

Theorem 3.7. *Under the assumptions of Theorem 3.6 suppose that the measure* (3.18) *is non-negative and that $M(\{-1\}) = \Sigma(\{0\}) = 0$. Then the following assertions hold.*

(i) *The form $h[f,f]$ defined on $\mathcal{D}_+$ is closable, and it is closed on the set of elements $f = (f_0, f_1, \ldots) \in \ell^2(\mathbb{Z}_+)$ such that*

$$h[f,f] = \int_{-1}^{1} \Big| \sum_{n=0}^{\infty} f_n \nu^n \Big|^2 dM(\nu) < \infty.$$

(ii) *The form $\mathbf{h}[\mathbf{f},\mathbf{f}]$ defined on $\mathbf{D}_+$ is closable, and it is closed on the set of functions $\mathbf{f} \in L^2(\mathbb{R}_+)$ such that*

$$\mathbf{h}[\mathbf{f},\mathbf{f}] = \int_0^\infty \Big| \int_0^\infty e^{-\lambda t} \mathbf{f}(t) dt \Big|^2 d\Sigma(\lambda) < \infty.$$

(iii) *The non-negative operators H and $\mathbf{H}$ corresponding to these quadratic forms are unitarily equivalent: $\mathbf{H}\mathcal{L} = \mathcal{L}H$.*

Sometimes the representations (1.3) and (1.4) are too restrictive. For example, if a Hankel operator $\mathbf{H}$ has kernel $a(t) = t^k e^{-\alpha t}$, $\operatorname{Re}\alpha > 0$ (such $\mathbf{H}$ has a finite rank), then representation (1.4) is formally satisfied with

$$d\Sigma(\zeta) = \delta^{(k)}(\zeta - \alpha) d\mathbf{M}_0(\zeta)$$

($d\mathbf{M}_0(\zeta)$ is the planar Lebesgue measure and $\delta(\zeta)$ is the delta-function) which is a measure for $k = 0$ only. A very general situation where $\mathbf{a} \in C_0^\infty(\mathbb{R}_+)'$ is an arbitrary distribution was considered in [16]. Then the role of the measure $d\Sigma(\zeta)$ is played also by a distribution which was called the sigma-function of the Hankel operator $\mathbf{H}$.

3.4. Although quite simple, Theorem 3.3 is very useful for relating the discrete and continuous representations. Let us now discuss the case of bounded Hankel operators. Let $D(z_0, r) = \{|z - z_0| < r\}$ be the disc in the complex plane $\mathbb{C}$. Recall that $dM(z)$ is called the *Carleson measure* on the unit disc $\mathbb{D}$ if

$$\sup_{\mu \in \mathbb{T}, r \in (0, r_0)} r^{-1} |M|(D(\mu, r) \cap \mathbb{D}) < \infty \tag{3.19}$$

where r_0 is some fixed small number. Similarly, $d\Sigma(\zeta)$ is called the Carleson measure on the right half-plane $\mathbb{C}^+$ if

$$\sup_{\lambda \in \mathbb{R}, R > 0} R^{-1} |\Sigma|(D(i\lambda, R) \cap \mathbb{C}^+) < \infty. \tag{3.20}$$

Theorem 3.8. *Let the measures $dM(z)$ and $d\Sigma(\zeta)$ be linked by the equation* (1.5)*. Then $dM(z)$ is a Carleson measure on the unit disc $\mathbb{D}$ if and and only if $d\Sigma(\zeta)$ is a Carleson measure on the half-plane $\mathbb{C}^+$.*

This assertion is checked in the Appendix by straightforward but rather tedious calculations. Theorem 3.8 can also be indirectly deduced from general results on analytic functions (see, e.g., Section E in Chapter VIII of [7]).

Theorem 3.9. *A Hankel operator H in the space $\ell^2(\mathbb{Z}_+)$ is bounded if and only if its matrix elements admit the representation* (1.3) *where $dM(z)$ is some Carleson measure on the unit disc $\mathbb{D}$.*

This result is stated as Theorem 7.4 in Chapter 1 of the book [12] and can be easily deduced from Theorem A2.12 of [12] where the representation of the symbol of a bounded Hankel operator as the Poisson balayage of some Carleson measure is given. For the proof of the latter result, we refer to Section G in Chapter X of the book [7] or pages 271, 272 of the book [3].

Putting together Theorems 3.3, 3.9 and Lemma 3.8, we get the following result.

Theorem 3.10. *A Hankel operator $\mathbf{H}$ in the space $L^2(\mathbb{R}_+)$ is bounded if and only if its kernel admits the representation* (1.4) *with some Carleson measure $d\Sigma(\zeta)$ on the right half-plane $\mathbb{C}^+$.*

It also follows from Corollary 3.4 that if $dM(z)$ and $d\Sigma(\zeta)$ in the equation (1.5) are Carleson measures, then the corresponding operators H and $\mathbf{H}$ are unitarily equivalent.

Of course the representations (1.3) and (1.4) are highly non-unique. Let us give a simple example.

