Opuscula Math. 37, no. 1 (2017), 81–107
http://dx.doi.org/10.7494/OpMath.2017.37.1.81 Opuscula Mathematica

SEMINORMAL SYSTEMS OF OPERATORS IN CLIFFORD ENVIRONMENTS

Mircea Martin

Communicated by P.A. Cojuhari

Abstract. The primary goal of our article is to implement some standard spin geometry techniques related to the study of Dirac and Laplace operators on Dirac vector bundles into the multidimensional theory of Hilbert space operators. The transition from spin geometry to operator theory relies on the use of Clifford environments, which essentially are Clifford algebra augmentations of unital complex C^*-algebras that enable one to set up counterparts of the geometric Bochner-Weitzenböck and Bochner-Kodaira-Nakano curvature identities for systems of elements of a C^*-algebra. The so derived self-commutator identities in conjunction with Bochner's method provide a natural motivation for the definitions of several types of seminormal systems of operators. As part of their study, we single out certain spectral properties, introduce and analyze a singular integral model that involves Riesz transforms, and prove some self-commutator inequalities.

Keywords: multidimensional operator theory, joint seminormality, Riesz transforms, Putnam inequality.

Mathematics Subject Classification: 47B20, 47A13, 47A63, 44A15.

1. INTRODUCTION

The study of *seminormal* – i.e., *hyponormal or cohyponormal* – Hilbert space operators started in the early 1950's. Though their definitions are deceptively simple, the theory of seminormal operators turned out to be quite intricate and far reaching. To begin with, suppose that $\mathcal{H}$ is a complex Hilbert space, let $\mathfrak{L}(\mathcal{H})$ be the C^*-algebra of continuous linear operators on $\mathcal{H}$, and for each $T \in \mathfrak{L}(\mathcal{H})$ let $T^* \in \mathfrak{L}(\mathcal{H})$ denote its adjoint. The interaction between T and T^*, which is controlled by their commutator in $\mathfrak{L}(\mathcal{H})$, plays an important part in deriving properties of T, and in this regard we can either form the *right self-commutator of* T, given by

$$\mathcal{C}_{\mathrm{R}}(T) = [T^*, T] = T^*T - TT^* \in \mathfrak{L}(\mathcal{H}), \tag{1.1}$$

or the *left self-commutator of* T, defined as

$$\mathcal{C}_{\mathrm{L}}(T) = [T, T^*] = TT^* - T^*T \in \mathfrak{L}(\mathcal{H}). \tag{1.2}$$

Following the standard terminology, $T \in \mathfrak{L}(\mathcal{H})$ is called a *seminormal operator*, provided its associated right or left self-commutators are either positive or negative semidefinite. Since obviously

$$\mathcal{C}_{\mathrm{L}}(T) = -\mathcal{C}_{\mathrm{R}}(T), \tag{1.3}$$

we end up with just two types of seminormal operators, namely, *hyponormal operators*, with the defining equivalent properties

$$\mathcal{C}_{\mathrm{R}}(T) \geq 0 \quad \text{or} \quad \mathcal{C}_{\mathrm{L}}(T) \leq 0, \tag{1.4}$$

or *cohyponormal operators*, for which we assume that

$$\mathcal{C}_{\mathrm{L}}(T) \geq 0 \quad \text{or} \quad \mathcal{C}_{\mathrm{R}}(T) \leq 0. \tag{1.5}$$

In addition, from yet another obvious identity,

$$\mathcal{C}_{\mathrm{L}}(T) = \mathcal{C}_{\mathrm{R}}(T^*), \tag{1.6}$$

we get that T is hyponormal, or cohyponormal, if and only if T^* is cohyponormal, or hyponormal, respectively. In spite of all such redundancies, we want to point out that as soon as we decide to distinguish between left or right, the definition of each class of seminormal operators can be stated by requiring just one of the two associated directed self-commutators to be positive semidefinite. It is merely a matter of fact, not at all surprising, that a change in direction – or a *spin* – yields a negative semidefinite self-commutator. Eventually, in a multidimensional setting, this observation will become quite relevant, and the apparently immaterial choice of a direction will turn out to make a difference. There is just one constraint we need to be aware of. The Hilbert space $\mathcal{H}$ must be infinite dimensional, because otherwise any semidefinite self-commutator – regardless its direction – equals the zero operator, and seminormality reduces to normality. As a second point, it would be worth to notice that all the previous equations and definitions make perfect sense if instead of operators in $\mathfrak{L}(\mathcal{H})$ we use elements of a unital complex C^*-algebra.

We should also mention that the theory of seminormal operators was initially developed for pairs (X, Y) of self-adjoint operators in $\mathfrak{L}(\mathcal{H})$ rather than a single operator $T \in \mathfrak{L}(\mathcal{H})$. Assuming that $X = \mathfrak{Re}(T)$ and $Y = \mathfrak{Im}(T)$, i.e., X is the real part of T and Y is the imaginary part of T, from

$$T = X + \sqrt{-1}\,Y, \quad T^* = X - \sqrt{-1}\,Y, \tag{1.7}$$

in conjunction with (1.1) and (1.2) we have

$$\mathcal{C}_{\mathrm{R}}(T) = \mathcal{C}(X, Y) \quad \text{and} \quad \mathcal{C}_{\mathrm{L}}(T) = -\mathcal{C}(X, Y), \tag{1.8}$$

where

$$\mathcal{C}(X, Y) = 2\sqrt{-1}\,[X, Y] = 2\sqrt{-1}\,(XY - YX) \in \mathfrak{L}(\mathcal{H}). \tag{1.9}$$

It seems quite reasonable to refer to $\mathcal{C}(X,Y)$ as the *self-commutator of the pair* (X,Y) – this time without distinguishing between left or right – and then to get the two concepts of *seminormal pairs of self-adjoint operators* by basically restating the definitions (1.4) or (1.5) above for the self-adjoint pair (X,Y).

For a comprehensive historical perspective on the development of the theory of seminormal operators and the relationship with the theory of *subnormal operators* we refer to the monographs by Putnam [36], Clancey [8], Xia [43], Martin and Putinar [28], and Conway [9]. Our goal is to single out and motivate what we believe to be the most natural counterparts of the previous equations and requirements in a multidimensional setting, i.e., for systems of operators. Significant early contributions to the development of the theory of *joint seminormality*, which prompted us to try and find a unifying framework, are due to Athavale [1], Cho, Curto, Huruya, and Zelazko [7], Curto [11], Curto and Jian [12], Curto, Muhly, and Xia [13], Douglas, Paulsen, and Yan [14], Martin and Salinas [29], McCullough and Paulsen [31], and Xia [44]. The viewpoint emphasized in our article was already outlined in Martin [19, 22–24] and is based on some techniques from Clifford analysis and spin geometry.

2. SPIN OPERATOR THEORY

This section outlines a spin geometry approach to multidimensional operator theory. A few prerequisites from *Clifford analysis* and *spin geometry* are briefly summarized in the first two subsections. As excellent introductions to these specific research fields concerned with the study of Dirac, Cauchy-Riemann, and Laplace operators in either a Euclidean, Hermitian, or a real or complex Dirac vector bundle setting, we refer to the monographs by Berline, Getzler, and Vergne [2], Brackx, Delanghe, and Sommen [5], Gilbert and Murray [15], and Lawson and Michelsohn [18].

2.1. EUCLIDEAN DIRAC AND LAPLACE OPERATORS

The *real* – or *Euclidean* – *Clifford algebra* $\mathfrak{A}_m(\mathbb{R})$, $m \geq 1$, is defined as the real unital C^*-algebra with generators $\{e_1, e_2, \dots, e_m\}$, called the *Clifford units of* $\mathfrak{A}_m(\mathbb{R})$, subject to the *Clifford relations*,

$$e_k e_l + e_l e_k = -2\,\delta_{kl}\,e_0, \text{ and } e_k^* = -e_k, \quad 1 \leq k, l \leq m, \tag{2.1}$$

where e_0 denotes the unit of $\mathfrak{A}_m(\mathbb{R})$, and δ_{kl}, $1 \leq k, l \leq m$, equals 1 or 0, according as $k = l$ or $k \neq l$, respectively. The Clifford units are identified with the standard orthonormal basis for $\mathbb{R}^m$, and consequently one gets an embedding of $\mathbb{R}^m$ into $\mathfrak{A}_m(\mathbb{R})$, such that if $\xi \in \mathbb{R}^m \subset \mathfrak{A}_m(\mathbb{R})$, then

$$\xi^2 = -\|\xi\|^2\,e_0, \text{ and } \quad \xi^* = -\xi, \tag{2.2}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^m$.

Suppose next that $\mathfrak{S}$ is an $\mathfrak{A}_m(\mathbb{R})$-module, i.e., a Hilbert space on which $\mathfrak{A}_m(\mathbb{R})$ is represented as an algebra of linear operators, and let $\mathcal{C}^\infty(\mathbb{R}^m, \mathfrak{S})$ be the space of smooth functions from $\mathbb{R}^m$ into $\mathfrak{S}$. The associated *Euclidean Dirac operator*,

$$D_{\mathrm{euc},m} : \mathcal{C}^\infty(\mathbb{R}^m, \mathfrak{S}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathfrak{S}),$$

is the constant coefficient first-order differential operator given by

$$D_{\mathrm{euc},m} = e_1\, \partial/\partial x_1 + e_2\, \partial/\partial x_2 + \cdots + e_m\, \partial/\partial x_m, \tag{2.3}$$

where $x_1, x_2, \ldots, x_m$ are the standard coordinate functions on $\mathbb{R}^m$. From (2.1) one gets that $D_{\mathrm{euc},m}$ is formally self-adjoint, and its square turns out to be the *Euclidean Laplace operator* on $\mathbb{R}^m$,

$$\Delta_{\mathrm{euc},m} : \mathcal{C}^\infty(\mathbb{R}^m, \mathfrak{S}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathfrak{S}),$$

defined as

$$\Delta_{\mathrm{euc},m} = -(\, \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_m^2\,). \tag{2.4}$$

Since $\Delta_{\mathrm{euc},m}$ is elliptic, we conclude that $D_{\mathrm{euc},m}$ is elliptic as well.

Moreover, if one starts with a differential operator $D_{\mathrm{euc},m}$ as in equation (2.3) and only assumes that the coefficients $\{e_1, e_2, \ldots, e_m\}$ are some continuous linear operators on a Hilbert space $\mathfrak{S}$, the properties

$$D^2_{\mathrm{euc},m} = \Delta_{\mathrm{euc},m} \quad \text{and} \quad D^*_{\mathrm{euc},m} = D_{\mathrm{euc},m} \tag{2.5}$$

would lead back to the Clifford relations (2.1).

The *complex* – or *Hermitian* – *Clifford algebras* $\mathfrak{A}_m(\mathbb{C}), m \geq 1$, are defined as complexifications of their real counterparts, i.e., $\mathfrak{A}_m(\mathbb{C}) = \mathfrak{A}_m(\mathbb{R}) \otimes \mathbb{C}$, $m \geq 1$. Actually, we will only be interested in the case when m is even. To make a point, if $m = 2n, n \geq 1$, instead of using the Clifford units $\{e_1, e_2, \ldots, e_{2n-1}, e_{2n}\}$ to generate $\mathfrak{A}_{2n}(\mathbb{C})$, a new set $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ of just n generators – which are referred to as the *Grassmann units* – can be introduced by setting

$$\varepsilon_i = (e_i - \sqrt{-1}\, e_{n+i})/2, \qquad 1 \leq i \leq n, \tag{2.6}$$

with the following set of *Grassmann relations*,

$$\varepsilon_i \varepsilon_j + \varepsilon_j \varepsilon_i = 0, \quad \varepsilon_i \varepsilon_j^* + \varepsilon_j^* \varepsilon_i = \delta_{ij} e_0, \qquad 1 \leq i, j \leq n. \tag{2.7}$$

The standard Clifford units can be easily recovered from

$$e_i = \varepsilon_i - \varepsilon_i^*, \quad e_{n+i} = \sqrt{-1}\, (\varepsilon_i + \varepsilon_i^*), \qquad 1 \leq i \leq n. \tag{2.8}$$

