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ON THE UNIFORM PERFECTNESS OF EQUIVARIANT DIFFEOMORPHISM GROUPS FOR PRINCIPAL G MANIFOLDS

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Abstract. We proved in [K. Abe, K. Fukui, *On commutators of equivariant diffeomorphisms*, Proc. Japan Acad. 54 (1978), 52–54] that the identity component $\mathrm{Diff}^r_{G,c}(M)_0$ of the group of equivariant C^r-diffeomorphisms of a principal G bundle M over a manifold B is perfect for a compact connected Lie group G and $1 \leq r \leq \infty$ ($r \neq \dim B + 1$). In this paper, we study the uniform perfectness of the group of equivariant C^r-diffeomorphisms for a principal G bundle M over a manifold B by relating it to the uniform perfectness of the group of C^r-diffeomorphisms of B and show that under a certain condition, $\mathrm{Diff}^r_{G,c}(M)_0$ is uniformly perfect if B belongs to a certain wide class of manifolds. We characterize the uniform perfectness of the group of equivariant C^r-diffeomorphisms for principal G bundles over closed manifolds of dimension less than or equal to 3, and in particular we prove the uniform perfectness of the group for the 3-dimensional case and $r \neq 4$.

Keywords: uniform perfectness, principal G manifold, equivariant diffeomorphism.

Mathematics Subject Classification: 58D05, 57R30.

1. INTRODUCTION

For a C^r-manifold M, let $\mathrm{Diff}^r_c(M)$ denote the group of C^r-diffeomorphisms of M with compact support($1 \leq r \leq \infty$). Let $\mathrm{Diff}^r_c(M)_0$ be the identity component of $\mathrm{Diff}^r_c(M)$ equipped with the compact open C^r-topology. Thurston ([9]) and Mather ([8]) proved that $\mathrm{Diff}^r_c(M)_0$ is perfect if $1 \leq r \leq \infty$ and $r \neq \dim M + 1$, that is, it coincides with its commutator subgroup.

Let G be a compact connected Lie group and M be the total space of a principal G bundle M over a smooth manifold B. Then we have a canonical smooth free G action on M and every smooth free G action on M induces a principal G bundle M over a smooth manifold B. Let $\mathrm{Diff}^r_{G,c}(M)$ denote the group of equivariant C^r-diffeomorphisms of M with compact support and with the relative topology

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as a subspace of $\mathrm{Diff}^r_c(M)$. Let $\mathrm{Diff}^r_{G,c}(M)_0$ be the identity component of $\mathrm{Diff}^r_{G,c}(M)$. Abe and the author proved in [1] (and also Banyaga in [3]) using the results of Thurston and Mather that $\mathrm{Diff}^r_{G,c}(M)_0$ is perfect if $1 \leq r \leq \infty$, $r \neq \dim M - \dim G + 1$ and $\dim M - \dim G \geq 1$.

Burago, Ivanov and Polterovich ([4]) and Tsuboi ([10, 11]) studied the uniform perfectness of $\mathrm{Diff}^r_c(M)_0$, where a group is uniformly perfect if any element in it can be represented by a product of a bounded number of commutators of its elements. Indeed, Tsuboi has proved that $\mathrm{Diff}^r_c(M)_0$ is uniformly perfect if $1 \leq r \leq \infty$ and $r \neq \dim M + 1$ and M belongs to a wide class $\mathcal{C}$ of manifolds (see §3 for $\mathcal{C}$).

In this paper we study the uniform perfectness of $\mathrm{Diff}^r_{G,c}(M)_0$ for a principal G bundle M over a manifold B by relating it to the uniform perfectness of the group of C^r-diffeomorphisms of B and show that under a certain condition, the neccesary and sufficient condition for $\mathrm{Diff}^r_{G,c}(M)_0$ to be uniformly perfect is that $\mathrm{Diff}^r_c(B)_0$ is uniformly perfect. As corollaries, (i) we have by the results of Tsuboi ([10,11]) that for $1 \leq r \leq \infty$, $r \neq \dim B + 1$, $\mathrm{Diff}^r_{G,c}(M)_0$ is uniformly perfect if $\dim B \geq 3$, $G = T^n$ and $B \in \mathcal{C}$, and (ii) we characterize the uniform perfectness of the group of equivariant C^r-diffeomorphisms for principal G bundles over closed manifolds of dimension ≤ 3, and in particular we prove the uniform perfectness of the group for the 3-dimensional case and $r \neq 4$.