Example 3.11. Let $d\mathbf{M}_0(z)$ be the Lebesgue measure on $\mathbb{D}$, and let the measure $dN(z)$ be supported by the point 0 so that $N(\{0\}) = 1$ and $N(\mathbb{D} \setminus \{0\}) = 0$. Put

$$dM(z) = d\mathbf{M}_0(z) - \pi dN(z).$$

Then

$$\int_{\mathbb{D}} z^n dM(z) = \int_{\mathbb{D}} z^n d\mathbf{M}_0(z) = \int_0^1 dr r^{n+1} \int_0^{2\pi} e^{in\theta} d\theta = 0 \quad \text{if} \quad n \geq 1,$$

and

$$\int_{\mathbb{D}} dM(z) = \int_{\mathbb{D}} d\mathbf{M}_0(z) - \pi = 0.$$

3.5. In particular, the measures $dM(z)$ and $d\Sigma(\zeta)$ may be carried by the intervals $(-1, 1)$ and $(0, \infty)$, respectively. Then conditions (3.19) and (3.20) mean that

$$|M|((1-\varepsilon, 1)) = O(\varepsilon) \quad \text{and} \quad |M|((-1, -1+\varepsilon)) = O(\varepsilon) \quad \text{as} \quad \varepsilon \to 0 \tag{3.21}$$

and

$$\sup_{R>0} R^{-1} |\Sigma|(0, R) < \infty. \tag{3.22}$$

For non-negative Hankel operators, the results of the previous subsection can be stated in a simpler and more definite form. Recall that the Hankel form $h[f, f]$ was defined by relation (3.1). The following assertion is the classical result of H. Widom.

Theorem 3.12 ([15, Theorem 3.1])**.** *Suppose that $h[f, f] \geq 0$ for all $f \in \mathcal{D}_+$. Then the following conditions are equivalent:*

(i) *The operator H is bounded.*
(ii) *The representation* (3.16) *holds with a non-negative measure $dM(\nu)$ satisfying condition* (3.21)*.*
(iii) *$a_n = O(n^{-1})$ as $n \to \infty$.*

Let us now state a continuous analogue of Theorem 3.12.

Theorem 3.13. *Let $\mathbf{a} \in C_0^\infty(\mathbb{R}_+)'$. Suppose that $\mathbf{h}[\mathbf{f}, \mathbf{f}] \geq 0$ for all $\mathbf{f} \in C_0^\infty(\mathbb{R}_+)$. Then the following conditions are equivalent:*

(i) *The operator $\mathbf{H}$ is bounded.*
(ii) *The representation* (3.17) *holds with a non-negative measure $d\Sigma(\lambda)$ satisfying condition* (3.22)*.*
(iii) *$\mathbf{a}(t) = O(t^{-1})$ as $t \to \infty$ and $t \to 0$.*

Proof. According to Theorem 5.1 in [16] the condition $\mathbf{h}[\mathbf{f}, \mathbf{f}] \geq 0$ implies that, for some non-negative measure $d\Sigma(\lambda)$ on $\mathbb{R}$,

$$\mathbf{a}(t) = \int_{-\infty}^{\infty} e^{-\lambda t} d\Sigma(\lambda) \tag{3.23}$$

where the integral converges for all $t > 0$. If $\Sigma(\mathbb{R} \setminus \mathbb{R}_+) > 0$, then $\mathbf{a}(t) \geq c > 0$ so that the operator $\mathbf{H}$ cannot be bounded. Thus representation (3.23) reduces to (3.17). Let the measures $d\Sigma(\lambda)$ and $dM(\nu)$ be linked by equation (3.18). By Lemma 3.8 (which is quite easy in this particular case) the conditions (3.21) and (3.22) are equivalent and hence, by Theorem 3.3, the operator $\mathbf{H}$ is bounded if and only if the operator H with matrix elements (3.16) is bounded. So the conditions (i) and (ii) are equivalent according to Theorem 3.12.

It follows from (3.23) that $\mathbf{a} \in C^\infty(\mathbb{R}_+)$. So under assumption (iii) the operator $\mathbf{H}$ is bounded because the Carleman operator (it has integral kernel $\mathbf{a}(t) = t^{-1}$) is bounded. Conversely, integrating in (3.17) by parts we see that

$$\mathbf{a}(t) = t \int_0^\infty e^{-\lambda t} \Sigma(\lambda) d\lambda.$$

Therefore condition (3.22) implies (iii). □

Of course under the assumptions of Theorems 3.12 and 3.13 the measures $dM(\nu)$ and $d\Sigma(\lambda)$ are unique.

