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FRACTIONAL BOUNDARY VALUE PROBLEMS ON THE HALF LINE

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Abstract. In this paper, we focus on the solvability of a fractional boundary value problem at resonance on an unbounded interval. By constructing suitable operators, we establish an existence theorem upon the coincidence degree theory of Mawhin. The obtained results are illustrated by an example.

Keywords: boundary value problem at resonance, existence of solution, unbounded interval, coincidence degree of Mawhin, fractional differential equation.

Mathematics Subject Classification: 34B40, 34B15.

1. INTRODUCTION

In this paper, we are concerned with the existence of solutions for the following fractional differential equation

$$D_{0^+}^{q}x(t) = f(t, x(t)) + e(t), \quad t \in (0, \infty), \tag{1.1}$$

subject to the nonlocal conditions

$$I_{0^+}^{2-q}x(0) = 0, \quad D_{0^+}^{q-1}x(\infty) = \frac{\Gamma(q)}{\eta^{q-1}}x(\eta), \tag{1.2}$$

where D_0^q denotes the Riemann Liouville fractional derivative of order q, $1 < q < 2$, $\eta > 0$, $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $e : [0, \infty) \to \mathbb{R}$ are given functions satisfying some conditions.

The problem (1.1)–(1.2) happens to be at resonance in the sense that the dimension of the kernel of the linear operator $Lx = D_{0^+}^{q}x$ is not less than one under boundary conditions (1.2).

It should be noted that in recent years, there have been many works related to boundary value problems at resonance for ordinary differential equations. We refer the reader to [7–9, 11, 13, 14, 16] and the references therein.

However, the articles on the existence of solutions of fractional differential equations on the half-line are still few in number, and most of them deal with problems under nonresonance conditions. For some recent articles investigating resonant and nonresonant fractional problems on the unbounded interval, see [1–5, 10, 17].

Recently fractional differential equations have been investigated by many researchers and different methods have been used to obtain such fixed point theory, upper and lower solutions method, coincidence degree theory, etc.

The rest of this paper is organized as follows. In Section 2, we give some necessary notations, definitions and lemmas. In Section 3, we study the existence of solutions of (1.1)–(1.2) by the coincidence degree theory due to Mawhin. Finally, we give an example to illustrate the obtained results.

2. PRELIMINARIES

First recall some notation, definitions and theorems which will be used later.

Let X and Y be two real Banach spaces and let $L : \operatorname{dom} L \subset X \to Y$ be a linear operator which is a Fredholm map of index zero, define the continuous projections P and Q, respectively, by $P : X \to X$, $Q : Y \to Y$ such that $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$. Then $X = \ker L \oplus \ker P$, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$, thus $L\,|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ is invertible, denote its inverse by K_P.

Let Ω be an open bounded subset of X such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the map $N : X \to Y$ is said to be L-compact on $\overline{\Omega}$ if $QN\left(\overline{\Omega}\right)$ is bounded and $K_P\left(I - QN\right) : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\ker L$ then, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$. It is known that the coincidence equation $Lx = Nx$ is equivalent to $x = (P + JQN)x + KP(I - Q)Nx$.

Theorem 2.1 ([15])**.** *Let L be a Fredholm operator of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(1) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0, 1)$,
(2) $Nx \notin \operatorname{Im} L$ *for every* $x \in \ker L \cap \partial\Omega$,
(3) $\deg\left(QN\,|_{\ker L}, \Omega \cap \ker L, 0\right) \neq 0$, *where* $Q : Y \to Y$ *is a projection such that* $\operatorname{Im} L = \ker Q$.

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Theorem 2.2 ([6])**.** *Let $C_\infty = \{y \in C\left([0, \infty)\right) : \lim_{t\to\infty} y\left(t\right)$ exists$\}$ equipped with the norm $\|y\|_\infty = \sup_{t\in[0,\infty)} |y(t)|$. Let $F \subset C_\infty$. Then F is relatively compact if the following conditions hold:*

(1) *F is bounded in C_∞,*
(2) *the functions belonging to F are equicontinuous on any compact subinterval of* $[0, \infty)$,
(3) *The functions from F are equiconvergent at ∞.*

Let X be the space

$$X = \Big\{ x \in C\left([0,\infty)\right) : D_{0^+}^{q-1} x \in C\left([0,\infty)\right), \\ \lim_{t\to\infty} e^{-t}\left|x(t)\right| \text{ and } \lim_{t\to\infty} e^{-t}\left|D_{0^+}^{q-1} x(t)\right| \text{ exist} \Big\}$$

endowed with the norm

$$\|x\| = \sup_{t\ge 0} e^{-t}\left|x(t)\right| + \sup_{t\ge 0} e^{-t}\left|D_0^{q-1} x(t)\right|,$$

then X is a Banach space.