As a special property of the complex Clifford algebra $\mathfrak{A}_{2n}(\mathbb{C})$ – that would eventually play an important part in our subsequent development of spin operator theory – we should recall that $\mathfrak{A}_{2n}(\mathbb{C})$ has a unique irreducible representation on the graded space of *complex spinors* $\mathfrak{S}_n^\# = \mathfrak{S}_n^\#(\mathbb{C})$. As a Hilbert space,

$$\mathfrak{S}_n^\#(\mathbb{C}) = \Lambda^\#[\mathbb{C}^n] = \bigoplus_{p=0}^{n} \Lambda^p[\mathbb{C}^n], \tag{2.9}$$

where $\Lambda^{\#}[\mathbb{C}^n]$ is the *complex exterior* — or *Grassmann – algebra* of $\mathbb{C}^n$. To get a simple description of $\mathfrak{S}_n^{\#}$, suppose that $\{\sigma_1, \sigma_2, \ldots \sigma_n\}$ is the standard orthonormal basis for $\mathbb{C}^n$, and let $\mathfrak{I}_n^p$, $0 \le p \le n$, be the set of all p-element subsets of $\{1, 2, \ldots, n\}$. Obviously $\mathfrak{I}_n^0 = \{\emptyset\}$. If $1 \le p \le n$, any $I \in \mathfrak{I}_n^p$ will be expressed as an ordered p-tuple,

$$I = (i_1, i_2, \ldots, i_p), \quad 1 \le i_1 < i_2 < \cdots < i_p \le n.$$

To each $I \in \mathfrak{I}_n^{\#} = \cup_{p=0}^n \mathfrak{I}_n^p$ we associate the element $\sigma_I \in \mathfrak{S}_n^{\#}$, which is given by $\sigma_\emptyset = 1 \in \mathbb{C} = \mathfrak{S}_n^0$ if $p = 0$, and by

$$\sigma_I = \sigma_{i_1} \wedge \sigma_{i_2} \wedge \cdots \wedge \sigma_{i_p} \quad \text{if} \quad I = (i_1, i_2, \ldots, i_p) \in \mathfrak{I}_n^p, \quad 1 \le p \le n. \tag{2.10}$$

The so defined elements $\{\, \sigma_I : I \in \mathfrak{I}_n^{\#} \,\}$ provide an orthonormal basis for $\mathfrak{S}_n^{\#}$. To each vector $\sigma \in \mathbb{C}^n \equiv \mathfrak{S}_n^1$, one associates the linear operator $\wedge(\sigma)$ of left exterior multiplication by σ on $\mathfrak{S}_n^{\#} = \Lambda^{\#}[\mathbb{C}^n]$, as well as its adjoint $\wedge(\sigma)^*$. The standard orthonormal basis $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ for $\mathbb{C}^n$ yields the operators

$$\varepsilon_i = \wedge(\sigma_i), \quad \varepsilon_i^* = \wedge(\sigma_i)^*, \quad 1 \le i \le n, \tag{2.11}$$

which satisfy the Grassmann relations (2.6), and consequently,

$$\mathfrak{A}_{2n}(\mathbb{C}) \equiv \mathcal{L}(\mathfrak{S}_n^{\#}). \tag{2.12}$$

We next identify $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$, $n \ge 1$, by assigning to each pair $(x, y) \in \mathbb{R}^{2n}$, where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, the point $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ given by

$$z = x + \sqrt{-1}\, y, \quad \text{i.e.,} \quad z_i = x_i + \sqrt{-1}\, y_i, \quad 1 \le i \le n. \tag{2.13}$$

Assuming that $\mathfrak{S}$ is an arbitrary $\mathfrak{A}_{2n}(\mathbb{C})$-module, we take the Euclidean Dirac operator $D_{\mathrm{euc},2n}$ – regarded now as on operator on $\mathcal{C}^\infty(\mathbb{C}^n, \mathfrak{S})$ – and then, using (2.3) and (2.8) we decompose it into a sum,

$$D_{\mathrm{euc},2n} = \mathcal{D}_{\mathrm{her},n} + \mathcal{D}_{\mathrm{her},n}^*, \tag{2.14}$$

of two *Hermitian semi-Dirac operators*,

$$\mathcal{D}_{\mathrm{her},n} = \sum_{i=1}^n \varepsilon_i \left(\partial/\partial x_i + \sqrt{-1}\, \partial/\partial y_i\right) = 2 \sum_{i=1}^n \varepsilon_i\, \partial/\partial \bar{z}_i, \tag{2.15}$$

$$\mathcal{D}_{\mathrm{her},n}^* = \sum_{i=1}^n \varepsilon_i^* \left(-\partial/\partial x_i + \sqrt{-1}\, \partial/\partial y_i\right) = -2 \sum_{i=1}^n \varepsilon_i^*\, \partial/\partial z_i. \tag{2.16}$$

Since

$$\mathcal{D}_{\mathrm{her},n}^2 = 0 \quad \text{and} \quad \mathcal{D}_{\mathrm{her},n}^{*2} = 0, \tag{2.17}$$

for the Euclidean Laplace operator $\Delta_{\mathrm{euc},2n}$ on $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ one gets the equation

$$\Delta_{\mathrm{euc},2n} = \mathcal{D}_{\mathrm{her},n} \mathcal{D}_{\mathrm{her},n}^* + \mathcal{D}_{\mathrm{her},n}^* \mathcal{D}_{\mathrm{her},n}. \tag{2.18}$$

Perhaps it should be noticed that if $\mathfrak{S} = \mathfrak{S}_n^{\#}$, the spaces $\mathcal{C}^\infty(\mathbb{C}^n, \mathfrak{S}_n^p)$, $0 \le p \le n$, coincide – up to an isomorphism – with the spaces of complex differential forms of type $(0, p)$ on $\mathbb{C}^n$, and subject to this identification one gets

$$\mathcal{D}_{\mathrm{her},n} = \sqrt{2}\, \bar{\partial} \quad \text{and} \quad \mathcal{D}_{\mathrm{her},n}^* = \sqrt{2}\, \bar{\partial}^*.$$

2.2. DIRAC AND LAPLACE OPERATORS ON DIRAC VECTOR BUNDLES

The theory of Euclidean Dirac and Laplace operators is an essential part of Clifford analysis. More general Dirac and Laplace operators are defined and studied in spin geometry, which could be regarded as Clifford analysis on smooth inner product vector bundles over Riemann manifolds.

To be specific, suppose that M is a Riemann manifold, and let $\mathrm{T}M$ and T^*M be the tangent and cotangent bundles of M. Further, assume that E is a smooth inner product vector bundle over M, and let $\Gamma^\infty(M,E)$ be the space of its smooth sections. On E one can introduce differential operators of various orders, define their principal symbols, and — more importantly — single out classes of operators with special properties. Of a particular interest is the class of *Laplace operators* on E, which consists of second order, formally self-adjoint, elliptic differential operators

$$\Delta : \Gamma^\infty(M,E) \to \Gamma^\infty(M,E),$$

with a principal symbol $\mathfrak{s}_2(\Delta)$ that assigns to each $x \in M$ a quadratic form

$$\mathfrak{s}_2(\Delta)_x : \mathrm{T}M_x \to \mathfrak{L}(E_x)$$

from the tangent space to M at x into the space of linear operators on the fiber E_x of E at x, such that

$$\mathfrak{s}_2(\Delta)_x(\tau_x) = -\,\|\,\tau_{\,x}\|_x^2\,\cdot\,\mathrm{Id}_{E_x}, \quad \tau_x \in \mathrm{T}M_x, \tag{2.19}$$

where $\|\cdot\|_x$ is the Riemann norm on $\mathrm{T}M_x$, and $\mathrm{Id}_{E_x} \in \mathfrak{L}(E_x)$ is the identity operator on E_x, $x \in M$.

By definition, a *Dirac operator* $\mathfrak{D}$ on E — if any — is a formally self-adjoint first order differential operator on E which is a square root of a Laplace operator. The existence of Dirac operators on E requires two additional geometric structures on E compatible with the inner product structure.

(i) First, a smooth Clifford action $\gamma : \mathrm{T}M \otimes E \to E$ of the tangent bundle $\mathrm{T}M$ on E, that assigns to each tangent vector $\tau_x \in \mathrm{T}M_x$, $x \in M$, a skew-adjoint linear operator $\gamma(\tau_x) \in \mathfrak{L}(E_x)$, such that

$$\gamma(\tau_x)^2 = -\,\|\,\tau_{\,x}\|_x^2\,\cdot\,\mathrm{Id}_{E_x}. \tag{2.20}$$

(ii) Second, a linear connection $\nabla : \Gamma^\infty(M,E) \to \Gamma^\infty(M,\mathrm{T}^*M \otimes E)$ on E that preserves both the inner product structure and the Clifford action γ on E.

Following the terminology employed in Lawson and Michelsohn [18], we will refer to such smooth vector bundles E as *Dirac bundles* over M. Each Dirac bundle E has a canonically associated *Dirac operator*,

$$\mathfrak{D} : \Gamma^\infty(M,E) \to \Gamma^\infty(M,E),$$

which is the first order differential operator defined as the composition of

$$\Gamma^\infty(M,E) \longrightarrow \Gamma^\infty(M,\mathrm{T}^*M \otimes E) \longrightarrow \Gamma^\infty(M,\mathrm{T}M \otimes E) \longrightarrow \Gamma^\infty(M,E), \tag{2.21}$$

where the left arrow stands for the linear connection ∇, the middle arrow is induced by the isometric identification of T^*M with $\mathrm{T}M$, and the right arrow is the mapping determined by the Clifford action γ. As required, $\mathfrak{D}$ is a formally self-adjoint first order differential operator on E, with a principal symbol $\mathfrak{s}_1(\mathfrak{D})$ that assigns to each $x \in M$ the linear operator valued form

$$\mathfrak{s}_1(\mathfrak{D})_x : \mathrm{T}M_x \to \mathfrak{L}(E_x),\ \mathfrak{s}_1(\mathfrak{D})_x(\tau_x) = \gamma(\tau_x), \quad \tau_x \in \mathrm{T}M_x. \tag{2.22}$$

From (2.21), (2.20), and (2.19) one easily gets that

$$\Delta = \mathfrak{D}^2 \ \text{ is a Laplace operator,} \tag{2.23}$$

called the *Dirac-Laplace operator* on the Dirac bundle E. Actually, there is another naturally defined Laplace operator Δ^c on E, called the *connection*, or *Bochner-Laplace operator*, which is given by

$$\Delta^c = \nabla^* \nabla, \tag{2.24}$$

where $\nabla^* : \Gamma^\infty(M, \mathrm{T}^*M \otimes E) \to \Gamma^\infty(M, E)$ is the formal adjoint of ∇. Since ∇ and $\mathfrak{D}$ uniquely determine each other, the two Laplace operators on a Dirac bundle E are related by an equation of the form

$$\Delta = \Delta^c + \mathfrak{R}, \tag{2.25}$$

referred to as the *Bochner-Weitzenböck identity*, where the remainder $\mathfrak{R}$ is an operator of order zero that only depends on the curvature operator of E associated with the linear connection ∇.

We next turn our attention to complex manifolds and complex vector bundles. Specifically, we assume that M is a Kähler manifold, and let E be a holomorphic hermitian Dirac bundle over M, with the property that the canonical Chern linear connection ∇ on E preserves the Clifford action. It is a basic fact that each complex differential one form on M decomposes into the sum of a (0,1) and a (1,0) form, and for that reason the linear connection ∇ has two components,

$$\nabla = \nabla^{0,1} + \nabla^{1,0}. \tag{2.26}$$

Consequently, the Dirac operator $\mathfrak{D}$ can be expressed as

$$\mathfrak{D} = \mathcal{D} + \mathcal{D}^*, \tag{2.27}$$

where $\mathcal{D}$ is a first order differential operator on E, $\mathcal{D}^*$ is its formal adjoint, and

$$\mathcal{D}^2 = 0, \quad \mathcal{D}^{*2} = 0. \tag{2.28}$$

As a result of these splittings, one gets two equations,

$$\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D} = 2(\nabla^{0,1})^*\nabla^{0,1} + \mathfrak{R}^{0,1}, \tag{2.29}$$

and

$$\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D} = 2(\nabla^{1,0})^*\nabla^{1,0} + \mathfrak{R}^{1,0}, \tag{2.30}$$

referred to as the Bochner-Kodaira-Nakano identities. The two remainders $\mathfrak{R}^{0,1}$ and $\mathfrak{R}^{1,0}$ are operators of order zero on E, and likewise $\mathfrak{R}$ in (2.25) they can be computed using the curvature operator on E.

The Bochner-Weitzenböck identity (2.25) is just the average of the two Bochner-Kodaira-Nakano identities (2.29) and (2.30). Specifically,

$$\Delta = \mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D}, \tag{2.31}$$

$$\Delta^c = (\nabla^{0,1})^*\nabla^{0,1} + (\nabla^{1,0})^*\nabla^{1,0}, \tag{2.32}$$

$$2\mathfrak{R} = \mathfrak{R}^{0,1} + \mathfrak{R}^{1,0}. \tag{2.33}$$

We would like to point out that though we assumed that ∇ is the Chern linear connection on E, there is an entire family of linear connections that make all equations (2.26)–(2.33) true.