3.6. By their definitions, Carleson measures are carried by the open sets $\mathbb{D}$ or $\mathbb{C}^+$. Let us now consider the opposite case when the measure $dM(z)$ is supported on the unit circle $\mathbb{T}$ and the corresponding measure $d\Sigma(\zeta)$ is supported on the line $\operatorname{Re}\zeta = 0$. Then relations (1.3), (1.4) and (1.5) read, respectively, as

$$a_n = \int_{\mathbb{T}} \mu^n dM(\mu), \quad n = 0, 1, \ldots, \tag{3.24}$$

$$\mathbf{a}(t) = \int_{\mathbb{R}} e^{-i\lambda t} d\Sigma(i\lambda), \quad t > 0, \tag{3.25}$$

and

$$dM(\mu) = -2\alpha(\lambda - i\alpha)^{-2} d\Sigma(i\lambda), \quad \mu = \frac{\lambda + i\alpha}{\lambda - i\alpha}. \tag{3.26}$$

In particular, if the measures $dM(\mu)$ on $\mathbb{T}$ and $d\Sigma(i\lambda)$ on $\mathbb{R}$ are absolutely continuous, that is, $dM(\mu) = \omega(\bar{\mu}) d\mathbf{m}_0(\mu)$ and $d\Sigma(i\lambda) = (2\pi)^{-1}\Omega(\lambda)d\lambda$, then (3.24), (3.25) give the representations of the matrix elements a_n of H and the integral kernel $\mathbf{a}(t)$ of $\mathbf{H}$ in terms of their symbols $\omega(\mu)$ and $\Omega(\lambda)$ (see formulas (3.5) and (3.7)). In this case (3.26) yields the standard relation (3.8) between these symbols. We recall that by the Nehari theorem [10], the operators H or $\mathbf{H}$ are bounded if and only if the symbols ω or Ω can be chosen as bounded functions, but the construction above does not require this condition.

4. SINGULAR CASE

4.1. In this section, we consider *signed* real measures $d\Sigma(\lambda)$ and $dM(\nu)$ satisfying the assumptions of Theorem 3.6, but we do not assume that they satisfy the Carleson conditions (3.21) or (3.22). We suppose that these measures are absolutely continuous. Then (3.16) and (3.17) read as

$$\mathbf{a}(t) = \int_0^\infty e^{-\lambda t}\sigma(\lambda)d\lambda \tag{4.1}$$

and

$$a_n = \int_{-1}^{1} \nu^n \eta(\nu)d\nu, \quad n = 0, 1, \dots, \tag{4.2}$$

with some real functions σ such that

$$\int_0^\infty (\lambda+1)^{-2}|\sigma(\lambda)|d\lambda < \infty$$

and $\eta \in L^1(-1, 1)$. The relation (3.18) yields

$$\eta(\nu) = \sigma\Big(\alpha\frac{1+\nu}{1-\nu}\Big). \tag{4.3}$$

It turns out that even this particular case leads to interesting examples. Our plan is to use the results obtained in [19, 21] for integral Hankel operators $\mathbf{H}$, and then with the help of Theorem 3.6 to state new results for matrix Hankel operators H.

Let us first illustrate our approach on the Carleman operator and the Hilbert matrix already discussed in Subsection 1.3. If $\sigma(\lambda) = 1$, then according to (4.1) we have $\mathbf{a}(t) = t^{-1}$ which yields the Carleman operator $\mathbf{H}$. It follows from (4.3) that $\eta(\nu) = 1$ and hence, by (4.2), the corresponding "discrete" Hankel operator $H = \mathcal{L}^*\mathbf{H}\mathcal{L}$ has matrix elements

$$a_n = (1 + (-1)^n)(n + 1)^{-1}.$$

Let $\mathbb{1}_X$ be the characteristic function of a set $X \subset \mathbb{R}$. Suppose that $\eta(\nu) = \mathbb{1}_{(0,1)}(\nu)$. Then according to (4.2) we have $a_n = (n + 1)^{-1}$ which yields the Hilbert matrix H. It follows from (4.3) that $\sigma(\lambda) = \mathbb{1}_{(\alpha,\infty)}(\lambda)$ so that the corresponding "continuous" Hankel operator $\mathbf{H} = \mathcal{L}H\mathcal{L}^*$ has integral hernel $\mathbf{a}(t) = t^{-1}e^{-\alpha t}$.

Our goal is to study the case when the sigma-function $\eta(\nu)$ is singular at the points $\nu = \pm 1$ and the Hankel operator H with matrix elements (4.2) is unbounded. We will consider two families of such Hankel operators. The first family admits an explicit spectral analysis (Theorem 4.3). Spectral information about the second family is more limited (Theorem 4.6).

4.2. First we consider functions $\eta(\nu)$ with arbitrary logarithmic singularities at the points $\nu = 1$ and $\nu = -1$. To be precise, we assume that

$$\eta(\nu) = \sum_{l=0}^{p} \gamma_l \ln^l\Big(\alpha\frac{1+\nu}{1-\nu}\Big), \quad p \ge 1, \tag{4.4}$$

where γ_l, $l = 0, 1, \dots, p$, are any real numbers. Without loss of generality, we set $\gamma_p = 1$. According to (4.2) the matrix elements of the corresponding Hankel operator (1.1) are given by the formulas

$$a_n = \sum_{l=0}^{p} \gamma_l \int_{-1}^{1} \nu^n \ln^l \Big(\alpha \frac{1+\nu}{1-\nu}\Big) d\nu. \tag{4.5}$$

Note (see Proposition 4.5, below) that

$$a_n = (1 + (-1)^{n+p}) \frac{\ln^p n}{n} \Big(1 + O\Big(\frac{1}{\ln n}\Big)\Big) \tag{4.6}$$

as $n \to \infty$. It can be expected that such Hankel operator H is unbounded because $\eta(\nu)d\nu$ is not the Carleson measure and $|a_n|n \to \infty$ as $n \to \infty$. Nevertheless, since $a_n = \bar{a}_n$ and $\{a_n\}_{n \in \mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+)$, the operator H is well defined and symmetric on the dense set $\mathcal{D}_+ \subset \ell^2(\mathbb{Z}_+)$ of sequences with only a finite number of non-zero components.