Let $Y = L^1[0,\infty)$ equipped with the norm

$$\|y\|_1 = \int_0^\infty |y(t)|dt.$$

Define the operator $L : \operatorname{dom} L \subset X \to Y$ by $Lx = D_{0^+}^q x$, where

$$\operatorname{dom} L = \Big\{ x \in X : D_{0^+}^q x \in Y, I_{0^+}^{2-q} x(0) = 0, D_{0^+}^{q-1} x(\infty) = \frac{\Gamma(q)}{\eta^{q-1}} x(\eta) \Big\} \subset X,$$

then L maps $\operatorname{dom} L$ into Y. Let $N : X \to Y$ be the operator

$$Nx(t) = f(t, x(t)) + e(t), \quad t \in [0,\infty),$$

then the problem (1.1)–(1.2) can be written as $Lx = Nx$.

We recall the definition of the Riemann-Liouville fractional derivative and Riemann-Liouville fractional integral, we can find their properties in [12].

Definition 2.3. The Riemann-Liouville fractional integral of order $p > 0$ is given by

$$I_{0^+}^p g(t) = \frac{1}{\Gamma(p)} \int_0^t \frac{g(s)}{(t-s)^{1-p}} ds, \quad t > 0,$$

provided that the right-hand side exists.

Definition 2.4. The Riemann-Liouville fractional derivative $D_{0^+}^p g$ of order $p > 0$ is given by

$$D_{0^+}^p g(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{p-n+1}} ds, \quad t > 0,$$

where $n = [p] + 1$ ($[p]$ is the integer part of p), provided that the right-hand side is point-wise defined on $(0,\infty)$.

Lemma 2.5. *The homogenous fractional differential equation* $D_{a^+}^q g(t) = 0$ *has a solution*

$$g(t) = c_1 t^{q-1} + c_2 t^{q-2} + \ldots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, \ldots, n$ *and* $n = [q] + 1$.

Lemma 2.6. *Let $p, q \geq 0$, $f \in L_1[0,\infty)$. Then the following assertions hold:*

1. $I_{0+}^{p} I_{0+}^{q} f(t) = I_{0+}^{p+q} f(t) = I_{0+}^{q} I_{0+}^{p} f(t)$ *and* $D_{0+}^{q} I_{0+}^{q} f(t) = f(t)$ *for all* $t \in [0,\infty)$.
2. *If* $p > q > 0$, *then the formula* $D_{0+}^{q} I_{0+}^{p} f(t) = I_{0+}^{p-q} f(t)$ *holds almost everywhere on* $t \in [0,\infty)$, *for* $f \in L_1[0,\infty)$ *and it is valid at any point* $t \in [0,\infty)$ *if* $f \in C[0,\infty)$.
3. *If* $q \geq 0$ *and* $p > 0$, *then* $\left(I_{0+}^{q} t^{p-1}\right)(x) = \frac{\Gamma(p)}{\Gamma(p+q)} x^{p+q-1}$, $\left(D_{0+}^{q} t^{p-1}\right)(x) = \frac{\Gamma(p)}{\Gamma(p-q)} x^{p-q-1}$, $\left(D_{0+}^{q} t^{q-j}\right)(x) = 0$, $j = 1, 2, \ldots, n$, *where* $n = [q] + 1$.

3. MAIN RESULTS

We give the first result on the existence of a solution for the problem (1.1)–(1.2).

Theorem 3.1. *Assume that the following conditions are satisfied:*

(H_1) *There exists nonnegative functions* α, $\beta \in L^1[0,\infty)$ *such that for all* $x \in \mathbb{R}$ $t \in [0,\infty)$ *we have*

$$|f(t,x) + e(t)| \leq e^{-t}\alpha(t)|x| + \beta(t). \tag{3.1}$$

(H_2) *There exists a constant* $M > 0$ *such that for* $x \in \operatorname{dom} L$ *if* $\left|D_0^{q-1} x(t)\right| > M$ *for all* $t \in [0,\infty)$

$$\int_0^\infty (f(s,x(s)) + e(s))\, ds - \frac{\Gamma(q)}{\eta^{q-1}} \int_0^\eta (\eta - s)^{q-1} (f(s,x(s)) + e(s))\, ds \neq 0. \tag{3.2}$$

(H_3) *There exists a constant* $M^* > 0$ *such that for any* $x(t) = c_0 t^{q-1} \in \ker L$ *with* $|c_0| > M^*$ *either*

$$c_0 \left[\int_0^\infty (f(s,x(s)) + e(s))\, ds - \frac{\Gamma(q)}{\eta^{q-1}} \int_0^\eta (\eta - s)^{q-1} (f(s,x(s)) + e(s))\, ds\right] < 0 \tag{3.3}$$

or

$$c_0 \left[\int_0^\infty (f(s,x(s)) + e(s))\, ds - \frac{\Gamma(q)}{\eta^{q-1}} \int_0^\eta (\eta - s)^{q-1} (f(s,x(s)) + e(s))\, ds\right] > 0. \tag{3.4}$$