The nice features of the reminders $\mathfrak{R}$, $\mathfrak{R}^{0,1}$, and $\mathfrak{R}^{1,0}$ provide the basis of a method discovered by Bochner [3] that yields various vanishing theorems under appropriate positivity assumptions on the curvature operator. Some of the monographs referred to above, as well as Bochner and Yano [4] and Goldberg [16], offer a good deal of relevant examples.

2.3. SELF-COMMUTATOR IDENTITIES IN OPERATOR THEORY

We will now move from spin geometry to operator theory. Suppose that $\mathcal{H}$ is a complex Hilbert space and let $T = (T_1, T_2, \dots, T_n), n \geq 1$, be an n-tuple of operators in $\mathfrak{L}(\mathcal{H})$. We form the adjoint $T^* = (T_1^*, T_2^*, \dots, T_n^*)$ of T, and then take the real and imaginary parts of T denoted by

$$X = \mathfrak{Re}(T) = (X_1, X_2, \dots, X_n) \text{ and } Y = \mathfrak{Im}(T) = (Y_1, Y_2, \dots, Y_n). \tag{2.34}$$

Their components are self-adjoint operators on $\mathcal{H}$, and for convenience we will refer to (X, Y) as a self-adjoint pair. The relationships between T, T^*, X, and Y are exactly as in equation (1.7), namely,

$$T = X + \sqrt{-1}\,Y, \quad T^* = X - \sqrt{-1}\,Y. \tag{2.35}$$

Next we introduce the Hilbert space $\mathfrak{S}_n^\#[\mathcal{H}]$ of $\mathcal{H}$-valued spinors given by

$$\mathfrak{S}_n^\#[\mathcal{H}] = \mathfrak{S}_n^\# \otimes \mathcal{H}, \tag{2.36}$$

where $\mathfrak{S}_n^\# = \mathfrak{S}_n^\#(\mathbb{C})$ is the $\mathfrak{A}_{2n}(\mathbb{C})$-module of complex spinors defined by equation (2.9). Obviously $\mathfrak{S}_n^\#[\mathcal{H}]$ is an $\mathfrak{A}_{2n}(\mathbb{C})$-module and from (2.12) we get that

$$\mathfrak{L}(\mathfrak{S}_n^\#[\mathcal{H}]) = \mathfrak{A}_{2n}(\mathbb{C}) \otimes \mathfrak{L}(\mathcal{H}). \tag{2.37}$$

The elements of $\mathfrak{L}(\,\mathfrak{S}_n^\#[\mathcal{H}]\,)$ will be subsequently referred to as *operator forms*.

Returning to the n-tuple T, or to the associated self-adjoint pair (X, Y), as a first step towards the development of spin operator theory we form the holomorphic hermitian product vector bundle $E = E(T)$, or $E = E(X, Y)$, given by

$$E = \mathbb{C}^n \times \mathfrak{S}_n^\#[\mathcal{H}],$$

on which we obviously have a Clifford action γ. The space of smooth sections of E equals the space $\mathcal{C}^\infty(\mathbb{C}^n, \mathfrak{S}_n^\#[\mathcal{H}])$ of smooth functions from $\mathbb{C}^n$ into $\mathfrak{S}_n^\#[\mathcal{H}]$. To make E a Dirac vector bundle over $\mathbb{C}^n$ we take the metric preserving linear connection ∇ which depends on (X, Y), and therefore on T, defined by

$$\nabla = \sum_{i=1}^{m} \left\{ \wedge(dx_i) \left(\frac{\partial}{\partial x_i} + \sqrt{-1}\, e_0 \otimes X_i \right) + \wedge(dy_i) \left(\frac{\partial}{\partial y_i} + \sqrt{-1}\, e_0 \otimes Y_i \right) \right\}.$$

Using the notation introduced in Subsections 2.1 and 2.2, direct calculations show that the Dirac operator $\mathfrak{D}$ associated with ∇ reduces to

$$\mathfrak{D} = D_{\mathrm{euc},2n} + \mathfrak{D}(X, Y), \tag{2.38}$$

where $\mathfrak{D}(X, Y) \in \mathfrak{L}(\, \mathfrak{S}_n^\#[\mathcal{H}]\,) = \mathfrak{A}_{2n}(\mathbb{C}) \otimes \mathfrak{L}(\mathcal{H})$ is the operator form given by

$$\mathfrak{D}(X, Y) = \sqrt{-1}\ \sum_{i=1}^{n} (e_i \otimes X_i + e_{n+i} \otimes Y_i), \tag{2.39}$$

which will be called the *Dirac operator form* of the pair (X, Y), or of T.

Though the linear connection ∇ is not the Chern connection of E, we can decompose it as in equation (2.26), and then split the Dirac operator $\mathfrak{D}$ as in equation (2.27). Actually, the resulting new form of equation (2.27) is just a combination of equations (2.14), (2.15), (2.16), and the next equation,

$$\mathfrak{D}(X, Y) = \mathcal{D}(T) + \mathcal{D}^*(T), \tag{2.40}$$

where

$$\mathcal{D}(T) = \sqrt{-1} \sum_{i=1}^{n} \varepsilon_i \otimes T_i, \quad \mathcal{D}^*(T) = -\sqrt{-1} \sum_{i=1}^{n} \varepsilon_i^* \otimes T_i^*. \tag{2.41}$$

The two components $\mathcal{D}(T)$ and $\mathcal{D}^*(T)$ of $\mathfrak{D}(X, Y)$ will be referred to as the *semi-Dirac operator forms* of T. As yet another related object, we are now in a position to define the *Laplace operator form* $\Delta(T)$ of T by setting

$$\Delta(T) = \mathcal{D}(T)\mathcal{D}^*(T) + \mathcal{D}^*(T)\mathcal{D}(T), \tag{2.42}$$

which obviously is consistent with equation (2.31). Finally, from the two Bochner-Kodaira-Nakano identities (2.29) and (2.30), we get the next *Bochner-Kodaira--Nakano self-commutator identities* in multidimensional operator theory.

Lemma 2.1. *For any n-tuple $T = (T_1, T_2, \ldots, T_n), n \geq 1,$ we have*

$$\Delta(T) = e_0 \otimes \Delta_{\mathrm{L}}^c(T) + \mathcal{R}_{\mathrm{L}}(T), \tag{$\mathrm{BKN_L}(T)$}$$

and

$$\Delta(T) = e_0 \otimes \Delta_{\mathrm{R}}^c(T) + \mathcal{R}_{\mathrm{R}}(T), \tag{$\mathrm{BKN_R}(T)$}$$

where

$$\Delta_{\mathrm{L}}^{c}(T) = \sum_{i=1}^{n} T_i^* T_i, \quad \Delta_{\mathrm{R}}^{c}(T) = \sum_{i=1}^{n} T_i T_i^*, \tag{2.43}$$

and

$$\mathcal{R}_{\mathrm{L}}(T) = \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j^* \otimes [T_i, T_j^*], \quad \mathcal{R}_{\mathrm{R}}(T) = \sum_{i,j=1}^{n} \varepsilon_i^* \varepsilon_j \otimes [T_i^*, T_j]. \tag{2.44}$$

The two operators $\Delta_{\mathrm{L}}^{c}(T)$ and $\Delta_{\mathrm{R}}^{c}(T)$, and the two operator forms $\mathcal{R}_{\mathrm{L}}(T)$ and $\mathcal{R}_{\mathrm{R}}(T)$ defined above will be referred to as the *left* and *right Laplace operators* of T, and the *left* and *right self-commutator operator forms* of T, respectively.

As a matter of fact, both self-commutator identities in Lemma 2.1 can be deduced in a straightforward way from (2.39), (2.40), (2.41) and (2.42) by relying on the Grassmann relations (2.27) in conjunction with (2.28). The main reason we pursued an indirect route rests upon our desire to decrypt the geometric origin of these identities. Moreover, instead of using Hilbert space operators, we can start with an arbitrary unital complex C^*-algebra $\mathfrak{A}$, and assume that $T = (T_1, T_2, \ldots, T_n)$, $n \geq 1$, is an n-tuple of elements of $\mathfrak{A}$. Consequently, T^*, as well as X and Y, are n-tuples of elements of $\mathfrak{A}$. The only change we need to make is that the operator forms associated with either (X, Y), or with T, are now elements of what we would like to call the *Clifford environment* $\mathfrak{C}_n(\mathfrak{A})$ of $\mathfrak{A}$, which is defined as the Clifford algebra augmentation of $\mathfrak{A}$ given by

$$\mathfrak{C}_n(\mathfrak{A}) = \mathfrak{A}_{2n}(\mathbb{C}) \otimes \mathfrak{A}, \quad n \geq 1. \tag{2.45}$$

We proceed with a couple of observations. First, we notice that the two Bochner-Kodaira-Nakano self-commutator identities for n-tuples of elements of a C^*-algebra $\mathfrak{A}$ are related to each other. To make the point, let $\theta \in \mathfrak{A}_{2n}(\mathbb{C})$ denote the unitary operator on $\mathfrak{S}_n^{\#}$ – a disguise of the the *Hodge $\star$-operator* – given by

$$\theta = (\varepsilon_1 + \varepsilon_1^*)(\varepsilon_2 + \varepsilon_2^*) \cdots (\varepsilon_n + \varepsilon_n^*) = (-\sqrt{-1})^n e_{n+1} e_{n+2} \cdots e_{2n}.$$

A repeated use of (2.27) shows that

$$\theta \varepsilon_i = (-1)^{n-1} \varepsilon_i^* \theta, \quad \varepsilon_j \theta = (-1)^{n-1} \theta \varepsilon_j^*, \qquad 1 \leq i, j \leq n.$$

We next introduce the unitary element $\Theta = \theta \otimes \mathrm{I}_{\mathfrak{A}} \in \mathfrak{A}_{2n}(\mathbb{C}) \otimes \mathfrak{A}$, where $\mathrm{I}_{\mathfrak{A}}$ is the unit of $\mathfrak{A}$, and observe that

$$\Theta^* \, \mathcal{D}(T^*) \, \Theta = (-1)^{n-1} \mathcal{D}^*(T), \quad \Theta^* \, \Delta(T^*) \, \Theta = \Delta(T), \quad \Theta^* \, \mathcal{R}_{\mathrm{R}}(T^*) \, \Theta = \mathcal{R}_{\mathrm{L}}(T).$$

Therefore, the left identity $\mathrm{BKN}_{\mathrm{L}}(T)$ for T coincides – up to a unitary equivalence – with the right identity $\mathrm{BKN}_{\mathrm{R}}(T^*)$ for T^*, so we may just choose one equation as the basic form of the Bochner-Kodaira-Nakano self-commutator identity. Our choice is the right identity $\mathrm{BKN}_{\mathrm{R}}(T)$ for T, and we rewrite it as

$$\Delta(T) = e_0 \otimes \Delta^c(T) + \mathcal{R}(T), \tag{BKN(T)}$$

where $\Delta^c(T) = \Delta^c_{\mathrm{R}}(T)$ and $\mathcal{R}(T) = \mathcal{R}_{\mathrm{R}}(T)$. We can now replace $\mathrm{BKN}_{\mathrm{L}}(T)$ with

$$\Delta(T^*) = e_0 \otimes \Delta^c(T^8) + \mathcal{R}(T^*). \qquad \mathrm{BKN}(T^*)$$

The second observation we want to make is about the Bochner-Weitzenböck identity (2.25), which in its turn has a counterpart in terms of the self-adjoint n-tuples X and Y, at least in the geometric setting when the entries of T are Hilbert space operators. Since the linear connection ∇ that we used in this setting was not the Chern connection, some of the nice properties pointed out in Subsection 2.2 are not true. However, it can be proved that we still have the important property (2.28), as well as (2.31), (2.32), or (2.33), provided T is a commuting n-tuple. For a complete proof we refer to Martin [19]. Nevertheless, we would like to have a Bochner-Weitzenböck self-commutator identity for the self-adjoint pair (X, Y) in the general setting of a Clifford environment, derived as the average of the Bochner-Kodaira-Nakano identities, and – more importantly – without imposing additional assumptions. To this end, we introduce the following combinations of the operators involved in $\mathrm{BKN}(T)$ and $\mathrm{BKN}(T^*)$,

$$\Delta(X, Y) = \frac{1}{2} \left\{ \Delta(T) - \Delta(T^*) \right\}, \tag{2.46}$$

$$\Delta^c(X, Y) = \frac{1}{2} \left\{ \Delta^c(T) - \Delta^c(T^*) \right\}, \tag{2.47}$$

$$\mathcal{R}(X, Y) = \frac{1}{2} \left\{ \mathcal{R}(T) - \mathcal{R}(T^*) \right\}. \tag{2.48}$$

From $\mathrm{BKN}(T)$ and $\mathrm{BKN}(T^*)$ we get the following Bochner-Weitzenböck self--commutator identity in multidimensional operator theory.