Our study of the Hankel operator H with matrix elements (4.5) relies on a combination of Theorem 3.6 with the results of [17, 21] on integral Hankel operators $\mathbf{H}$ in the space $L^2(\mathbb{R}_+)$. In view of (4.3) the sigma-function of $\mathbf{H} = \mathcal{L} H \mathcal{L}^*$ equals

$$\sigma(\lambda) = \sum_{l=0}^{p} \gamma_l \ln^l \lambda, \quad p \geq 1,$$

and its integral kernel $\mathbf{a}(t)$ is given by formula (4.1). Let us define a differential operator

$$L = v \sum_{l=0}^{p} \gamma_l D^l v, \quad D = id/d\xi, \tag{4.7}$$

of order p in the space $L^2(\mathbb{R})$, where v is the operator of multiplication by the universal function

$$v(\xi) = \frac{\sqrt{\pi}}{\sqrt{\cosh(\pi\xi)}}.$$

Let us introduce also a unitary transformation $F : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ by the formula

$$(F\mathbf{f})(\xi) = (2\pi)^{-1/2} \frac{\Gamma(1/2 + i\xi)}{|\Gamma(1/2 + i\xi)|} \int_0^\infty t^{-1/2 - i\xi} \mathbf{f}(t) dt$$

(so it is the Mellin transform, up to an insignificant phase factor). Note that $F\mathbf{f} \in \mathcal{S}$ if $\mathbf{f} \in \mathbf{D}_+ = \mathcal{L}\mathcal{D}_+$.

Putting together Theorem 3.6 and Theorem 3.2 of [17], we can state the following result.

Proposition 4.1. *Let matrix elements of the Hankel operator H be defined by formula* (4.5)*, where the function $\eta(\nu)$ is given by formula* (4.4)*. Then for all $f \in \mathcal{D}_+$, the identity*

$$(Hf, f) = (LF\mathcal{L}f, F\mathcal{L}f)$$

holds.

It is shown in Theorem 3.13 of [17] that the operator L defined on the set $C_0^p(\mathbb{R}_+)$ is essentially self-adjoint. The same arguments work if L is defined on the set $F\mathbf{D}_+$. Therefore Proposition 4.1 allows us to obtain a similar result for the operator H.

Proposition 4.2. *The Hankel operator H with matrix elements* (4.5) *is essentially self-adjoint on the set $\mathcal{D}_+$.*

Our goal is to obtain rather a complete information about the spectral structure of the closure of H which will also be denoted H.

Theorem 4.3. *Let matrix elements of the Hankel operator H be defined by formula* (4.5) *where $\gamma_p = 1$ and γ_ℓ for $\ell \leq p-1$ are arbitrary real numbers. Then the following assertions hold.*

(i) *The spectrum of the operator H is absolutely continuous except eigenvalues that may accumulate to zero and infinity only.*
(ii) *The absolutely continuous spectrum of the operator H covers $\mathbb{R}$ and is simple for p odd. It coincides with $[0,\infty)$ and has multiplicity 2 for p even.*
(iii) *The point 0 is not an eigenvalue of the operator H. If p is odd, then the multiplicities of eigenvalues of the operator H are bounded by $(p-1)/2$. If p is even, then the multiplicities of positive eigenvalues are bounded by $p/2-1$, and the multiplicities of negative eigenvalues are bounded by $p/2$.*

Given Proposition 4.1, this result is a consequence (except the assertion about the point 0 which is proven in Theorem 4.7 of [17]) of the corresponding statement, Theorem 4.8 in [21], for the differential operator L. We emphasize that the method of [21] yields a sufficiently explicit spectral analysis of the operator L and, in particular, information about its eigenfunctions of the continuous spectrum. In view of the unitary equivalence, this yields the corresponding results for the Hankel operator H (and $\mathbf{H} = \mathcal{L} H \mathcal{L}^*$), but we will not dwell upon them.

It remains to justify asymptotic relation (4.6).

Lemma 4.4. *The asymptotic relation*

$$\int_0^1 \nu^n \ln^l (1-\nu) d\nu = \frac{(-\ln n)^l}{n} \Big(1 + O\Big(\frac{1}{\ln n}\Big)\Big). \tag{4.8}$$

as $n \to \infty$ holds.