Then for every $e \in Y$, *the problem* (1.1)–(1.2) *has at least one solution in* X, *provided* $(1 - 2A_1 \|\alpha\|_1) > 0$, *where*

$$A_1 = M_1 + 1, \quad M_1 = \max_{t \geq 0} \left(\frac{e^{-t} t^{q-1}}{\Gamma(q)}\right) = \frac{\left[e^{-1}(q-1)\right]^{q-1}}{\Gamma(q)}.$$

In order to prove Theorem 3.1 we need some lemmas.

Lemma 3.2. *The operator* $L : \operatorname{dom} L \subset X \to Y$ *is a Fredholm operator of index zero. Furthermore, the linear projector operator* $Q : Y \to Y$ *is defined by*

$$Qy(t) = Ae^{-t}\left(\int_0^\infty y(s)ds - \frac{\Gamma(q)}{\eta^{q-1}}\int_0^\eta (\eta - s)^{q-1}y(s)ds\right) \tag{3.5}$$

and the linear operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ *can be written as*

$$K_p y(t) = I_{0+}^q y(t), \quad y \in \operatorname{Im} L.$$

Morover,

$$\|K_p y\| \leq A_1 \|y\|_1, \quad y \in \operatorname{Im} L, \tag{3.6}$$

where

$$A^{-1} = 1 - \frac{\Gamma(q)}{\eta^{q-1}}\int_0^\eta (\eta - s)^{q-1} e^{-s} ds.$$

Proof. Using Lemma 2.5, we get

$$x(t) = at^{q-1} + bt^{q-2}, \quad t > 0, \tag{3.7}$$

as a solution of $Lx(t) = 0$. Applying I_{0+}^{2-q} to both sides of the equality (3.7), then using condition $I_{0+}^{2-q}x(0) = 0$, we get $b = 0$. Thanks to condition $D_{0+}^{q-1}x(\infty) = \frac{\Gamma(q)}{\eta^{q-1}}x(\eta)$, we conclude

$$\ker L = \left\{x \in \operatorname{dom} L : x(t) = at^{q-1},\ a \in \mathbb{R},\ t \in [0, \infty)\right\}.$$

Now the problem

$$D_{0+}^q x(t) = y(t) \tag{3.8}$$

has a solution x that satisfies the conditions $I_{0+}^{2-q}x(0) = 0, D_{0+}^{q-1}x(\infty) = \frac{\Gamma(q)}{\eta^{q-1}}x(\eta)$ if and only if

$$\int_0^\infty y(s)ds - \frac{\Gamma(q)}{\eta^{q-1}}\int_0^\eta (\eta - s)^{q-1}y(s)ds = 0. \tag{3.9}$$

In fact from (3.8) and together with the boundary condition $I_{0+}^{2-q}x(0) = 0$ we get

$$x(t) = I_{0+}^q y(t) + at^{q-1}.$$

According to $D_{0+}^{q-1}x(\infty) = \frac{\Gamma(q)}{\eta^{q-1}}x(\eta)$, we obtain

$$\int_0^\infty y(s)ds = \frac{\Gamma(q)}{\eta^{q-1}} I_{0+}^q y(\eta).$$

On the other hand, if (3.9) holds, setting

$$x(t) = I_{0+}^q y(t) + at^{q-1},$$

where a is an arbitrary constant, then $x(t)$ is a solution of (3.8). Hence

$$\operatorname{Im} L = \left\{ y \in Y : \int_0^\infty y(s)ds - \frac{\Gamma(q)}{\eta^{q-1}} \int_0^\eta (\eta - s)^{q-1} y(s)ds = 0 \right\}.$$

Now taking (3.5) into account, it yields

$$\begin{aligned} Q^2 y = Q\left(Qy\right) = A^2 e^{-t} &\left(\int_0^\infty y\left(s\right) ds - \frac{\Gamma(q)}{\eta^{q-1}} \int_0^\eta (\eta - s)^{q-1} y(s)ds \right) \\ &\times \left(\int_0^\infty e^{-s} ds - \frac{\Gamma(q)}{\eta^{q-1}} \int_0^\eta (\eta - s)^{q-1} e^{-s} ds \right) = Qy, \end{aligned}$$

thus Q is a continuous projector and $\operatorname{Im} L = \ker Q$. Rewrite $y = (y - Qy) + Qy$, then $y - Qy \in \ker Q = \operatorname{Im} L$, $Qy \in \operatorname{Im} Q$ and $\operatorname{Im} Q \cap \operatorname{Im} L = \{0\}$, then $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. Thus $\dim \ker L = 1 = \dim \operatorname{Im} Q = co \dim \operatorname{Im} L = 1$, this means that L is a Fredholm operator of index zero. Now define a projector P from X to X by setting

$$Px(t) = \frac{D_{0^+}^{q-1} x(0)}{\Gamma\left(q\right)} t^{q-1},$$

and the generalized inverse K_P: $\operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ of L as

$$K_p y(t) = I_{0^+}^q y(t) = \frac{1}{\Gamma\left(q\right)} \int_0^t \left(t - s\right)^{q-1} y(s)ds.$$