Lemma 2.2. *For any pair (X, Y) of self-adjoint n-tuples, $n \geq 1$, we have*

$$\Delta(X, Y) = e_0 \otimes \Delta^c(X, Y) + \mathcal{R}(X, Y), \qquad \mathrm{BW}(X, Y)$$

where

$$\Delta^c(X, Y) = \sqrt{-1} \sum_{i=1}^{n} [X_i, Y_i], \tag{2.49}$$

and

$$\mathcal{R}(X, Y) = \sqrt{-1} \sum_{i,j=1}^{n} \varepsilon_i^* \varepsilon_j \otimes ([X_i, Y_j] + [X_j, Y_i]). \tag{2.50}$$

3. SEMINORMALITY IN HIGHER DIMENSION

This section introduces several types of seminormal systems of operators in the general setting of a Clifford environment. The subsequent definitions are motivated by Bochner's method in spin geometry, and our goal is to uncover the geometric significance of some of the existing concepts of joint seminormality.

3.1. SEMINORMAL SYSTEMS OF OPERATORS

Suppose $\mathfrak{A}$ is a unital complex C^*-algebra. and let $\mathfrak{E}_n(\mathfrak{A}) = \mathfrak{A}_{2n}(\mathbb{C}) \otimes \mathfrak{A}$, $n \geq 1$, denote its associated Clifford environments.

Definition 3.1. An n-tuple T of elements of $\mathfrak{A}$ is called *one-sided seminormal* provided either the right self-commutator operator form $\mathcal{R}_{\mathrm{R}}(T)$ in identity $\mathrm{BKN}_{\mathrm{R}}(T)$, or the left self-commutator operator form $\mathcal{R}_{\mathrm{L}}(T)$ in identity $\mathrm{BKN}_{\mathrm{L}}(T)$, is semidefinite as an element of $\mathfrak{A}_{2n}(\mathbb{C}) \otimes \mathfrak{A}$. To be more specific,
(i) T is called *right hyponormal*, or *right cohyponormal*, if

$$\mathcal{R}_{\mathrm{R}}(T) \geq 0, \quad \text{or} \quad \mathcal{R}_{\mathrm{R}}(T) \leq 0, \quad \text{respectively;} \tag{3.1}$$

(ii) T is called *left cohyponormal*, or *left hyponormal*, if

$$\mathcal{R}_{\mathrm{L}}(T) \geq 0, \quad \text{or} \quad \mathcal{R}_{\mathrm{L}}(T) \leq 0, \quad \text{respectively.} \tag{3.2}$$

Strictly speaking, since Bochner's method in spin geometry requires positivity assumptions, only two classes of one-sided seminormal n-tuples seem to qualify as appropriate for the implementation of Bochner's method in spin operator theory, namely, the right hyponormal and the left cohyponormal n-tuples. Apparently the two other classes should be either ruled out, or regarded as pathological. This would be wrong, and the following theorem due to Athavale [1], as well as some of our subsequent results will make the point.

In a nutshell, according to our terminology, *Athavale's theorem* states that *any commuting subnormal n-tuple of Hilbert space operators is left hyponormal.*

Proof. For n-tuples of elements of a C^*-algebra $\mathfrak{A}$, by using the Clifford environment $\mathfrak{E}_n(\mathfrak{A})$ of $\mathfrak{A}$, Athavale's theorem and its proof amount to the following. We start by taking a *commuting* n-tuple $N = (N_1, N_2, \ldots, N_n)$ of normal elements of $\mathfrak{A}$, and assume that $P \in \mathfrak{A}$ is a projection, i.e., $P = P^* = P^2$, such that

$$N_i P = P N_i P, \quad 1 \leq i \leq n. \tag{3.3}$$

Associated with N and P, we define the commuting subnormal n-tuple

$$T = (T_1, T_2, \ldots, T_n), \quad T_i = N_i P = P N_i P, \quad 1 \leq i \leq n, \tag{3.4}$$

whose adjoint is given by

$$T^* = (T_1^*, T_2^*, \ldots, T_n^*), \quad T_j^* = P N_j^* = P N_j^* P, \quad 1 \leq j \leq n. \tag{3.5}$$

By the well known Fuglede's theorem, the assumption $N_i N_j = N_j N_i$ implies $N_i N_j^* = N_j^* N_i$, $1 \leq i, j \leq n$, and therefore from (3.3), (3.4), (3.5) we have

$$[T_i, T_j^*] = P N_i P N_j^* P - P N_j^* N_i P = P N_i P N_j^* P - P N_i N_j^* P = -P N_i (\mathrm{I} - P) N_j^* P,$$

for all $1 \leq i, j \leq n$, where $\mathrm{I} = \mathrm{I}_{\mathfrak{A}}$ is the unit of $\mathfrak{A}$. Consequently, the n-tuple

$$S = (S_1, S_2, \ldots, S_n), \quad S_i = P N_i (\mathrm{I} - P), \quad 1 \leq i \leq n,$$

has the property

$$[T_i, T_j^*] = -S_i S_j^*, \quad 1 \leq i, j \leq n.$$

Equivalently, if we now take the semi-Dirac form $\mathcal{D}(S)$ associated with S and its adjoint $\mathcal{D}^*(S)$ defined as in equations (2.41), from (2.24) we get

$$\mathcal{R}_{\mathrm{L}}(T) = -\mathcal{D}(S)\mathcal{D}^*(S), \tag{3.6}$$

so, according to Definition 3.1, T is left hyponormal. The proof is complete. □

We proceed with two more definitions.

Definition 3.2. An n-tuple T of elements of $\mathfrak{A}$ is called *two-sided hyponormal*, or *two-sided cohyponormal*, provided T is simultaneously right and left hyponormal, or simultaneously right and left cohyponormal, respectively.

Definition 3.3. A pair (X, Y) of self-adjoint n-tuples of elements of $\mathfrak{A}$ is called *seminormal* provided the self-commutator operator form $\mathcal{R}(X, Y)$ in identity $\mathrm{BW}(X, Y)$ is semidefinite as an element of $\mathfrak{A}_{2n}(\mathbb{C}) \otimes \mathfrak{A}$. To be more specific, (X, Y) is called *hyponormal*, or *cohyponormal*, if

$$\mathcal{R}(X, Y) \geq 0, \quad \text{or} \quad \mathcal{R}(X, Y) \leq 0, \quad \text{respectively.} \tag{3.7}$$

We will always regard the two self-adjoint n-tuples X and Y as the real and imaginary parts of an n-tuple T. A quick inspection of the definitions of $\mathcal{R}_{\mathrm{R}}(T)$, $\mathcal{R}_{\mathrm{L}}(T)$, and $\mathcal{R}(X, Y)$ leads to the following direct consequences.

Proposition 3.4. *Let T be an n-tuple of elements of $\mathfrak{A}$ with the associated self-adjoint pair (X, Y).*

(i) *T is left cohyponormal, or left hyponormal, if and only if its adjoint T^* is right hyponormal, or right cohyponormal, respectively.*
(ii) *If T is two-sided hyponormal, or two-sided cohyponormal, then (X, Y) is hyponormal, or cohyponormal, respectively.*
(iii) *(X, Y) is hyponormal, or cohyponormal, if and only if either of the pairs (Y, X), $(X, -Y)$, or $(-X, Y)$ is cohyponormal, or hyponormal, respectively.*

Proof. Let $\mathcal{R}(T) = \mathcal{R}_{\mathrm{R}}(T)$ and recall that $\mathcal{R}_{\mathrm{L}}(T)$ is unitarily equivalent to $\mathcal{R}(T^*)$. Property (i) follows from Definition 3.1 by observing that T is left cohyponormal, or left hyponormal, if and only $\mathcal{R}(T^*) \geq 0$, or $\mathcal{R}(T^*) \leq 0$. Property (ii) is a consequence of the same remark and Definitions 3.2 and 3.3 in conjunction with equation (2.48). Finally, property (iii) follows from the identities

$$\mathcal{R}(Y, X) = \mathcal{R}(-X, Y) = \mathcal{R}(X, -Y) = -\mathcal{R}(X, Y). \qquad \square$$

The next result is a test for two-sided seminormal systems.

Proposition 3.5. *Let T be an n-tuple of elements of $\mathfrak{A}$.*

(i) *T is two-sided hyponormal if and only if*

$$0 \leq \mathcal{R}_{\mathrm{R}}(T) \leq e_0 \otimes \sum_{i=1}^{n} [T_i^*, T_i]. \tag{3.8}$$

(ii) *T is two-sided cohyponormal if and only if*

$$0 \leq \mathcal{R}_{\mathrm{L}}(T) \leq e_0 \otimes \sum_{i=1}^{n} [T_i, T_i^*]. \tag{3.9}$$

Proof. From (2.43) and (2.44) we notice that $\mathrm{BKN}_{\mathrm{R}}(T)$ and $\mathrm{BKN}_{\mathrm{L}}(T)$ imply

$$\mathcal{R}_{\mathrm{R}}(T) - \mathcal{R}_{\mathrm{L}}(T) = e_0 \otimes (\, \Delta_{\mathrm{L}}^c(T) - \Delta_{\mathrm{R}}^c(T)\,) = e_0 \otimes \sum_{i=1}^{n} [T_i^*, T_i],$$

an equation that leads to both properties. □

3.2. JOINTLY SEMINORMAL SYSTEMS OF OPERATORS

Our next goal is to find new equivalent forms of Definitions 3.1 and 3.3, which will show how some of the existing concepts of *jointly seminormal* n-tuples of Hilbert space operators can be reconciled with our Clifford environment approach. We will let $\mathfrak{M}_n(\mathbb{C}) \equiv \mathfrak{L}(\mathbb{C}^n)$, $n \geq 1$, denote the algebra of $n \times n$ complex matrices and, given a complex unital C^*-algebra $\mathfrak{A}$, form the algebra $\mathfrak{M}_n(\mathfrak{A})$ defined by

$$\mathfrak{M}_n(\mathfrak{A}) = \mathfrak{M}_n(\mathbb{C}) \otimes \mathfrak{A}.$$

The elements of $\mathfrak{M}_n(\mathfrak{A})$ are referred to as *operator matrices.* Assuming that T is an n-tuple of elements of $\mathfrak{A}$ we introduce the *right* and *left self-commutator operator matrices* of T, given by

$$\mathcal{C}_{\mathrm{R}}(T) = ([T_i^*, T_j])_{i,j=1}^n, \quad \mathcal{C}_{\mathrm{L}}(T) = ([T_i, T_j^*])_{i,j=1}^n, \tag{3.10}$$

as well as the *self-commutator operator matrix* of the self-adjoint pair (X, Y) associated with T, defined as

$$\mathcal{C}(X, Y) = \sqrt{-1}([X_i, Y_j] + [X_j, Y_i])_{i,j=1}^n. \tag{3.11}$$

If $n = 1$, we recover the self-commutators in equations (1.1), (1.2), and (1.9), which according to equations (1.3), (1.6), and (1.8) are related to each other. In sharp contrast, if $n \geq 2$, $\mathcal{C}_{\mathrm{R}}(T)$ and $\mathcal{C}_{\mathrm{L}}(T)$ are in general linearly independent, though we still have (1.6), i.e.,

$$\mathcal{C}_L(T) = \mathcal{C}_R(T^*). \tag{3.12}$$

Nevertheless, from a formal viewpoint we may regard $\mathcal{C}_{\mathrm{L}}(T)$ as the *transpose* of $-\mathcal{C}_{\mathrm{R}}(T)$, and also get the following weaker form of (1.8),

$$\mathcal{C}(X, Y) = \frac{1}{2}\left\{\mathcal{C}_{\mathrm{R}}(T) - \mathcal{C}_{\mathrm{L}}(T)\right\}. \tag{3.13}$$