2. EQUIVARIANT DIFFEOMORPHISMS OF A MANIFOLD WITH TRIVIAL G ACTION

Let M be a smooth manifold without boundary on which a compact connected Lie group G acts smoothly and freely. Then the orbit map $\pi : M \to M/G$ is a principal G bundle over a smooth manifold $B = M/G$. Let $\mathrm{Diff}^r_{G,c}(M)_0$ denote the group of equivariant C^r-diffeomorphisms of M with compact support, which are G-isotopic to the identity through equivariant C^r-diffeomorphisms with compact support.

By using the results of Thurston ([9]) and Mather ([8]), Abe and the author in [1] (and also Banyaga in [3]) proved the following.

Theorem 2.1. *If* $1 \leq r \leq \infty$, $r \neq \dim M - \dim G + 1$ *and* $\dim M - \dim G \geq 1$, *then* $\mathrm{Diff}^r_{G,c}(M)_0$ *is perfect.*

In this section we consider the uniform perfectness of $\mathrm{Diff}^r_{G,c}(M)_0$ for the case $M = \mathbf{R}^m \times G$. Let $\pi : \mathbf{R}^m \times G \to \mathbf{R}^m$ be the projection, which induces the group epimorphism $P : \mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0 \to \mathrm{Diff}^r_c(\mathbf{R}^m)_0$ defined by $P(f) = \bar{f}$, where $f(x,g) = (\bar{f}(x), h(x,g))$ for $x \in \mathbf{R}^m$ and $g \in G$.

Theorem 2.2.

1. *If* $1 \leq r \leq \infty$, $r \neq m+1$ *and* $m \geq 1$, *then* $\mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0$ *is uniformly perfect. In fact, any* $f \in \mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0$ *can be represented by a product of two commutators of elements in* $\mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0$.
2. *If* $1 \leq r \leq \infty$ *and* $m \geq 1$, *then any* $f \in \ker P$ *can be represented by a product of two commutators of elements in* $\ker P$ *and* $\mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0$.

Proof. (1) The proof follows from the proof of [10, Theorem 2.1] of Tsuboi but we write the proof for the completeness. Take $f \in \mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0$. By Theorem 2.1, f can be represented by a product of commutators as

$$f = \prod_{i=1}^{k} [a_i, b_i], \ \text{ where } \ a_i, b_i \in \mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0.$$

Let U be an bounded open set of $\mathbf{R}^m$ satisfying that $\pi^{-1}(U)$ contains the supports of a_i and b_i. Take $\bar{\phi} \in \mathrm{Diff}^r_c(\mathbf{R}^m)_0$ satisfying that $\{\bar{\phi}^i(U)\}_{i=1}^k$ are disjoint. Define $\phi : \mathbf{R}^m \times G \to \mathbf{R}^m \times G$ by $\phi(x, g) = (\bar{\phi}(x), g)$ for $(x, g) \in \mathbf{R}^m \times G$. Then $\phi \in \mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0$. We put

$$F = \prod_{j=1}^{k} \phi^j \Big(\prod_{i=j}^{k} [a_i, b_i] \Big) \phi^{-j}$$

which is in $\mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0$. Then we have

$$\phi^{-1} \circ F \circ \phi \circ F^{-1} = f \circ \Big(\prod_{j=1}^{k} \phi^j [a_j, b_j]^{-1} \phi^{-j} \Big) = f \circ \Big[\prod_{j=1}^{k} \phi^j b_j \phi^{-j}, \prod_{j=1}^{k} \phi^j a_j \phi^{-j} \Big].$$

Thus we have

$$f = [\phi^{-1}, F] \circ \Big[\prod_{j=1}^{k} \phi^j a_j \phi^{-j}, \prod_{j=1}^{k} \phi^j b_j \phi^{-j} \Big].$$

That is, any $f \in \mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0$ can be represented by two commutators of elements in $\mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0$.