Obviously, $\operatorname{Im} P = \ker L$ and $P^2 x = Px$. It follows from $x = (x - Px) + Px$ that $X = \ker P + \ker L$. By simple calculation, we can get $\ker L \cap \ker P = \{0\}$. Hence $X = \ker L \oplus \ker P$.

Let us show that the generalized inverse of L is K_P. In fact, for $y \in \operatorname{Im} L$, we have

$$\left(LK_p\right) y(t) = D_{0^+}^q \left(K_p y\left(t\right)\right) = D_{0^+}^q I_{0^+}^q y(t) = y\left(t\right),$$

and for $x \in \operatorname{dom} L \cap \ker P$, we obtain

$$\begin{aligned} \left(K_p L\right) x(t) &= \left(K_p\right) D_{0^+}^q x(t) = I_{0^+}^q D_{0^+}^q x(t) \\ &= x(t) - \frac{D_{0^+}^{q-1} x(0)}{\Gamma\left(q\right)} t^{q-1} - \frac{I_{0^+}^{2-q} x(0)}{\Gamma\left(q-1\right)} t^{q-2}. \end{aligned}$$

In view of $x \in \operatorname{dom} L \cap \ker P$, then $I_{0^+}^{2-q} x\left(0\right) = 0$ and $Px = 0$, thus

$$(K_p L) x(t) = x(t).$$

This shows that $K_p = \left(L|_{\operatorname{dom} L \cap \ker P} \right)^{-1}$. From the definition of K_p, we have

$$\begin{aligned} e^{-t} \left| (K_p y)(t) \right| &\le \frac{e^{-t}}{\Gamma(q)} \int_0^t (t-s)^{q-1} |y(s)| \, ds \\ &\le \frac{e^{-t} t^{q-1}}{\Gamma(q)} \int_0^\infty |y(s)| \, ds \le \max_{t \ge 0} \left(\frac{e^{-t} t^{q-1}}{\Gamma(q)} \right) \|y\|_1 = M_1 \|y\|_1 . \end{aligned}$$

$$\left| D_{0^+}^{q-1} (K_p y)(t) \right| = \left| I_{0^+}^1 y(t) \right| \le \int_0^\infty |y(s)| \, ds = \|y\|_1 ,$$

that leads to

$$\|K_p y\| \le (M_1 + 1) \|y\|_1 = A_1 \|y\|_1 .$$

This completes the proof. □

Lemma 3.3. *Let*

$$\Omega_1 = \{ x \in \operatorname{dom} L \backslash \ker L : Lx = \lambda N x \quad \textit{for some } \lambda \in [0,1] \} .$$

Then Ω_1 is bounded.

Proof. Suppose that $x \in \Omega_1$ and $Lx = \lambda Nx$. Thus $\lambda \neq 0$ and $QNx = 0$, so

$$\int_0^\infty (f(s,x(s)) + e(s)) \, ds - \frac{\Gamma(q)}{\eta^{q-1}} \int_0^\eta (\eta - s)^{q-1} (f(s,x(s)) + e(s)) \, ds = 0.$$

Thus, by condition (H_2), there exists $t_0 \in \mathbb{R}_+$ such that $\left| D_{0^+}^{q-1} x(t_0) \right| \le M$. It follows from $D_{0^+}^q x(t) = D D_{0^+}^{q-1} x(t)$ and the fact that $D_{0^+}^q x \in Y$, that

$$\left| D_{0^+}^{q-1} x(0) \right| = \left| D_{0^+}^{q-1} x(t_0) - \int_0^{t_0} D_{0^+}^q x(s) \, ds \right| .$$

Then, we have

$$\left| D_{0^+}^{q-1} x(0) \right| \le M + \int_0^\infty |Lx(s)| \, ds \le M + \int_0^\infty |Nx(s)| \, ds = M + \|Nx\|_1 . \tag{3.10}$$