Proof. Differentiating the formula

$$\int_0^1 \nu^n (1-\nu)^\epsilon d\nu = \frac{\Gamma(n+1)\Gamma(\epsilon+1)}{\Gamma(\epsilon+n+2)} \tag{4.9}$$

l times in ϵ and then putting $\epsilon = 0$, we see that

$$\int_0^1 \nu^n \ln^l (1-\nu) d\nu = \frac{\partial^l}{\partial \epsilon^l} \Big(\frac{\Gamma(n+1)\Gamma(\epsilon+1)}{\Gamma(\epsilon+n+2)}\Big)\Big|_{\epsilon=0}. \tag{4.10}$$

According to formula (1.18.4) in [2] we have

$$\frac{\Gamma(n+1)}{\Gamma(\epsilon+n+2)} = n^{-\epsilon-1}\big(1+O(n^{-1})\big), \tag{4.11}$$

and this asymptotic formula can be infinitely differentiated in ϵ. In view of (4.10), this yields (4.8). □

Integrating separately over the intervals $(-1,0)$, $(0,1)$ and making the change of variables $\nu \mapsto -\nu$ in the integral over $(-1,0)$, we see that

$$\int_{-1}^{1} \nu^n \ln^l \Big(\alpha \frac{1+\nu}{1-\nu}\Big) d\nu = (-1)^l \int_0^1 \nu^n \big(\ln(1-\nu) - \ln(1+\nu) - \ln\alpha\big)^l d\nu$$
$$+ (-1)^n \int_0^1 \nu^n \big(\ln(1-\nu) - \ln(1+\nu) + \ln\alpha\big)^l d\nu.$$

The asymptotic behavior of the integrals on the right is determined by a neighborhood of the point $\nu = 1$ where the function $\ln(1-\nu)$ is singular. The terms with $\ln(1+\nu)$ and $\ln\alpha$ do not give a contribution to the leading term of the asymptotics. Therefore putting together formulas (4.5) (where $\gamma_p = 1$) and (4.8) we obtain the following result.

Proposition 4.5. *Matrix elements* (4.5) *of the Hankel operator H satisfy the asymptotic relation* (4.6)*.*

We finally note that for $p = 1$, the differential operator (4.7) reduces by an explicit unitary transformation to the operator $id/d\xi$. Therefore the same is true for the corresponding Hankel operators $\mathbf{H}$ and H, see [17] for details.

4.3. Now we consider even more singular compared to (4.4) case when the function $\eta(\nu)$ has power singularities at the points $\nu = 1$ or $\nu = -1$. Let

$$\eta(\nu) = \Big(\frac{1+\nu}{1-\nu}\Big)^q, \quad q \in (-1,1), \quad q \neq 0, \tag{4.12}$$

and let the sequence a_n be defined by formula (4.2). Then according to Theorem 3.7 the non-negative Hankel operator H with such matrix elements is correctly defined via its quadratic form. Our goal here is to describe its spectral structure.

Theorem 4.6. *The spectrum of the Hankel operator $H = H(q)$ with the matrix elements*

$$a_n = a_n(q) = \int_{-1}^{1} \nu^n \Big(\frac{1+\nu}{1-\nu}\Big)^q d\nu, \quad q \in (-1,1), \quad q \neq 0, \tag{4.13}$$

is absolutely continuous, coincides with the half-axis $[0,\infty)$ and has constant multiplicity.

In view of Theorem 3.7 this result can be deduced from the corresponding assertion for the Hankel operator $\mathbf{H} = \mathbf{H}(q)$ in the space $L^2(\mathbb{R}_+)$. Indeed, it follows from formula (4.3) that the sigma-function of the operator $\mathbf{H} = \mathcal{L}H\mathcal{L}^*$ is $\sigma(\lambda) = 2^q\lambda^q$, and hence according to relation (4.1) its integral kernel equals

$$\mathbf{a}(t) = \mathbf{a}_q(t) = 2^q \int_0^\infty e^{-t\lambda}\lambda^q d\lambda = 2^q\Gamma(q+1)t^{-q-1}. \tag{4.14}$$

Hankel operators $\mathbf{H}$ with such kernels were studied in [19]. It was shown in Theorem 1.2 that for all $q > -1$, $q \neq 0$, the spectra of the operators $\mathbf{H}$ are absolutely continuous, coincide with the half-axis $[0,\infty)$ and have constant multiplicity. In particular, for $q < 1$, this result yields Theorem 4.6.

Remark 4.7.

1. The proof of Theorem 1.2 in [19] relies only on the invariance of the Hankel operator with kernel (4.14) with respect to the group of dilations. So, spectral information about the Hankel operators with matrix elements (4.13) is more limited than under the assumptions of Theorem 4.3. Even the spectral multiplicity of $\mathbf{H}$ is unknown. We recall that according to the fundamental results of [9] the spectral multiplicity of a positive bounded Hankel operator does not exceed 2, but, strictly speaking, this result is not applicable because the operator H in Theorem 4.6 is unbounded. Actually, we expect that the spectrum of H is simple since the kernel (4.14) has only one singular point. This point is $t = 0$ if $q > 0$ and $t = \infty$ if $q < 0$.
2. It is by no means obvious how to prove Theorem 4.6 directly in $\ell^2(\mathbb{Z}_+)$ because the realization in this space of the group of dilations in the space $L^2(\mathbb{R}_+)$ is not transparent.
3. It follows from definition (4.13) that

$$a_n(q) = (-1)^n a_n(-q) \tag{4.15}$$

   and hence $H(-q) = V^*H(q)V$ where the unitary operator V is defined on sequences $f = (f_0, f_1, \dots)$ by the formula $(Vf)_n = (-1)^n f_n$. Thus the Hankel operators $\mathbf{H}(-q)$ and $\mathbf{H}(q)$ with kernels (4.14) in the space $L^2(\mathbb{R}_+)$ are also unitarily equivalent. This fact does not look obvious in the continuous representation.
4. For $q \geq 1$, equality (4.13) makes no sense. In this case there is no reasonable interpretation of the Hankel operator H with sigma-function (4.12) in the space $\ell^2(\mathbb{Z}_+)$ although the Hankel operator $\mathbf{H}$ with integral kernel (4.14) is well defined in the space $L^2(\mathbb{R}_+)$.