Next, using the graded Hilbert space of complex spinors $\mathfrak{S}_n^{\#} = \bigoplus_{p=0}^{n} \mathfrak{S}_n^p$ introduced in Subsection 2.1, let $P_{\mathrm{R}}, P_{\mathrm{L}} \in \mathfrak{L}(\mathfrak{S}_n^{\#}) \equiv \mathfrak{A}_{2n}(\mathbb{C})$ be the orthogonal projections of $\mathfrak{S}_n^{\#}$

onto the subspaces $\mathfrak{S}_n^{n-1}$ and $\mathfrak{S}_n^1$, respectively. We already identified $\mathfrak{S}_n^1$ with $\mathbb{C}^n$, by assuming that the orthonormal basis $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ for $\mathfrak{S}_n^1$ is the standard orthonormal basis for $\mathbb{C}^n$. As an orthonormal basis for the subspace $\mathfrak{S}_n^{n-1}$, which we will use to identify $\mathfrak{S}_n^{n-1}$ with $\mathbb{C}^n$, we take the spinors $\{\star\sigma_1, \star\sigma_2, \ldots, \star\sigma_n\}$ given by

$$\star\sigma_i = (-1)^{i-1}\sigma_1 \wedge \cdots \wedge \sigma_{i-1} \wedge \sigma_{i+1} \wedge \cdots \wedge \sigma_n, \quad 1 \leq i \leq n.$$

Consequently, we have two ways of realizing the matrix algebra $\mathfrak{M}_n(\mathbb{C})$ as a compression of the Clifford algebra $\mathfrak{A}_{2n}(\mathbb{C})$, namely,

$$\mathfrak{M}_n(\mathbb{C}) \equiv \mathfrak{L}(\mathfrak{S}_n^{n-1}) = P_{\mathrm{R}}\, \mathfrak{A}_{2n}(\mathbb{C})\, P_{\mathrm{R}}, \quad \mathfrak{M}_n(\mathbb{C}) \equiv \mathfrak{L}(\mathfrak{S}_n^{1}) = P_{\mathrm{L}}\, \mathfrak{A}_{2n}(\mathbb{C})\, P_{\mathrm{L}},$$

and therefore we get the next two identifications of the matrix operator algebra $\mathfrak{M}_n(\mathfrak{A})$ as a compression of the Clifford environment $\mathfrak{E}_n(\mathfrak{A}) = \mathfrak{A}_{2n}(\mathbb{C}) \otimes \mathfrak{A}$,

$$\mathfrak{M}_n(\mathfrak{A}) \equiv (P_{\mathrm{R}} \otimes \mathrm{I}_{\mathfrak{A}})\, \mathfrak{E}_n(\mathfrak{A})\, (P_{\mathrm{R}} \otimes \mathrm{I}_{\mathfrak{A}}), \tag{3.14}$$

$$\mathfrak{M}_n(\mathfrak{A}) \equiv (P_{\mathrm{L}} \otimes \mathrm{I}_{\mathfrak{A}})\, \mathfrak{E}_n(\mathfrak{A})\, (P_{\mathrm{L}} \otimes \mathrm{I}_{\mathfrak{A}}). \tag{3.15}$$

The two above identifications make it possible to assign operator matrices to operator forms. In particular – as expected – for the operator forms $\mathcal{R}_{\mathrm{R}}(T)$, $\mathcal{R}_{\mathrm{L}}(T)$, and $\mathcal{R}(X, Y)$ of an n-tuple T, or of the corresponding self-adjoint pair (X, Y), we have the following associated operator matrices.

Lemma 3.6. *The self-commutator matrices* $\mathcal{C}_{\mathrm{R}}(T)$*,* $\mathcal{C}_{\mathrm{L}}(T)$*,* $\mathcal{C}(X, Y)$ *are given by*

$$\mathcal{C}_{\mathrm{R}}(T) \equiv (P_{\mathrm{R}} \otimes \mathrm{I}_{\mathfrak{A}})\, \mathcal{R}_{\mathrm{R}}(T)\, (P_{\mathrm{R}} \otimes \mathrm{I}_{\mathfrak{A}}),$$

$$\mathcal{C}_{\mathrm{L}}(T) \equiv (P_{\mathrm{L}} \otimes \mathrm{I}_{\mathfrak{A}})\, \mathcal{R}_{\mathrm{L}}(T)\, (P_{\mathrm{L}} \otimes \mathrm{I}_{\mathfrak{A}}),$$

$$\mathcal{C}(X, Y) \equiv (P_{\mathrm{R}} \otimes \mathrm{I}_{\mathfrak{A}})\, \mathcal{R}(X, Y\, (P_{\mathrm{R}} \otimes \mathrm{I}_{\mathfrak{A}}).$$

Prompted by this result we introduce two new definitions.

Definition 3.7. An n-tuple T of elements of $\mathfrak{A}$ is called *jointly one-sided seminormal* provided either its right, or its left self-commutator operator matrix is semidefinite as an element of $\mathfrak{M}_n(\mathfrak{A})$. To be more specific,

(i) T is called *jointly right hyponormal*, or *jointly right cohyponormal*, if

$$\mathcal{C}_{\mathrm{R}}(T) \geq 0, \quad \text{or} \quad \mathcal{C}_{\mathrm{R}}(T) \leq 0, \quad \text{respectively;} \tag{3.16}$$

(ii) T is called *jointly left cohyponormal*, or *jointly left hyponormal*, if

$$\mathcal{C}_{\mathrm{L}}(T) \geq 0, \quad \text{or} \quad \mathcal{C}_{\mathrm{L}}(T) \leq 0, \quad \text{respectively.} \tag{3.17}$$

Definition 3.8. A pair (X, Y) of self-adjoint n-tuples of elements of $\mathfrak{A}$ is called *jointly seminormal* provided the self-commutator operator matrix $\mathcal{C}(X, Y)$ is semidefinite as an element of $\mathfrak{M}_n(\mathfrak{A})$. To be specific, (X, Y) is called *jointly hyponormal*, or *jointly cohyponormal*, if

$$\mathcal{C}(X, Y) \geq 0, \quad \text{or} \quad \mathcal{C}(X, Y) \leq 0, \quad \text{respectively.} \tag{3.18}$$

For the benefit of our reader, we should mention that the concepts of *joint hyponormality* and *joint t-hyponormality* introduced by Athavale [1] and Xia [44], and eventually adopted and studied by other researchears, correspond in our terminology to *joint left hyponormality* and *joint right hyponormality,* respectively. To get a common ground, we just need to observe that according to Lemma 3.6, each of the self-commutator operator matrices defined by equations (3.10) and (3.11) is derived as a compression of a certain specific self-commutator operator form. From Definitions 3.1, 3.3, 3.7, and 3.8 we get the following two properties.

Corollary 3.9. *Any one-sided seminormal n-tuple T of elements of $\mathfrak{A}$ is jointly one-sided seminormal, and any seminormal pair (X, Y) of self-adjoint n-tuples of elements of $\mathfrak{A}$ is jointly seminormal, with the preservation of direction.*

Corollary 3.10. *If $T = (T_1, T_2, \dots, T_n)$, $n \geq 1$, is a one-sided hyponormal, or a one-sided cohyponormal n-tuple of elements of $\mathfrak{A}$, then each of its entries T_i, $1 \leq i \leq n$, is a hyponormal, or a cohyponormal element of $\mathfrak{A}$, respectively.*

Corollary 3.10 implies that when the C^*-algebra $\mathfrak{A}$ is finite dimensional, all components of a seminormal n-tuple T are normal. In particular, if T is two-sided seminormal, from Proposition 3.5 we conclude that both its self-commutator operator forms are equal to 0. We get the same conclusion if T is seminormal and commuting. Therefore, seminormality as a relevant concept in multidimensional operator theory only makes sense in an infinite dimensional framework.

3.3. SEMIDEFINITE QUADRATIC OPERATOR FORMS

Motivated by our last comments, for the remaining parts of our article we will assume that $\mathfrak{A} = \mathfrak{L}(\mathcal{H})$, where $\mathcal{H}$ is an *infinite dimensional* complex Hilbert space. Consequently, the associated Clifford environments are given by equation (2.37), i.e., $\mathfrak{E}_n(\mathfrak{L}(\mathcal{H})) = \mathfrak{L}(\mathfrak{S}_n^{\#}[\mathcal{H}])$, where $\mathfrak{S}_n^{\#}[\mathcal{H}]$, $n \geq 1$, is the Hilbert space of $\mathcal{H}$-valued spinors given by equation (2.36). Moreover, from equations (3.15) and (3.16) we get that the left and right identifications of the matrix operator algebra $\mathfrak{M}_n(\mathfrak{L}(\mathcal{H})) = \mathfrak{L}(\mathbb{C}^n \otimes \mathcal{H})$ as compressions of $\mathfrak{L}(\mathfrak{S}_n^{\#}[\mathcal{H}])$ reduce to

$$\mathfrak{M}_n(\mathfrak{L}(\mathcal{H})) \equiv \mathfrak{L}(\mathfrak{S}_n^1 \otimes \mathcal{H}), \quad \text{or} \quad \mathfrak{M}_n(\mathfrak{L}(\mathcal{H})) \equiv \mathfrak{L}(\mathfrak{S}_n^{n-1} \otimes \mathcal{H}). \tag{3.19}$$

Suppose next that $\mathcal{C} = (C_{ij})_{i,j=1}^n \in \mathfrak{M}_n(\mathfrak{L}(\mathcal{H}))$ is an operator matrix, and let $\mathcal{R}_{\mathrm{L}}(\mathcal{C})$, $\mathcal{R}_{\mathrm{R}}(\mathcal{C}) \in \mathfrak{L}(\mathfrak{S}_n^{\#}[\mathcal{H}])$ denote the operator forms defined by

$$\mathcal{R}_{\mathrm{L}}(\mathcal{C}) = \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j^* \otimes C_{ij}, \quad \mathcal{R}_{\mathrm{R}}(\mathcal{C}) = \sum_{i,j=1}^{n} \varepsilon_i^* \varepsilon_j \otimes C_{ij}, \tag{3.20}$$

which will be referred to as the *left* and *right quadratic operator forms with coefficient operator matrix $\mathcal{C}$*. Since with respect to the grading of the space $\mathfrak{S}_n^{\#}[\mathcal{H}]$ the two so defined quadratic operator forms are homogeneous of degree 0, their coefficient operator matrix $\mathcal{C}$ can be easily recovered based on the identifications (3.19) by using appropriate restrictions. Specifically,

$$\mathcal{C} = \mathcal{R}_{\mathrm{L}}(\mathcal{C})|\mathfrak{S}_n^1 \otimes \mathcal{H} : \mathfrak{S}_n^1 \otimes \mathcal{H} \to \mathfrak{S}_n^1 \otimes \mathcal{H}, \tag{3.21}$$

$$\mathcal{C} = \mathcal{R}_{\mathrm{R}}(\mathcal{C})|\mathfrak{S}_n^{n-1} \otimes \mathcal{H} : \mathfrak{S}_n^{n-1} \otimes \mathcal{H} \to \mathfrak{S}_n^{n-1} \otimes \mathcal{H}. \tag{3.22}$$

In particular, if $\mathcal{R}_{\mathrm{L}}(\mathcal{C})$ or $\mathcal{R}_{\mathrm{R}}(\mathcal{C})$ is semidefinite, then one gets that the coefficient operator matrix $\mathcal{C}$ is semidefinite, with the preservation of direction. The following technical result will be used to show that the converse of the previous remark is also true when $\mathcal{H}$ is an infinite dimensional complex Hilbert space.