On the other hand, since $x \in \operatorname{dom} L \backslash \ker L$, then $(I-P)x \in \operatorname{dom} L \cap \ker P$ and $LPx = 0$, thus from Lemma 3.2, we get

$$\begin{aligned} \|(I-P)x\| &= \|K_p L(I-P)x\| \le A_1 \|L(I-P)x\|_1 \\ &= A_1 \|Lx\|_1 \le A_1 \|Nx\|_1. \end{aligned} \tag{3.11}$$

So
$$\|x\| \leq \|Px\| + \|(I-P)\,x\| = A_1 \left|D_{0^+}^{q-1}x(0)\right| + A_1\|Nx\|_1, \tag{3.12}$$
thanks to inequalities (3.10) and (3.11), (3.12) becomes
$$\|x\| \leq A_1 M + 2A_1\|Nx\|_1. \tag{3.13}$$
On the other hand, by (3.1), we have
$$\|Nx\|_1 = \int_0^\infty |f(s, x(s)) + e(s)|\, ds \leq \|x\|\, \|\alpha\|_1 + \|\beta\|_1. \tag{3.14}$$
Thus
$$\|x\| \leq A_1 M + 2A_1 \|x\|\, \|\alpha\|_1 + 2A_1\|\beta\|_1.$$
Since $(1 - 2A_1\|\alpha\|_1) > 0$, we obtain
$$\|x\| \leq \frac{A_1 M}{1 - 2A_1\|\alpha\|_1} + \frac{2A_1\|\beta\|_1}{1 - 2A_1\|\alpha\|_1} < \infty,$$
which proves that Ω_1 is bounded. □

Lemma 3.4. *The set $\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}$ is bounded.*

Proof. Let $x \in \Omega_2$, then $x \in \ker L$ implies $x(t) = ct^{q-1}$, $c \in \mathbb{R}$ and $QNx = 0$, therefore
$$\int_0^\infty \left(f\left(s, cs^{q-1}\right) + e(s)\right) ds - \frac{\Gamma(q)}{\eta^{q-1}} \int_0^\eta (\eta - s)^{q-1} \left(f\left(s, cs^{q-1}\right) + e(s)\right) ds = 0.$$
From condition (H_2) there exists $t_1 \geq 0$ such that $\left|D_{0^+}^{q-1}x(t_1)\right| \leq M$ and so $|c| \leq \frac{M}{\Gamma(q)}$. On the other hand,
$$\begin{aligned}\|x\| &= \sup_{t\geq 0} e^{-t}|x(t)| + \sup_{t\geq 0}\left|D_{0^+}^{q-1}x(t)\right| \leq |c| \left(\sup_{t\in[0\infty)} e^{-t}t^{q-1} + \Gamma(q)\right) \\ &\leq |c|\,\Gamma(q) A_1 \leq M A_1 < \infty,\end{aligned}$$
so Ω_2 is bounded. □

Lemma 3.5. *Suppose that the first part of Condition (H_3) of Theorem 3.1 holds. Let*
$$\Omega_3 = \{x \in \ker L : -\lambda Jx + (1-\lambda)\, QNx = 0, \lambda \in [0,1]\},$$
where $J : \ker L \to \operatorname{Im} Q$ is the linear isomorphism given by $J\left(ct^{q-1}\right) = ce^{-t}$ for all $c \in \mathbb{R}$, $t \geq 0$. Then Ω_3 is bounded.

Proof. Let $x_0 \in \Omega_3$, then $x_0(t) = c_0 t^{q-1}$ and $\lambda J x_0 = (1-\lambda)\, QNx_0$ that is equivalently written as

$$\lambda c_0 = (1-\lambda)\, A\Bigg(\int_0^\infty \left(f\left(s, c_0 s^{q-1}\right) + e(s)\right) ds - \frac{\Gamma(q)}{\eta^{q-1}} \int_0^\eta (\eta - s)^{q-1} \left(f\left(s, c_0 s^{q-1}\right) + e(s)\right) ds\Bigg).$$

If $\lambda = 1$, then $c_0 = 0$. Otherwise, if $|c_0| > M^*$, in view of (3.3) one has

$$\lambda c_0^2 = (1-\lambda) A c_0 \Bigg(\int_0^\infty \left(f\left(s, c_0 s^{q-1}\right) + e(s)\right) ds - \frac{\Gamma(q)}{\eta^{q-1}} \int_0^\eta (\eta - s)^{q-1} \left(f\left(s, c_0 s^{q-1}\right) + e(s)\right)\Bigg) < 0,$$

which contradicts the fact that $\lambda c_0^2 \geq 0$. So $|c_0| \leq M^*$, moreover

$$\|x_0\| = |c_0| \left(\sup_{t \geq 0} e^{-t} t^{q-1} + \Gamma(q)\right) \leq |c_0|\, \Gamma(q) A_1 \leq M^* \Gamma(q)\, A_1,$$

thus Ω_3 is bounded. □

Lemma 3.6. *Under the second part of Condition (H_3) of Theorem 3.1, the set Ω_3 is bounded.*