(2) By Proposition 6 of [1], any $f \in \ker P$ can be represented by a product of commutators of elements in $\ker P$ and $\mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0$ as

$$f = \prod_{i=1}^{k} [c_i, d_i], \ \text{ where } \ c_i \in \ker P \ \text{ and } \ d_i \in \mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0.$$

Note that it also holds for $r = m + 1$. By the similar way as in (1), we can prove that $f \in \ker P$ is represented by two commutators of elements in $\ker P$ and $\mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0$. This completes the proof. □

3. UNIFORM PERFECTNESS OF $\mathrm{Diff}^r_{G,c}(M)_0$

Let G be a compact connected Lie group and $\pi : M \to B$ be a principal G bundle over an m-dimensional closed C^r manifold B ($m \geq 1$), where "closed" means "compact and without boundary". Let $P : \mathrm{Diff}^r_G(M)_0 \to \mathrm{Diff}^r(B)_0$ be the map defined by $P(f)(x) = \pi(f(\hat{x}))$ for $f \in \mathrm{Diff}^r_G(M)_0$ and $x \in B$, $\hat{x} \in M$ with $\pi(\hat{x}) = x$. Curtis in [5] proved that P is a surjective homomorphism and a local trivial fibration.

In this section we study the uniform perfectness of $\mathrm{Diff}^r_G(M)_0$ by relating it to the uniform perfectness of $\mathrm{Diff}^r(B)_0$. Then we have the following.

Theorem 3.1.

1. *If* $\mathrm{Diff}^r_G(M)_0$ *is uniformly perfect, then* $\mathrm{Diff}^r(B)_0$ *is uniformly perfect.*
2. *If the number of connected components of* $\ker P$ *is finite and* $\mathrm{Diff}^r(B)_0$ *is uniformly perfect, then* $\mathrm{Diff}^r_G(M)_0$ *is uniformly perfect.*

Proof. (1) Take any $\bar{f} \in \mathrm{Diff}^r(B)_0$. Then from the result of Curtis ([5]), we have $f \in \mathrm{Diff}^r_G(M)_0$ satisfying $P(f) = \bar{f}$. From the assumption, f can be represented as a product of a bounded number, say k, of commutators;

$$f = \prod_{j=1}^{k} [g_j, h_j], \text{ where } g_j, h_j \in \mathrm{Diff}^r_G(M)_0.$$

Then we have

$$\bar{f} = P(f) = P\Big(\prod_{j=1}^{k} [g_j, h_j]\Big) = \prod_{j=1}^{k} [P(g_j), P(h_j)].$$

(2) Take any $f \in \mathrm{Diff}^r_G(M)_0$. Then from the assumption, we have $\bar{f} = P(f) = \prod_{j=1}^{k} [\bar{g}_j, \bar{h}_j]$, where $\bar{g}_j, \bar{h}_j \in \mathrm{Diff}^r(B)_0$ and k is a bounded number. By using the result of Curtis ([5]) again, we can take g_j and h_j in $\mathrm{Diff}^r_G(M)_0$ satisfying $P(g_j) = \bar{g}_j$ and $P(h_j) = \bar{h}_j$. Then we have $(\prod_{j=1}^{k} [g_j, h_j])^{-1} \circ f \in \ker P$.

First we consider the case that $\psi = (\prod_{j=1}^{k} [g_j, h_j])^{-1} \circ f$ is G-isotopic to the identity in $\ker P$. We have $f = \prod_{j=1}^{k} [g_j, h_j] \circ \psi$ and $\psi \in \ker P$.

Let $\{U_i\}_{i=1}^{\ell+1}$ be an open covering of B such that each U_i is a disjoint union of open balls, where ℓ is the category number of $B\,(\ell \le m)$. Let $\{\lambda_i\}_{i=1}^{\ell+1}$ be a partition of unity subordinate to the covering $\{U_i\}_{i=1}^{\ell+1}$. Let $\psi_t (0 \le t \le 1)$ be an isotopy in $\ker P$ from $\psi_0 =$ identity to $\psi_1 = \psi$. Define $h_i \in \ker P$ $(i = 1, 2, \ldots, \ell + 1)$ as follows:

$$h_1(p) = \psi_{\lambda_1 \circ \pi(p)}(p) \ for\ p \in M,$$
$$h_2(p) = h_1^{-1} \circ \psi_{\lambda_1 \circ \pi(p) + \lambda_2 \circ \pi(p)}(p) \ for\ p \in M,$$

and in general

$$h_i(p) = (h_1 \circ \ldots \circ h_{i-1})^{-1} \circ \psi_{\sum_{j=1}^{i} \lambda_j \circ \pi(p)}(p) \ for\ p \in M \ (i = 3, \ldots, \ell + 1).$$