Lemma 3.11. *Let $\mathcal{C} = (C_{ij})_{i,j=1}^n$ be an operator matrix, i.e., a linear operator on $\mathbb{C}^n \otimes \mathcal{H}$. The following two properties are equivalent:*

(i) $\mathcal{C} \geq 0$.
(ii) *There exists an n-tuple $(C_1, C_2, \ldots, C_n)$ of operators on $\mathcal{H}$ such that*

$$C_{ij} = C_i^* C_j, \qquad 1 \leq i, j \leq n. \tag{3.23}$$

Proof. Since obviously (ii) implies (i), it suffices to show that (i) implies (ii). To this end, we first choose a unitary operator $U : \mathbb{C}^n \otimes \mathcal{H} \to \mathcal{H}$. Such unitary operators exist because $\mathcal{H}$ is infinite dimensional. Next, we define $C_i : \mathcal{H} \to \mathcal{H}$, $1 \leq i \leq n$, by setting $C_i \xi = U\left(\mathcal{C}^{1/2}(\sigma_i \otimes \xi)\right)$, $\xi \in \mathcal{H}$, where $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ is the standard orthonormal basis for $\mathbb{C}^n$ and $\mathcal{C}^{1/2} : C^n \otimes \mathcal{H} \to C^n \otimes \mathcal{H}$ denotes the square root of $\mathcal{C}$. It remains to observe that

$$\begin{aligned} \langle C_i^* C_j \xi, \xi \rangle_{\mathcal{H}} &= \langle C_j \xi, C_i \xi \rangle_{\mathcal{H}} = \left\langle U\left(\mathcal{C}^{1/2}(\sigma_j \otimes \xi)\right), U\left(\mathcal{C}^{1/2}(\sigma_i \otimes \xi)\right)\right\rangle_{\mathcal{H}} \\ &= \left\langle \mathcal{C}^{1/2}(\sigma_j \otimes \xi), \mathcal{C}^{1/2}(\sigma_i \otimes \xi)\right\rangle_{\mathbb{C}^n \otimes \mathcal{H}} = \langle \mathcal{C}(\sigma_j \otimes \xi), \sigma_i \otimes \xi \rangle_{\mathbb{C}^n \otimes \mathcal{H}} \\ &= \langle C_{ij} \xi, \xi \rangle_{\mathcal{H}}, \end{aligned}$$

for any $1 \leq i, j \leq n$ and all $\xi \in \mathcal{H}$. The proof of (3.23) is complete. □

Proposition 3.12. *Let $\mathcal{R}_{\mathrm{L}}(\mathcal{C})$ and $\mathcal{R}_{\mathrm{R}}(\mathcal{C})$ be the two quadratic operator forms with the coefficient operator matrix $\mathcal{C}$. The following properties are equivalent:*

(i) $\mathcal{C} \geq 0$.
(ii) *There exists an n-tuple $S = (S_1, S_2, \ldots, S_n)$ of operators on $\mathcal{H}$ such that*

$$\mathcal{R}_{\mathrm{L}}(\mathcal{C}) = \mathcal{D}(S)\mathcal{D}^*(S). \tag{3.24}$$

(iii) *There exists an n-tuple $S = (S_1, S_2, \ldots, S_n)$ of operators on $\mathcal{H}$ such that*

$$\mathcal{R}_{\mathrm{R}}(\mathcal{C}) = \mathcal{D}^*(S)\mathcal{D}(S). \tag{3.25}$$

Proof. First, let us recall that $\mathcal{D}(S)$ and $\mathcal{D}^*(S)$ in parts (ii) and (iii) above are the semi-Dirac operator forms of an n-tuple S defined according to equation (2.41). We already noticed that the assumptions $\mathcal{R}_{\mathrm{L}}(\mathcal{C}) \geq 0$, or $\mathcal{R}_{\mathrm{R}}(\mathcal{C}) \geq 0$, imply property (i).

Assuming now that (i) is true, we take an n-tuple $(C_1, C_2, \ldots, C_n)$ as in Lemma 3.11. Equation (3.24) follows from the first equation in (3.20) and from (3.23), by setting $S = (C_1^*, C_2^*, \ldots, C_n^*)$. In its turn, equation (3.25) follows from the second equation in (3.20) and from (3.23), by setting $S = (C_1, C_2, \ldots, C_n)$. The proof is complete. □

If $\mathcal{C} \leq 0$, we use Proposition 3.12 for the positive semidefite operator martix $-\mathcal{C}$ and get the equivalent properties $\mathcal{R}_{\mathrm{L}}(\mathcal{C}) \leq 0$ and $\mathcal{R}_{\mathrm{R}}(\mathcal{C}) \leq 0$.

The previous results have some direct consequences for n-tuples of operators on $\mathcal{H}$. If $T = (T_1, T_2, \ldots, T_n)$ is such an n-tuple, and (X, Y) is its associated self-adjoint pair, from (2.44), (2.50), (3.10), (3.11), and (3.20), we get that the self-commutator operator matrices $\mathcal{C}_{\mathrm{L}}(T)$, $\mathcal{C}_{\mathrm{R}}(T)$, $\mathcal{C}(X, Y)$, and the quadratic self-commutator operator forms $\mathcal{R}_{\mathrm{L}}(T)$, $\mathcal{R}_{\mathrm{R}}(T)$, $\mathcal{R}(X, Y)$ are related to each other by the following equations,

$$\mathcal{R}_{\mathrm{L}}(T) = \mathcal{R}_{\mathrm{L}}(\mathcal{C}_{\mathrm{L}}(T)),\ \mathcal{R}_{\mathrm{R}}(T) = \mathcal{R}_{\mathrm{R}}(\mathcal{C}_{\mathrm{R}}(T)), \quad \mathcal{R}(X, Y) = \mathcal{R}_{\mathrm{R}}(\mathcal{C}(X, Y)). \tag{3.26}$$

From Proposition 3.12 and the brief comment after it in conjunction with Corollary 3.9 we derive the next result.

Theorem 3.13. *If $\mathfrak{A} = \mathfrak{L}(\mathcal{H})$ with $\mathcal{H}$ an infinite dimensional complex Hilbert space, Definitions 3.1 and 3.3 are equivalent to Definitions 3.7 and 3.8, respectively, i.e., seminormality – in each of its specific forms – is equivalent to joint seminormality.*

The following result is yet another consequence of Proposition 3.12.

Proposition 3.14. *If $\mathcal{C} = (C_{ij})_{i,j=1}^n$ is a semidefinite operator matrix, i.e., a positive or negative semidefinite linear operator on $\mathbb{C}^n \otimes \mathcal{H}$, then*

$$\|\mathcal{C}\| \leq \sum_{i=1}^n \|C_{ii}\|.$$

Proof. It would be enough to prove (3.14) when $\mathcal{C} \geq 0$. Under this assumption, from Proposition 3.12 we have the factorization (3.24), where $S = (S_1, S_2, \ldots, S_n)$ is an n-tuple of operators on $\mathcal{H}$ such that

$$C_{ij} = S_i S_j^*, \quad 1 \leq i, j \leq n.$$

Since, according to equation (3.21), $\mathcal{C} = \mathcal{R}_{\mathrm{L}}(\mathcal{C})|\mathfrak{S}_n^1 \otimes \mathcal{H}$, from (3.24) we get that $\mathcal{C} = \mathcal{D}^{(0)}(S)\mathcal{D}^{(0)*}(S)$, where $\mathcal{D}^{(0)}(S) = \mathcal{D}(S)|\mathfrak{S}_n^0 \otimes \mathcal{H} : \mathfrak{S}_n^0 \otimes \mathcal{H} \to \mathfrak{S}_n^1 \otimes \mathcal{H}$. Therefore,

$$\|\mathcal{C}\| = \|\mathcal{D}^{(0)}(S)\|^2. \tag{3.27}$$

On the other hand, if $\xi \in \mathcal{H} \equiv \mathfrak{S}_n^0 \otimes \mathcal{H}$, then

$$\|\mathcal{D}^{(0)}(S)\xi\|^2_{\mathfrak{S}_n^1 \otimes \mathcal{H}} = \left\| \sum_{i=1}^n \sigma_i \otimes S_i \xi \right\|^2_{\mathfrak{S}_n^1 \otimes \mathcal{H}} = \sum_{i=1}^n \|S_i \xi\|^2_{\mathcal{H}},$$

whence

$$\|\mathcal{D}^{(0)}(S)\|^2 \leq \sum_{i=1}^n \|S_i\|^2 = \sum_{i=1}^n \|S_i S_i^*\| = \sum_{i=1}^n \|C_{ii}\|. \tag{3.28}$$

Inequality (3.14) follows from (3.27) and (3.28). □

In the particular case when the semidefinite operator matrix $\mathcal{C}$ in Proposition 3.14 is either the right, or the left self-commutator operator matrix of a seminormal n-tuple of operators T, (3.14) yields a self-commutator inequality.

Corollary 3.15. *If $T = (T_1, T_2, \dots, T_n)$ is a one-sided seminormal n-tuple of operators on an infinite dimensional Hilbert space $\mathcal{H}$, then*

$$\|\mathcal{C}(T)\| \leq \sum_{i=1}^{n} \|[T_i^*, T_i]\|, \tag{3.29}$$

where $\mathcal{C}(T) = \mathcal{C}_{\mathrm{R}}(T)$ or $\mathcal{C}(T) = \mathcal{C}_{\mathrm{L}}(T)$.

4. SPECTRAL PROPERTIES OF SEMINORMAL SYSTEMS

For the specific purposes of this section, we assume that $\mathcal{H}$ is an infinite dimensional complex Hilbert space, and $T = (T_1, T_2, \dots, T_n), n \geq 1$, is a *commuting* n-tuple of operators on $\mathcal{H}$, i.e.,

$$[T_i, T_j] = 0, \quad 1 \leq i, j \leq n. \tag{4.1}$$

In terms of X and Y, the real and imaginary parts of T, (4.1) amounts to

$$[X_i, X_j] = [Y_i, Y_j], \quad 1 \leq i, j \leq n, \tag{4.2}$$

$$[X_i, Y_j] = [X_j, Y_i], \quad 1 \leq i, j \leq n. \tag{4.3}$$

Moreover, direct calculations show that T is a commuting n-tuple if and only if its associated semi-Dirac operator form $\mathcal{D}(T) : \mathfrak{S}_n^{\#}[\mathcal{H}] \to \mathfrak{S}_n^{\#}[\mathcal{H}]$ defined by equation (2.41) has the property

$$\mathcal{D}(T)^2 = 0. \tag{4.4}$$

Consequently, using the space $\mathfrak{S}_n^{\#}[\mathcal{H}]$ of $\mathcal{H}$-valued spinors with the co-boundary operator $\mathcal{D}(T)$ one gets the co-chain complex $\mathcal{K}^\star(T) = \{\mathfrak{S}_n^{\#}[\mathcal{H}], \mathcal{D}(T)\}$ given by

$$\{0\} \longrightarrow \mathfrak{S}_n^0[\mathcal{H}] \xrightarrow{\mathcal{D}(T)} \cdots \xrightarrow{\mathcal{D}(T)} \mathfrak{S}_n^p[\mathcal{H}] \xrightarrow{\mathcal{D}(T)} \mathfrak{S}_n^{p+1}[\mathcal{H}] \xrightarrow{\mathcal{D}(T)} \cdots \xrightarrow{\mathcal{D}(T)} \mathfrak{S}_n^n[\mathcal{H}] \longrightarrow \{0\},$$

which is just the *Koszul complex* associated with T. The cohomology spaces of $\mathcal{K}^\star(T)$ are denoted by $\mathrm{H}^p\,\mathcal{K}^\star(T)$, $0 \leq p \leq n$, and their direct sum $\mathrm{H}^\star\,\mathcal{K}^\star(T) = \bigoplus_{p=0}^n \mathrm{H}^p\,\mathcal{K}^\star(T)$ is the total cohomology space of $\mathcal{K}^\star(T)$. For more details on this complex and its role in multivariable spectral theory we refer to Taylor [39].

We next take the Laplace operator form $\Delta(T) : \mathfrak{S}_n^{\#}[\mathcal{H}] \to \mathfrak{S}_n^{\#}[\mathcal{H}]$ of T defined by equation (2.42), which with respect to the grading of $\mathfrak{S}_n^{\#}[\mathcal{H}]$ is homogeneous of degree 0, and consider its homogeneous components

$$\Delta^{(p)}(T) = \Delta(T)|\mathfrak{S}_n^p[\mathcal{H}] : \mathfrak{S}_n^p[\mathcal{H}] \to \mathfrak{S}_n^p[\mathcal{H}], \quad 0 \leq p \leq n.$$

A quick inspection of the Bochner-Kodaira-Nakano identities $\mathrm{BKN}_{\mathrm{L}}(T)$ and $\mathrm{BKN}_{\mathrm{R}}(T)$ in Lemma 2.1, and the Hilbert space isomorphisms $\mathfrak{S}_n^0[\mathcal{H}] \equiv \mathcal{H}$ and $\mathfrak{S}_n^n[\mathcal{H}] \equiv \mathcal{H}$, lead to the identifications

$$\Delta^{(0)}(T) = \Delta_{\mathrm{L}}^c(T), \quad \Delta^{(n)}(T) = \Delta_{\mathrm{R}}^c(T), \tag{4.5}$$

with $\Delta^c_{\mathrm{L}}(T)$ and $\Delta^c_{\mathrm{R}}(T)$ the Laplace operators of T given by equations (2.43). We are now in a position to prove the following result.