Proof. The proof is similar to the one of Lemma 3.5. □

Lemma 3.7. *Suppose that Ω is an open bounded subset of X such that* $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$. *Then N is L-compact on $\overline{\Omega}$.*

Proof. Suppose that $\Omega \subset X$ is a bounded set. Without loss of generality, we may assume that $\Omega = B(0, r)$, then for any $x \in \overline{\Omega}$, $\|x\| \leq r$. For $x \in \overline{\Omega}$, and by condition (3.1), we obtain

$$
\begin{aligned}
|QNx| &\leq Ae^{-t}\left(\int_0^{\infty}|f(s,x(s))+e(s)|\,ds\right.\\
&\quad\left.+\frac{\Gamma(q)}{\eta^{q-1}}\int_0^{\eta}(\eta-s)^{q-1}\,|f(s,x(s))+e\,(s)|\,ds\right)\\
&\leq Ae^{-t}\left[\int_0^{\infty}e^{-s}\alpha(s)\,|x(s)|+\beta(s)ds\right.\\
&\quad\left.+\frac{\Gamma(q)}{\eta^{q-1}}\int_0^{\eta}(\eta-s)^{q-1}\left(e^{-s}\alpha(s)\,|x\,(s)|+\beta(s)\right)ds\right]\\
&\leq Ae^{-t}\left(r\int_0^{\infty}\alpha(s)ds+\int_0^{\infty}\beta(s)ds+r\frac{\Gamma\,(q)}{\eta^{q-1}}\int_0^{\eta}(\eta-s)^{q-1}\alpha\,(s)\,ds\right.\\
&\quad\left.+\frac{\Gamma(q)}{\eta^{q-1}}\int_0^{\eta}(\eta-s)^{q-1}\,\beta(s)ds\right)\\
&\leq Ae^{-t}\,(1+\Gamma(q))\,(r\,\|\alpha\|_1+\|\beta\|_1)\,.
\end{aligned}
$$

Thus

$$
\|QNx\|_1\leq A\,(1+\Gamma(q))\,(r\,\|\alpha\|_1+\|\beta\|_1)\,, \tag{3.15}
$$

which implies that $QN\left(\overline{\Omega}\right)$ is bounded. Next, we show that $K_P\,(I-Q)\,N\left(\overline{\Omega}\right)$ is compact. For $x\in\overline{\Omega}$, by (3.1) we have

$$
\|Nx\|_1=\int_0^{\infty}|f\,(s,x\,(s))+e(s)|\,ds\leq r\,\|\alpha\|_1+\|\beta\|_1. \tag{3.16}
$$

On the other hand, from the definition of K_P and together with (3.6), (3.15) and (3.16) one gets

$$
\begin{aligned}
\|K_P\,(I-Q)\,Nx\| &\leq A_1\,\|(I-Q)Nx\|_1\leq A_1\,[\|Nx\|_1+\|QNx\|_1]\\
&\leq A_1\,(1+A\,(1+\Gamma(q)))\,(r\,\|\alpha\|_1+\|\beta\|_1)\,.
\end{aligned}
$$

It follows that $K_P\,(I-Q)\,N\left(\overline{\Omega}\right)$ is uniformly bounded.

Let us prove that $K_P(I-Q)N\left(\overline{\Omega}\right)$ is equicontinuous. For any $x \in \overline{\Omega}$, and any $t_1, t_2 \in [0,T]$, $t_1 < t_2$ with $T > 0$, we have

$$\begin{aligned}
&\left|e^{-t_1}\left(K_P(I-Q)Nx\right)(t_1) - e^{-t_2}\left(K_P(I-Q)Nx\right)(t_2)\right| = \\
&= \frac{1}{\Gamma(q)}\left|\int_0^{t_1} e^{-t_1}(t_1-s)^{q-1}(I-Q)Nx(s)ds\right. \\
&\quad \left.- \int_0^{t_2} e^{-t_2}(t_2-s)^{q-1}(I-Q)Nx(s)ds\right| \\
&\le \frac{1}{\Gamma(q)}\left[\int_0^{t_1}\left(e^{-t_2}(t_2-s)^{q-1} - e^{-t_1}(t_1-s)^{q-1}\right)|(I-Q)Nx(s)|\,ds\right. \\
&\quad \left.+ \int_{t_1}^{t_2} e^{-t_2}(t_2-s)^{q-1}|(I-Q)Nx(s)|\,ds\right] \\
&\le \frac{1}{\Gamma(q)}\left[e^{-t_1}\left(t_2^{q-1} - t_1^{q-1}\right)\int_0^{t_1}|(I-Q)Nx(s)|\,ds\right. \\
&\quad \left.+ e^{-t_2}t_2^{q-1}\int_{t_1}^{t_2}|(I-Q)Nx(s)|\,ds\right] \to 0,
\end{aligned}$$