Finally, we find the asymptotics of the matrix elements (4.13) as $n \to \infty$. In view of formula (4.15) we may suppose that $q \in (0,1)$. Then the asymptotics of the integral

(4.13) is determined by a neighborhood of the point $\nu = 1$. So, we write formula (4.13) as

$$a_n = 2^q \int_0^1 \nu^n (1-\nu)^{-q} d\nu + \int_0^1 \nu^n (1-\nu)^{-q}\big((1+\nu)^q - 2^q\big) d\nu + (-1)^n \int_0^1 \nu^n \Big(\frac{1-\nu}{1+\nu}\Big)^q d\nu.$$

The first integral on the right coincides with expression (4.9) for $\epsilon = -q$, and its asymptotics as $n \to \infty$ is given by formula (4.11). The second and third integrals are $O(n^{-2+q})$ and $O(n^{-1-q})$, respectively. This yields the following result.

Proposition 4.8. *The sequence* (4.13) *satisfies the asymptotic relation*

$$a_n = (\operatorname{sgn} q)^n 2^{|q|} \Gamma(1-|q|) n^{|q|-1} \big(1 + O(n^{-\varepsilon})\big), \quad n \to \infty,$$

where $\varepsilon = \min\{1, 2|q|\}$.

As could be expected, $a_n \to 0$ as $n \to \infty$ but much slower than sequence (4.6).

A. PROOF OF THEOREM 3.8

A.1. Suppose, for definiteness, that the equations (1.5) are satisfied with $\alpha = 1$. Then

$$\zeta = \frac{1+z}{1-z} =: \omega(z) \quad \text{and} \quad z = \omega^{-1}(\zeta) = \frac{\zeta - 1}{\zeta + 1}.$$

A standard calculation shows that if $r < |z_0 - 1|$, then

$$\omega(D(z_0, r)) = D(\zeta_0, R), \tag{A.1}$$

where

$$\zeta_0 = \frac{(1+z_0)(1-\bar{z}_0) + r^2}{|1-z_0|^2 - r^2}, \quad R = \frac{2r}{|1-z_0|^2 - r^2}. \tag{A.2}$$

Conversely, supposing that $R < |\zeta_0 + 1|$, we see that relation (A.1) holds true with

$$z_0 = \frac{(\zeta_0 - 1)(\bar{\zeta}_0 + 1) - R^2}{|\zeta_0 + 1|^2 - R^2}, \quad r = \frac{2R}{|\zeta_0 + 1|^2 - R^2}. \tag{A.3}$$

In particular, if $z_0 = e^{i\theta}$, $\theta \in [0, 2\pi)$, then $|z_0 - 1| = 2\sin(\theta/2)$ and (A.2) reads as

$$\zeta_0 = \frac{2i \sin\theta + r^2}{4\sin^2(\theta/2) - r^2}, \quad R = \frac{2r}{4\sin^2(\theta/2) - r^2}. \tag{A.4}$$

Similarly, if $\zeta_0 = i\lambda$, $\lambda \in \mathbb{R}$, then $|\zeta_0 + 1| = \sqrt{\lambda^2+1}$ and (A.3) reads as

$$z_0 = \frac{\lambda^2 + 2i\lambda - 1 - R^2}{\lambda^2 + 1 - R^2}, \quad r = \frac{2R}{\lambda^2+1-R^2}. \tag{A.5}$$

A.2. By the proof of Theorem 3.8, we may suppose that the measures $dM(z)$ and $d\Sigma(\zeta)$ are non-negative. It is also convenient to extend these measures onto the whole complex plane $\mathbb{C}$ setting $M(\mathbb{C}\setminus\mathbb{D}) = 0$ and $\Sigma(\mathbb{C}\setminus\mathbb{C}_+) = 0$. Below C (sometimes with indices) are different positive constants whose precise values are of no importance.

Let us define a measure $d\widetilde{M}(\zeta)$ on $\mathbb{C}$ by the relation $\widetilde{M}(Y) = M(\omega^{-1}(Y))$ for an arbitrary set $Y \subset \mathbb{C}$. In particular, according to (A.1) we have

$$M(D(z_0, r)) = \widetilde{M}(D(\zeta_0, R)). \tag{A.6}$$

It follows from (1.5) that

$$d\widetilde{M}(\zeta) = 2|\zeta+1|^{-2} d\Sigma(\zeta). \tag{A.7}$$

It is convenient to state the conditions (3.19) and (3.20) in an equivalent way.