Theorem 4.1. *Let $T = (T_1, T_2, \ldots, T_n)$, $n \geq 1$, be a commuting n-tuple of operators on $\mathcal{H}$, and suppose that T is either right hyponormal, or left cohyponormal.*

(i) *$\mathcal{K}^\star(T)$ is an exact complex if and only if either $\Delta^c_{\mathrm{R}}(T)$, or $\Delta^c_{\mathrm{L}}(T)$ is an invertible operator on $\mathcal{H}$, respectively.*
(ii) *$\mathcal{K}^\star(T)$ is a Fredholm complex if and only if either $\operatorname{Ker}\Delta^c_{\mathrm{R}}(T)$, or $\operatorname{Ker}\Delta^c_{\mathrm{L}}(T)$ is a finite dimensional subspace of $\mathcal{H}$, respectively.*

Proof. Our subsequent reasonings are based on some results due to Vasilescu [40, 41] and Curto [10]. Both the exactness and Fredholmness of $\mathcal{K}^\star(T)$ can be characterized in terms of the Laplace operator form $\Delta(T)$. Specifically, $\mathcal{K}^\star(T)$ is exact if and only if $\Delta(T)$ is an invertible operator on $\mathfrak{S}^\#_n[\mathcal{H}]$, and $\mathcal{K}^\star(T)$ is Fredholm if and only if $\operatorname{Ker}\Delta(T)$ is a finite dimensional subspace of $\mathfrak{S}^\#_n[\mathcal{H}]$. Therefore, if $\mathcal{K}^\star(T)$ is exact, or Fredholm, then both Laplace operators $\Delta^c_{\mathrm{R}}(T)$ and $\Delta^c_{\mathrm{L}}(T)$ of T are invertible, or both $\operatorname{Ker}\Delta^c_{\mathrm{R}}(T)$ and $\operatorname{Ker}\Delta^c_{\mathrm{L}}(T)$ are finite dimensional, without any other assumptions.There are the converse properties that require supplementary assumptions and in this regard Bochner's method turns out to be relevant.

Let us first examine the case when T is right hyponormal, i.e., $\mathcal{R}_{\mathrm{R}}(T) \geq 0$. To complete the proof of statements (i) and (ii) in this case we rely on the Bochner-Kodaira-Nakano identity $\mathrm{BKN}_{\mathrm{R}}(T)$. If $\Delta^c_{\mathrm{R}}(T)$ is invertible, then $\Delta(T)$ equals the sum of an invertible positive definite operator and a positive semidefinite operator, and as such $\Delta(T)$ is invertible. This simple observation concludes the proof of statement (i). We next notice that under the same assumption on $\mathcal{R}_{\mathrm{R}}(T)$ we have

$$\operatorname{Ker}\Delta(T) = \mathfrak{S}^\#_n(\mathbb{C}) \otimes \operatorname{Ker}\Delta^c_{\mathrm{R}}(T) \cap \operatorname{Ker}\mathcal{R}_{\mathrm{R}}(T), \tag{4.6}$$

an equation that concludes the proof of statement (ii). The second case when T is left cohyponormal, i.e., $\mathcal{R}_{\mathrm{L}}(T) \geq 0$, and $\Delta^c_{\mathrm{L}}(T)$ is invertible, or $\operatorname{Ker}\Delta^c_{\mathrm{L}}(T)$ is finite dimensional, can be handled in a similar way by using the Bochner-Kodaira-Nakano identity $\mathrm{BKN}_{\mathrm{L}}(T)$. The proof is complete. □

As specific spectral properties of seminormal n-tuples of Hilbert space operators we have the following consequence of Theorem 4.1.

Corollary 4.2. *Assume that $T = (T_1, T_2, \ldots, T_n)$, $n \geq 1$, is a commuting n-tuple of operators on $\mathcal{H}$, and let $\sigma(T)$, $\sigma_{\mathrm{R}}(T)$, and $\sigma_{\mathrm{L}}(T)$ be the spectrum, the right spectrum, and the left spectrum of T. If T is right hyponormal, or left cohyponormal, then*

$$\sigma(T) = \sigma_{\mathrm{R}}(T), \quad \text{or} \quad \sigma(T) = \sigma_{\mathrm{L}}(T), \tag{4.7}$$

respectively. A similar conclusion holds if one replaces $\sigma(T)$, $\sigma_{\mathrm{R}}(T)$, and $\sigma_{\mathrm{L}}(T)$ by the essential spectrum $\sigma_{\mathrm{ess}}(T)$, the right essential spectrum $\sigma_{\mathrm{ess,R}}(T)$, and the left essential spectrum $\sigma_{ess,\mathrm{L}}(T)$ of T.

The reader may find details on the spectral sets specified above in Bunce [6] and Curto [10]. For different proofs of Corollary 3.15 when T is left cohyponormal we refer to Xia [44] and Curto and Jian [12].

5. RIESZ TRANSFORMS AND JOINT HYPONORMALITY

The goal of this section is to introduce Riesz transforms models of hyponormal pairs of self-adjoint n-tuples of Hilbert space operators, i.e., a special type of singular integral models that make use of Riesz transforms. Such models generalize to higher dimension the Hilbert transform models of hyponormal operators with a self-commutator of rank 1 discovered by Xia [42] and Pincus [33], analyzed in more detail by Pincus and Xia [34] and Picus and Xia and Xia [35], and afterwards set up in full generality for pure hyponormal operators by Kato [17] and Muhly [32]. We should point out that the natural framework for developing Riesz transforms models is provided by n-tuples of decomposable linear operators on a direct integral Hilbert space

$$\mathcal{H} = \int_{\Omega}^{\oplus} \mathfrak{H}(x)\, dx,$$

where $\Omega \subset \mathbb{R}^n$ is a compact set. For more details in this regard we refer to Martin [23, 34]. However, the basic features of the models can be fully illustrated by assuming that each of the spaces $\mathfrak{H}(x)$, $x \in \Omega$, is one-dimensional, i.e., by taking the Lebesgue space $\mathcal{H} = L^2(\Omega)$, and that is exactly what we will be doing in this section.

5.1. COMMUTATORS INVOLVING RIESZ TRANSFORMS

We begin by recalling that the Riesz transforms of a function u in the Schwartz space $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$ of complex-valued functions on $\mathbb{R}^n, n \geq 1$, are defined by

$$R_i u(x) = \text{p.v.} \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \int_{\mathbb{R}^n} \frac{x_i - y_i}{|x-y|^{n+1}}\, u(y)\, dy, \quad 1 \leq i \leq n,\ x \in \mathbb{R}^n. \tag{5.1}$$

For convenience, we will let k_i denote the kernels

$$k_i(x) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \cdot \frac{x_i}{|x|^{n+1}}, \quad 1 \leq i \leq n,\ x \in \mathbb{R}^n_0 = \mathbb{R}^n \setminus \{0\}, \tag{5.2}$$

and express each R_i as a convolution operator, i.e., $R_i u = k_i * u$, $1 \leq i \leq n$.

Let M_i denote the multiplication operators on $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$ given by

$$M_i u(x) = x_i u(x), \quad 1 \leq i \leq n,\ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n. \tag{5.3}$$

A direct calculation shows that the commutators $[M_i, R_j] = M_i\, R_j - R_j\, M_i$, $1 \leq i, j \leq n$, are also convolution operators on $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$, namely,

$$[M_i, R_j] u(x) = k_{ij} * u(x), \quad u \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}),\ x \in \mathbb{R}^n, \tag{5.4}$$

with the kernels k_{ij} given by

$$k_{ij}(x) = x_i k_j(x) = x_j k_i(x), \quad x \in \mathbb{R}^n_0, \tag{5.5}$$

where k_i and k_j are defined by equation (5.2). From (5.5) we obviously get

$$[M_i, R_j] = [M_j, R_i], \quad 1 \le i, j \le n. \tag{5.6}$$

We next form the commutator operator matrix $\mathcal{C} = ([M_i, R_j])_{i,j=1}^n$ regarded as an operator on the Schwartz space $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^n) \equiv \mathbb{C}^n \otimes \mathcal{S}(\mathbb{R}^n, \mathbb{C})$ of $\mathbb{C}^n$-valued functions on $\mathbb{R}^n$. Actually, using the notation introduced in Subsection 2.1 and the conventions made there, we can represent each function $u \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^n)$ as

$$u = \sigma_1 \otimes u_1 + \sigma_2 \otimes u_2 + \ldots + \sigma_n \otimes u_n \in \mathfrak{S}_n^1 \otimes \mathcal{S}(\mathbb{R}^n, \mathbb{C}),$$

and interpret $\mathcal{C}$ as the coefficient operator matrix of the left quadratic operator form $\mathcal{R}_{\mathrm{L}}(\mathcal{C})$ on the space $\mathfrak{S}_n^{\#} \otimes \mathcal{S}(\mathbb{R}^n, \mathbb{C})$ of $S(\mathbb{R}^n, \mathbb{C})$-valued spinors given by

$$\mathcal{R}_{\mathrm{L}}(\mathcal{C}) = \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j^* \otimes [M_i, R_j]. \tag{5.7}$$

On the space $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$ we now take the inner product inherited from the standard Lebesgue space $L^2(\mathbb{R}^n)$ of square integrable complex-valued functions on $\mathbb{R}^n$, and convert $\mathfrak{S}_n^{\#} \otimes \mathcal{S}(\mathbb{R}^n, \mathbb{C})$ and $\mathfrak{S}_n^1 \otimes \mathcal{S}(\mathbb{R}^n, \mathbb{C})$ into inner product spaces. We have the following technical result.

Lemma 5.1. *The commutator operator matrix $\mathcal{C} = ([M_i, R_j])_{i,j=1}^n$ is positive semidefinite, i.e.,*

$$\langle \mathcal{C}u, u \rangle \ge 0 \tag{5.8}$$

for any $u \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^n) \equiv \mathfrak{S}_n^1 \otimes \mathcal{S}(\mathbb{R}^n, \mathbb{C})$.

A complete proof of Lemma 5.1 based on a Fourier transform argument can be found in Martin [22]. To make a point, we note that by taking the Fourier transforms of the kernels k_i given in equation (5.2), we get

$$(R_i u)\widehat{\ }(\xi) = -\sqrt{-1}\, \xi_i \, \|\xi\|^{-1} \, \widehat{u}(\xi), \quad 1 \le i \le n, \, u \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}), \quad \xi \in \mathbb{R}_0^n, \tag{5.9}$$

a set of equations that imply several basic properties of the Riesz transforms. For instance, one gets that each R_i, $1 \le i \le n$, extends to a skew-adjoint operator on $L^2(\mathbb{R}^n)$, such that

$$\sum_{i=1}^{n} R_i^* R_i = \mathrm{Id}_{L^2(\mathbb{R}^n)}. \tag{5.10}$$

More properties of Riesz transforms are discussed in Stein [38]. As far as Lemma 5.1 is concerned, we need the Fourier transforms of the kernels k_{ij} defined by (5.5), which are given by

$$\widehat{k}_{ij}(\xi) = \delta_{ij} |\xi|^{-1} - \xi_i \xi_j |\xi|^{-3}, \quad 1 \le i, j \le n, \; \xi \in \mathbb{R}_0^n. \tag{5.11}$$

Lemma 5.1 proves essential in setting up Riesz transforms models of jointly hyponormal self-adjoint pairs in higher dimension.

5.2. RIESZ TRANSFORMS MODELS

We take the Lebesgue space $\mathcal{H} = L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a compact set, and identify it with the subspace of $L^2(\mathbb{R}^n)$ consisting of functions equal to 0 on $\mathbb{R}^n \setminus \Omega$. Let P_Ω denote the orthogonal projection of $L^2(\mathbb{R}^n)$ onto $L^2(\Omega)$. The first step in setting up a singular integral model of jointly hyponormal pairs of self-adjoint n-tuples of operators on $\mathcal{H} = L^2(\Omega)$ consists in choosing $n+1$ functions $a_1, a_2, \ldots, a_n$, $b \in L^\infty(\Omega)$, where each $a_1, a_2, \ldots, a_n$ is a real-valued function and b is different from 0 almost everywhere on Ω. For reasons that will eventually transpire, we refer to function b as a *primary parameter*, and to the functions $a_1, a_2, \ldots, a_n$ as *secondary parameters*. As a second step, we let $A_1, A_2, \ldots, A_n$, $B \in \mathfrak{L}(L^2(\Omega))$ stand for the multiplication operators given by

$$A_i\, u(x) = a_i(x)\, u(x), \quad u \in L^2(\Omega),\, 1 \leq i \leq n, \quad x \in \Omega, \tag{5.12}$$

$$B\, u(x) = b(x)\, u(x), \quad u \in L^2(\Omega),\ x \in \Omega. \tag{5.13}$$

The operators $A_1, A_2, \ldots, A_n$ are self-adjoint. We next define two self-adjoint n-tuples of operators on $L^2(\Omega)$, denoted by $X = (X_1, X_2, \ldots, X_n)$ and $Y = (Y_1, Y_2, \ldots, Y_n)$, by setting

$$X_i = M_i \mid L^2(\Omega), \quad 1 \leq i \leq n, \tag{5.14}$$

with M_i given by equation (5.3), and

$$Y_i = A_i - \sqrt{-1}\, B^*\, P_\Omega\, R_i\, P_\Omega\, B, \quad 1 \leq i \leq n,$$

where R_i are the Riesz operators defined by equation (5.1) considered as bounded linear operators on $L^2(\mathbb{R}^n)$.