as $t_1 \to t_2$. On the other hand, we have

$$\begin{aligned}
&\left|e^{-t_1}D_{0^+}^{q-1}\left(K_P(I-Q)Nx\right)(t_1) - e^{-t_1}D_{0^+}^{q-1}\left(K_P(I-Q)Nx\right)(t_2)\right| \\
&= \left|\int_0^{t_1} e^{-t_1}(I-Q)Nx(s)ds - \int_0^{t_2} e^{-t_2}(I-Q)Nx(s)ds\right| \\
&\le \int_0^{t_1}\left(e^{-t_1} - e^{-t_2}\right)|(I-Q)Nx(s)|\,ds + \int_{t_1}^{t_2} e^{-t_2}|(I-Q)Nx(s)|\,ds \\
&\le (t_2-t_1)\int_0^{t_1}|(I-Q)Nx(s)|\,ds + \int_{t_1}^{t_2}|(I-Q)Nx(s)|\,ds \to 0,
\end{aligned}$$

as $t_1 \to t_2$. So $K_P(I-Q)N\left(\overline{\Omega}\right)$ is equicontinuous on every compact subinterval of $[0,\infty)$. In addition, we claim that $K_P(I-Q)N\left(\overline{\Omega}\right)$ is equiconvergent at infinity. In fact,

$$\left|e^{-t}\left(K_p(I-Q)Nx\right)(t)\right| \le \frac{1}{\Gamma(q)}\int_0^t e^{-t}(t-s)^{q-1}|(I-Q)Nx(s)|\,ds$$

$$\leq \frac{e^{-t}t^{q-1}}{\Gamma(q)} \int_0^t |(I-Q)\, Nx(s)|\, ds \leq \frac{e^{-t}t^{q-1}}{\Gamma(q)} \|(I-Q)\, Nx\|_1$$
$$\leq \frac{e^{-t}t^{q-1}}{\Gamma(q)} \left(\|Nx\|_1 + \|QNx\|_1\right) \leq \frac{e^{-t}t^{q-1}}{\Gamma(q)} (1+2A) \left[r\|\alpha\|_1 + \|\beta\|_1\right].$$

On the other hand, we have

$$\left| e^{-t} D_{0^+}^{q-1} \left(K_p (I-Q)\, Nx\right)(t)\right| \leq e^{-t} \int_0^t |(I-Q)\, Nx(s)|\, ds$$
$$\leq e^{-t} \left(\|Nx\|_1 + \|QNx\|_1\right) \leq e^{-t} \left(1 + A\left(1+\Gamma(q)\right)\right)\left(r\|\alpha\|_1 + \|\beta\|_1\right).$$

Thus

$$\lim_{t\to\infty} e^{-t} \left(K_p (I-Q)\, Nx\right)(t) = 0, \lim_{t\to\infty} e^{-t} D_{0^+}^{q-1} \left(K_p (I-Q)\, Nx\right)(t) = 0,$$

consequently $K_P(I-Q)\, N\left(\overline{\Omega}\right)$ is equiconvergent at ∞. □

Now we are able to give the proof of Theorem 3.1.

Proof of Theorem 3.1. In what follows, we shall prove that all conditions of Theorem 2.1 are satisfied. Let Ω to be an open bounded subset of X such that $\cup_{i=1}^3 \overline{\Omega}_i \subset \Omega$. From Lemma 3.2, we know that L is a Fredholm operator of index zero. By Lemma 3.7, we have N is L-compact on $\overline{\Omega}$. By virtue of the definition of Ω, it yields

(i) $Lx \neq \lambda Nx$ for all $(x,\lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,1)$,
(ii) $Nx \notin \operatorname{Im} L$ for all $x \in \ker L \cap \partial\Omega$.

Now we prove that condition (iii) of Theorem 2.1 is satisfied. Let $H(x,\lambda) = \pm\lambda Jx + (1-\lambda)\, QNx$. Since $\overline{\Omega}_3 \subset \Omega$, then $H(x,\lambda) \neq 0$ for every $x \in \ker L \cap \partial\Omega$. By the homotopy property of degree, we get

$$\begin{aligned}\deg\left(QN\,|_{\ker L}, \Omega\cap\ker L, 0\right) &= \deg\left(H(\cdot,0), \Omega\cap\ker L, 0\right)\\ &= \deg\left(H(\cdot,1), \Omega\cap\ker L, 0\right)\\ &= \deg\left(\pm J, \Omega\cap\ker L, 0\right) \neq 0.\end{aligned}$$