Lemma A.1. *Condition* (3.19) *is satisfied if*

$$\sup_{\theta\in[0,2\pi)} \sup_{r\leq 2\gamma_0 \sin(\theta/2)} r^{-1} M(D(e^{i\theta}, r)) < \infty \tag{A.8}$$

for some $\gamma_0 \in (0,1)$ *and, for some* $r_0 > 0$,

$$\sup_{r\in(0,r_0)} r^{-1} M(D(1,r)) < \infty. \tag{A.9}$$

Condition (3.20) *is satisfied if*

$$\sup_{\lambda\in\mathbb{R}} \sup_{R\leq \delta_0\sqrt{\lambda^2+1}} R^{-1}\Sigma(D(i\lambda, R)) < \infty \tag{A.10}$$

for some $\delta_0 \in (0,1)$ *and, for some* $R_0 > 0$,

$$\sup_{R\geq R_0} R^{-1}\Sigma(D(0,R)) < \infty. \tag{A.11}$$

Proof. Indeed, if $r \geq 2\gamma_0 \sin(\theta/2)$, then $D(e^{i\theta}, r) \subset D(1, \gamma r)$ for $\gamma = 1 + \gamma_0^{-1}$. Therefore (3.19) for such θ and r follows from (A.9).

Similarly, if $R \geq \delta_0\sqrt{\lambda^2+1}$, then $R \geq \delta_0$ and $D(i\lambda, R) \subset D(0, \delta R)$ for $\delta = 1 + \delta_0^{-1}$. Therefore (3.20) for such λ and R follows from (A.11). □

First, we consider the conditions (A.8) and (A.10).

Lemma A.2. *Condition* (3.20) *on* $d\Sigma$ *implies condition* (A.8) *on* dM.

Proof. Let $r \leq 2\gamma_0 \sin(\theta/2)$, $\lambda = \cot(\theta/2)$, and let ζ_0 and R be given by formulas (A.4). Since $i\lambda \in D(\zeta_0, R)$, we have $D(\zeta_0, R) \subset D(i\lambda, 2R)$, and hence according to (3.20)

$$\Sigma(D(\zeta_0, R)) \leq CR. \tag{A.12}$$

Put $z_0 = e^{i\theta}$, then $|1 - z_0| = 2\sin(\theta/2)$. If $\zeta \in D(\zeta_0, R)$, then $z = \omega^{-1}(\zeta) \in D(z_0, r)$ and hence $|z_0 - z| \leq r \leq 2\gamma_0 \sin(\theta/2)$. So, we have

$$2|1 + \zeta|^{-1} = |1 - z| \leq |1 - z_0| + |z_0 - z| \leq 2(1 + \gamma_0)\sin(\theta/2).$$

In view of (A.7), it now follows from (A.12) that

$$M(D(z_0, r)) = \widetilde{M}(D(\zeta_0, R)) \leq 2(1 + \gamma_0)^2 \sin^2(\theta/2)\Sigma(D(\zeta_0, R)) \leq CR\sin^2(\theta/2). \tag{A.13}$$

Since, according to the second formula (A.4),

$$R\sin^2(\theta/2) = 2^{-1}r\Big(1 - \frac{r^2}{4\sin^2(\theta/2)}\Big)^{-1} \leq 2^{-1}\big(1 - \gamma_0^2\big)^{-1} r,$$

estimate (A.13) yields (A.8). □

Next, we prove the converse assertion.

Lemma A.3. *Condition* (3.19) *on dM implies condition* (A.10) *on $d\Sigma$.*

Proof. Consider the disc $D(i\lambda, R)$ where $R \leq \delta_0\sqrt{\lambda^2 + 1}$ with $\delta_0 < 1$. In view of relations (A.6) and (A.7), we have

$$\Sigma(D(i\lambda, R)) \leq C(\lambda^2 + 1)\widetilde{M}(D(i\lambda, R)) = C(\lambda^2 + 1)M(D(z_0, r)) \tag{A.14}$$

where z_0 and r are given by formulas (A.5). Since $e^{i\theta} = z_0|z_0|^{-1} \in D(z_0, r)$ we see that $D(z_0, r) \subset D(e^{i\theta}, 2r)$. Thus, it follows from condition (3.19) that the right-hand side of (A.14) is bounded by

$$C(\lambda^2 + 1)r = 2CR\Big(1 - \frac{R^2}{\lambda^2 + 1}\Big)^{-1} \leq 2\big(1 - \delta_0^2\big)^{-1} CR.$$

Therefore (A.14) implies condition (A.10). □

A.3. It remains to compare the conditions (A.9) and (A.11).