Theorem 5.2. *The self-adjoint pair (X, Y) of n-tuples of operators on $L^2(\Omega)$ defined by equations* (5.14) *and* (5.2) *is jointly hyponormal.*

Proof. We need to show that the self-commutator operator matrix $\mathcal{C}(X, Y)$ of the self-adjoint pair (X, Y) given by equation (3.11) is positive semidefinite. We first notice that

$$[X_i, Y_j] = [X_j, Y_i] = -\sqrt{-1}\, B^*\, P_\Omega\, [M_i, R_j]\, P_\Omega\, B, \quad 1 \leq i, j \leq n. \tag{5.15}$$

Substituting these equations into (3.11) we have

$$\mathcal{C}(X, Y) = 2\, \left(B^*\, P_\Omega\, [M_i, R_j]\, P_\Omega\, B\right)_{i,j=1}^n . \tag{5.16}$$

It remains to observe that whenever $u \in \mathfrak{S}_n^1 \otimes L^2(\Omega) \subset \mathfrak{S}_n^1 \otimes L^2(\mathbb{R}^n)$ we get

$$\langle\, \mathcal{C}(X, Y)u, u\,\rangle = 2\langle\, \mathcal{C}_\Omega\, Bu, Bu\,\rangle, \tag{5.17}$$

where $\mathcal{C}_\Omega$ is the compression of the commutator operator matrix $\mathcal{C}$ defined by equation (5.7) to the subspace $\mathfrak{S}_n^1 \otimes L^2(\Omega)$. Since any $u \in \mathfrak{S}_n^1 \otimes L^2(\Omega)$ can be approximated by restrictions to Ω of functions from $\mathfrak{S}_n^1 \otimes \mathcal{S}(\mathbb{R}^n, \mathbb{C})$, Lemma 5.1 implies $\mathcal{C}_\Omega \geq 0$, whence $\mathcal{C}(X, Y) \geq 0$. ☐

5.3. FURTHER RESULTS AND COMMENTS

We conclude this section with a few remarks and additional results. First, we should observe that the jointly hyponormal self-adjoint pair (X,Y) of n-tuples of operators on $L^2(\Omega)$ which we defined and analyzed in Subsection 5.2 has several peculiar properties. In addition to (5.15), both X and Y are commuting n-tuples, so X and Y satisfy equations (4.2) and (4.3) in Section 4. Consequently, the n-tuple

$$T = X + \sqrt{-1}\,Y \tag{5.18}$$

of operators on $L^2(\Omega)$ is commuting. Moreover, direct calculations show that the right and left self-commutator operator matrices of T defined in equation (3.10) are given by, respectively,

$$\mathcal{C}_{\mathrm{R}}(T) = \mathcal{C}(X,Y), \quad \mathcal{C}_{\mathrm{L}}(T) = -\mathcal{C}(X,Y). \tag{5.19}$$

Therefore, according to Definitions 3.7 and 3.8, and based on Theorem 3.13, T is a two-sided hyponormal n-tuple. By Corollary 4.2 the spectrum $\sigma(T)$ and the essential spectrum $\sigma_{\mathrm{ess}}(T)$ of T coincide with the right spectrum $\sigma_{\mathrm{R}}(T)$ and the essential right spectrum $\sigma_{\mathrm{ess,R}}(T)$, so presumably one can find a simple description of these spectra in terms of Ω and the parameters of the model. It seems also appropriate to ask ourselves whether any n-tuple T of Hilbert space operators for which the associated self-adjoint pair (X,Y) has all the above mentioned additional properties is unitarily equivalent to a Riesz transforms model on a direct integral Hilbert space.

Returning to the Riesz transforms model (X,Y) of n-tuples of operators on $L^2(\Omega)$ we want to mention the following result which is a Putnam type inequality in higher dimension.

Theorem 5.3. *The operator form* $\mathcal{R}(X,Y)$ *in the Bochner-Weitzenböck self--commutator identity* $\mathrm{BW}(X,Y)$ *of the self-adjoint pair* (X,Y) *satisfies the inequality*

$$\|\mathcal{R}(X,Y)\| \le 2\,n\,\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}\,|\,\mathbb{B}^n\,|^{(n-1)/n}\,|\,\Omega\,|^{1/n}\,\|b\|^2_{L^\infty(\Omega)}, \tag{5.20}$$

where $\mathbb{B}^n$ *is the closed unit ball in* $\mathbb{R}^n$, $|\,\mathbb{B}^n\,|$ *and* $|\,\Omega\,|$ *are the Lebesgue measures of* $\mathbb{B}^n$ *and* Ω, *and* $b \in L^\infty(\Omega)$ *is the primary parameter of the model.*

For a complete proof of (5.1) and for some other more general related results we refer to Martin [20–24, 26, 27], and [30].

REFERENCES

[1] A. Athavale, *On joint hyponormal operators*, Proc. Amer. Math. Soc. **103** (1988), 417–423.

[2] N. Berline, E. Getzler, M. Vergne, *Heat kernels and Dirac operators*, [in:] Grundlehren der mathematischen Wissenschaften, vol. 298, Springer Verlag, Berlin, Heidelberg, New York, 1992.

[3] S. Bochner, *Curvature and Betti numbers. I, II*, Ann. of Math. **49, 50** (1948, 1949), 379–390, 79–93.

[4] S. Bochner, K. Yano, *Curvature and Betti Numbers*, Ann. of Math. Studies **32**, Princeton University Press, Princeton, 1953.

[5] F. Brachx, R. Delanghe, F. Sommen, *Clifford Analysis*, Pitman Research Notes in Mathematics Series, **76**, 1982.

[6] J. Bunce, *The joint spectrum of commuting non-normal operators*, Proc. Amer. Math. Soc. **29** (1971), 499–505.

[7] M. Cho, R.E. Curto, T. Huruya, W. Zelazko, *Cartesian form of Putnam's inequality for doubly commuting n-tuples*, Indiana Univ. Math. J. **49** (2000), 1437–1448.

[8] K.F. Clancey, *Seminormal Operators*, Lecture Notes in Math. **742**, Springer Verlag, Berlin, Heidelberg, New York, 1979.

[9] J.B. Conway, *Subnormal Operators*, Pitman, Boston, 1981.

[10] R.E. Curto, *Applications of several complex variables to multiparameter spectral theory*, [in:] J.B. Conway, B.B. Morrel (eds), Surveys of Recent Results in Operator Theory, Vol. II, Longman Publishing Co., London, **278** (1988), 25–90.

[11] R.E. Curto, *Joint hyponormality: a bridge between hyponormality and subnormality*, [in:] W.B. Arveson, R.G. Douglas (eds), Operator Theory, Operator Algebras and Applications, Part II, Proc. Sympos. Pure Math. **51** (1990), 69–91.

[12] R.E. Curto, R. Jian, *A matricial identity involving the self-commutator of a commuting n-tuple*, Proc. Amer. Math. Soc. **121** (1994), 461–464.

[13] R.E. Curto, P.A. Muhly, J. Xia, *Hyponormal pairs of commuting operators*, [in:] Operator Theory: Advances and Applications, vol. 35, Birkhäuser Verlag, 1988, 1–22.

[14] R.G. Douglas, V. Paulsen, K. Yan, *Operator theory and algebraic geometry*, Bull. Amer. Math. Soc. **20** (1988), 67–71.

[15] J.E. Gilbert, M.A.M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge Studies in Advanced Mathematics 26, Cambridge University Press, Cambridge, 1991.

[16] R. Goldberg, *Curvature and Homology*, Academic Press, New York, London, 1962.

[17] T. Kato, *Smooth operators and commutators*, Studia Math. **31** (1968), 531–546.

[18] H.B. Lawson, M.-L. Michelsohn, *Spin Geometry*, Princeton Mathematical Series 38, Princeton University Press, Princeton, 1989.

[19] M. Martin, *Joint seminormality and Dirac operators*, Integral Equations Operator Theory **30** (1998), 101–121.

[20] M. Martin, *Higher-dimensional Ahlfors-Beurling inequalities in Clifford analysis*, Proc. Amer. Math. Soc. **126** (1998), 2863–2871.

[21] M. Martin, *Convolution and maximal operator inequalities in Clifford analysis*, [in:] Clifford Algebras and Their Applications in Mathematical Physics, Vol. II, Clifford Analysis, Progress in Physics 9, Birkhäuser Verlag, Basel, 2000, 95–113.

[22] M. Martin, *Self-commutator inequalities in higher dimension*, Proc. Amer. Math. Soc. **130** (2002), 2971–2983.

[23] M. Martin, *Spin geometry, Clifford analysis, and joint seminormality*, [in:] Advances in Analysis and Geometry, Vol. I, Trends in Mathematics Series, Birkhäuser Verlag, Basel, 2004, 227–255.

[24] M. Martin, *Uniform approximation by solutions of elliptic equations and seminormality in higher dimensions*, Operator Theory: Advances and Applications, vol. 149, Birkhäuser Verlag, Basel, 2004, 387–406.

[25] M. Martin, *Deconstructing Dirac operators. I: Quantitative Hartogs–Rosenthal theorems*, Proceedings of the 5th International Society for Analysis, Its Applications and Computation Congress, ISAAC 2005, Catania, Italy, 2005, More Progress in Analysis, World Scientific, 2009, 1065–1074.

[26] M. Martin, *Deconstructing Dirac operators. III: Dirac and semi-Dirac pairs of differential operators*, Operator Theory: Advances and Applications, vol. 203, Birkhäuser Verlag, Basel, 2009, 347–362.

[27] M. Martin, *Deconstructing Dirac operators. II: Integral representation formulas*, Proceedings of the 7th International Society for Analysis, Its Applications and Computation Congress, ISAAC 2009, London, United Kingdom, 2009, Trends in Mathematics: Hypercomplex Analysis and Applications, Springer Verlag, Berlin, Heidelberg, New York, 2011, 195–211.

[28] M. Martin, M. Putinar, *Lectures on Hyponormal Operators*, Operator Theory: Advances and Applications, vol. 39, Birkhäuser Verlag, Basel, 1989.

[29] M. Martin, N. Salinas, *Weitzenböck type formulas and joint seminormality*, [in:] Contemporary Math., Amer. Math. Soc. **212** (1998), 157–167.

[30] M. Martin, P. Szeptycki, *Sharp inequalities for convolution operators with homogeneous kernels and applications*, Indiana Univ. Math. J. **46** (1997), 975–988.

[31] S. McCullough, V. Paulsen, *A note on joint hyponormality*, Proc. Amer. Math. Soc. **107** (1989), 187–195.

[32] P.S. Muhly, *A note on commutators and singular integrals*, Proc. Amer. Math. Soc. **54** (1976), 117–121.

[33] J.D. Pincus, *Commutators and systems of singular integral equations*, Acta Math. **121** (1968), 219–249.

[34] J.D. Pincus, D. Xia, *The analytic model of a hyponormal operator with rank one self-commutator*, Integral Equations Operator Theory **4** (1981), 134–150.

[35] J.D. Pincus, D. Xia, J. Xia, *The analytic model of a hyponormal operator with rank one self-commutator*, Integral Equations Operator Theory **7** (1984), 516–535.

[36] C.R. Putnam, *Commutation Properties of Hilbert Space Operators and Related Topics*, Springer Verlag, Berlin, Heidelberg, New York, 1967.

[37] C.R. Putnam, *An inequality for the area of hyponormal spectra*, Math. Z. **116** (1970), 323–330.

[38] E.M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, 1993.

[39] J.L. Taylor, *A joint spectrum for several commuting operators*, J. Funct. Anal. **6** (1970), 172–191.

[40] F.-H. Vasilescu, *A characterization of the joint spectrum in Hilbert spaces*, Rev. Roumaine Math. Pures Appl. **22** (1977), 1003–1009.

[41] F.-H. Vasilescu, *A multidimensional spectral theory in C^*-algebras*, Banach Center Publ. **8** (1982), 471–491.

[42] D. Xia, *On non-normal operators. I*, Chinese J. Math. **3** (1963), 232–246; *II*, Acta Math. Sinica **21** (1987), 103–108.

[43] D. Xia, *Spectral Theory of Hyponormal Operators*, Birkhäuser Verlag, Basel, Boston, Stuttgart, 1983.

[44] D. Xia, *On some classes of hyponormal tuples of commuting operators*, Operator Theory: Advances and Applications, vol. 48, Birkhäuser Verlag, 1990, 423–448.

Mircea Martin
mircea.martin@bakeru.edu

Baker University
Department of Mathematics
Baldwin City, KS 66006, USA

Received: October 2, 2016.
Accepted: November 2, 2016.