So, the third assumption of Theorem 2.1 is fulfilled and $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, i.e. (1.1)–(1.2) has at least one solution in X. The proof is completed. □

Example 3.8. Consider the fractional boundary value problem

$$\begin{cases} D_{0^+}^q x(t) = f(t, x(t)) + e(t), \ t \in (0,\infty), \\ I_{0^+}^{2-q} x(0) = 0, \ D_{0^+}^{q-1} x(\infty) = \frac{\Gamma(q)}{\eta^{q-1}} x(\eta), \end{cases} \qquad (P)$$

where $q = \frac{3}{2}$, $f(t,x) = \frac{1}{4}e^{-t}\left(e^{-t}|x| - 8e^{-t}\right)$, $e(t) = 3e^{-t}$. Choosing $\alpha(t) = \frac{1}{4}e^{-t}$ and $\beta(t) = 3e^{-t} - 2e^{-2t}$, then α, β are nonnegative and belong to $L^1[0,\infty)$. For all $x \in \mathbb{R}$, $t \in [0,\infty)$ we have

$$|f(t,x) + e(t)| \leq e^{-t}\alpha(t)|x| + \beta(t),$$

so hypothesis (H_1) of Theorem 3.1 is satisfied. We claim that condition (H_3) is satisfied, indeed, since

$$\int_0^\infty (f(s,x(s)) + e(s))\,ds - \frac{\Gamma(q)}{\eta^{q-1}} \int_0^\eta (\eta - s)^{q-1} (f(s,x(s)) + e(s))\,ds$$
$$\geq \int_\eta^\infty (f(s,x(s)) + e(s))\,ds,$$

then for $M^* = 8$, $\eta = 1$ and any $x(t) = c_0 t^{q-1} \in \ker L$ with $|c_0| > M^*$, we have

$$\int_0^\infty (f(s,x(s)) + e(s))\,ds - \frac{\Gamma(q)}{\eta^{q-1}} \int_0^\eta (\eta - s)^{q-1} (f(s,x(s)) + e(s))\,ds$$
$$\geq \int_1^\infty \left(\frac{1}{4} e^{-s} \left(e^{-s} \left| c_0 s^{\frac{3}{2}-1} \right| - 8e^{-s} \right) + 3e^{-s} \right) ds$$
$$\geq \int_1^\infty \left(2e^{-2s}\sqrt{s} - 2e^{-s} + 3e^{-s} \right) ds = 0.531\,73 > 0.$$

As a result, if $c_0 > 0$, then

$$c_0 \left[\int_0^\infty (f(s,x(s)) + e(s))\,ds - \frac{\Gamma(q)}{\eta^{q-1}} \int_0^\eta (\eta - s)^{q-1} (f(s,x(s)) + e(s))\,ds \right] > 0,$$

or if $c_0 < 0$, then

$$c_0 \left[\int_0^\infty (f(s,x(s)) + e(s))\,ds - \frac{\Gamma(q)}{\eta^{q-1}} \int_0^\eta (\eta - s)^{q-1} (f(s,x(s)) + e(s))\,ds \right] < 0.$$

Now, we have

$$\begin{aligned}
&\int_0^\infty (f(s,x(s))+e(s))\,ds-\frac{\Gamma(q)}{\eta^{q-1}}\int_0^\eta (\eta-s)^{q-1}\left(f(s,x(s))+e(s)\right)ds\\
&=\int_0^\infty \left(\frac{1}{4}e^{-s}\left(e^{-s}\left|x(s)\right|-8e^{-s}\right)+3e^{-s}\right)ds\\
&\quad-\Gamma\left(\frac{3}{2}\right)\int_0^1 (1-s)^{\frac{3}{2}-1}\left(\frac{1}{4}e^{-s}\left(e^{-s}\left|x(s)\right|-8e^{-s}\right)+3e^{-s}\right)ds\\
&\geq\int_1^\infty \frac{1}{4}e^{-s}\left(e^{-s}\left|x(s)\right|-8e^{-s}\right)+3e^{-s}ds\geq\int_1^\infty\left(3e^{-s}-2e^{-2s}\right)ds\\
&=0.968\,3\neq 0,
\end{aligned}$$

consequently, (H_2) is satisfied for any constant $M>0$. Finally, a simple calculus gives

$$\left(1-2A_1\left\|\alpha\right\|_1\right)=0.353\,24>0,$$

where

$$A_1=1+\frac{\left[e^{-1}(q-1)\right]^{q-1}}{\Gamma(q)}=1.293\,5\quad\text{and}\quad\left\|\alpha\right\|_1=\int_0^\infty \frac{1}{4}e^{-t}ds=\frac{1}{4}.$$

We conclude from Theorem 3.1 that the problem (P) has at least one solution in X.

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