Lemma A.4. *Conditions* (A.9) *and* (A.11) *are equivalent.*

Proof. Let (A.11) be satisfied. Observe that

$$\omega(D(1, r)) = \mathbb{C} \setminus D(-1, 2r^{-1}) \tag{A.15}$$

and set $\phi(R) = \Sigma(D(-1, R))$. In view of (A.7), we have

$$\widetilde{M}(D(-1, R') \setminus D(-1, R)) = \int_R^{R'} d\widetilde{M}(D(-1, \rho)) = 2\int_R^{R'} \rho^{-2} d\phi(\rho).$$

Integrating here by parts and then passing to the limit $R' \to \infty$, we see that

$$2^{-1}\widetilde{M}(\mathbb{C}\setminus D(-1,R)) = -R^{-2}\phi(R) + 2\int_R^\infty \rho^{-3}\phi(\rho)d\rho.$$

Under assumption (A.11) the right- and hence the left-hand sides here are $O(R^{-1})$ as $R \to \infty$. Thus, according to (A.6) and (A.15), we have

$$M(D(1,r)) = \widetilde{M}(\mathbb{C}\setminus D(-1,2r^{-1})) = O(r), \quad r \to 0.$$

Conversely, let (A.9) be satisfied. Again in view of (A.7), we have

$$2\Sigma(D(-1,R)) = 2\int_0^R d\Sigma(D(-1,\rho)) = \int_0^R \rho^2 d\widetilde{M}(D(-1,\rho)).$$

Since, by (A.15),

$$\widetilde{M}(\mathbb{C}\setminus D(-1,R)) = M(D(1,2R^{-1})) =: \psi(R),$$

it follows that

$$2\Sigma(D(-1,R)) = -\int_0^R \rho^2 d\psi(\rho) = -R^2\psi(R) + 2\int_0^R \rho\psi(\rho)d\rho.$$

Under assumption (A.9), $\psi(R) = O(R^{-1})$ as $R \to \infty$. So, the right- and hence the left-hand sides here are $O(R)$ as $R \to \infty$ which proves (A.11). □

Now it easy to conclude the proof of Theorem 3.8. Let condition (3.20) on $d\Sigma$ be true. Then, by Lemma A.2, dM satisfies (A.8) and, by Lemma A.4, it satisfies (A.9) . So, Lemma A.1 implies that condition (3.19) is fulfilled. Similarly, let condition (3.19) on dM be true. Then, by Lemma A.3, $d\Sigma$ satisfies (A.10) and, by Lemma A.4, it satisfies (A.11). So, Lemma A.1 implies that condition (3.20) is fulfilled. □

REFERENCES

[1] A. Böttcher, B. Silbermann, *Analysis of Toeplitz operators*, Springer Verlag, 2006.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, vol. 1, 2, McGraw-Hill, New York, Toronto, London, 1953.

[3] J.B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[4] I.M. Gel'fand, G.E. Shilov, *Generalized Functions*, vol. 1, Academic Press, New York, London, 1964.

[5] I. Gohberg, S. Goldberg, M. Kaashoek, *Classes of Linear Operators*, vol. 1, Birkhäuser, 1990.

[6] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, New York, 1962.

[7] P. Koosis, *Introduction to H_p Spaces*, LMS, Lecture Note Series 40, Cambridge Univ. Press, 1980.

[8] W. Magnus, *On the spectrum of Hilbert's matrix*, Amer. J. Math. **72** (1950), 405–412.

[9] A.V. Megretskii, V.V. Peller, S.R. Treil, *The inverse spectral problem for self-adjoint Hankel operators*, Acta Math. **174** (1995), 241–309.

[10] Z. Nehari, *On bounded bilinear forms*, Ann. Math. **65** (1957), 153–162.

[11] N.K. Nikolski, *Operators, Functions, and Systems: an Easy Reading*, vol. I: Hardy, Hankel, and Toeplitz, Math. Surveys and Monographs, vol. 92, Amer. Math. Soc., Providence, Rhode Island, 2002.

[12] V.V. Peller, *Hankel Operators and Their Applications*, Springer Verlag, 2003.

[13] M. Rosenblum, *On the Hilbert matrix*, I, II, Proc. Amer. Math. Soc. **9** (1958), 137–140, 581–585.

[14] M. Rosenblum, J. Rovnyak, *Hardy Classes and Operator Theory*, Oxford Univ. Press, 1985.

[15] H. Widom, *Hankel matrices*, Trans. Amer. Math. Soc. **121** (1966), 1–35.

[16] D.R. Yafaev, *Quasi-diagonalization of Hankel operators*, preprint, arXiv:1403.3941 (2014); to appear in J. d'Analyse Mathématique.

[17] D.R. Yafaev, *Diagonalizations of two classes of unbounded Hankel operators*, Bulletin Math. Sciences **4** (2014), 175–198.

[18] D.R. Yafaev, *Criteria for Hankel operators to be sign-definite*, Analysis & PDE **8** (2015), 183–221.

[19] D.R. Yafaev, *Quasi-Carleman operators and their spectral properties*, Integral Equations Operator Theory **81** (2015), 499–534.

[20] D.R. Yafaev, *On finite rank Hankel operators*, J. Funct. Anal. **268** (2015), 1808–1839.

[21] D.R. Yafaev, *Spectral and scattering theory for differential and Hankel operator*, arXiv: 1511.04683 (2015).

[22] D.R. Yafaev, *Unbounded Hankel operator and moment problems*, Integral Equations Operator Theory **85** (2016), 289–300.

[23] D.R. Yafaev, *On semibounded Toeplitz operators*, J. Oper. Theory **77** (2017), 101–112.

[24] D.R. Yafaev, *On semibounded Wiener-Hopf operators*, arXiv: 1606.01361 (2016).

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