Then we have the support of h_i is contained in U_i $(i = 1, 2, \ldots, \ell + 1)$ and $h_i \in \ker P$. For, any element $\psi \in \ker P$ has locally (say, on $\pi^{-1}(U)$ for an open ball U in B) the form of $\psi(x, g) = (x, g \cdot L(\psi)(x))$, where $L : \ker P \to C^r(U, G_0)$ is the map defined by $(x, L(\psi)(x)) = \psi(x, e)$ (see [1]). Thus the isotopy $\psi_t (0 \le t \le 1)$ has the form $(x, g \cdot L(\psi)_t(x))$, where $L(\psi)_t(x) (0 \le t \le 1)$ is a homotopy in $C^r(U, G)$ from $L(\psi)_0(x) = e$ to $L(\psi)_1(x) = L(\psi)(x)$. Hence each h_i is in $\ker P$. Furthermore we have $\psi = h_1 \circ h_2 \circ \ldots \circ h_{\ell+1}$.

As U_i is a disjoint union of open balls diffeomorphic to the unit open ball $\mathrm{int}D^m$, we have only to prove the case that U_i is $\mathrm{int}D^m$ in order to prove Theorem 3.1(2). Since π is trivial over U_i, $\pi^{-1}(U_i)$ is G-diffeomorphic to $U_i \times G$. Thus we may assume that each

h_i is contained in $\ker P$ for the homomorphism $P : \mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0 \to \mathrm{Diff}^r_c(\mathbf{R}^m)_0$ in §2. From Theorem 2.2(2), each h_i can be represented by a product of two commutators of elements in $\ker P$ and $\mathrm{Diff}^r_{G,c}(\mathbf{R}^m \times G)_0$ if $1 \le r \le \infty$. Thus ψ can be represented by a product of $2(\ell+1)$ commutators of elements in $\ker P$ and $\mathrm{Diff}^r_G(M)_0$. Hence f can be represented by a product of $k + 2(\ell+1)$ commutators of elements in $\mathrm{Diff}^r_G(M)_0$, where k and ℓ are bounded numbers.

Next we consider the case that ψ is not connected to the identity in $\ker P$. Let a be the number of the connected components of $\ker P$. Take elements, say $g_1, \ldots, g_a$, from each connected component of $\ker P$ and fix them. Then from Theorem of [1], each g_i can be written by t_i commutators of elements in $\mathrm{Diff}^r_G(M)_0$. Put $t = \max\{t_1, \ldots, t_a\}$. For any element $g \in \ker P$, there exists some i $(i = 1, \ldots, a)$ satisfying that g and g_i are in the same connected component of $\ker P$. Since $g \circ (g_i)^{-1}$ is in the identity component of $\ker P$, g can be written by at most $2(\ell+1) + t$ commutators. Thus for any element $f \in \mathrm{Diff}^r_G(M)_0$, above ψ can be written by $2(\ell+1)+t$ commutators. Hence $f \in \mathrm{Diff}^r_G(M)_0$ can be written by $k + 2(\ell+1) + t$ commutators of elements in $\mathrm{Diff}^r_G(M)_0$. Since k, ℓ and t are bounded numbers, this completes the proof. □

The fibration map $P : \mathrm{Diff}^r_G(M)_0 \to \mathrm{Diff}^r(B)_0$ induces the homomorphism between the fundamental groups $P_* : \pi_1(\mathrm{Diff}^r_G(M)_0, 1) \to \pi_1(\mathrm{Diff}^r(B)_0, 1)$.

Corollary 3.2. *Suppose that the cokernel of the homomorphism*

$$P_* : \pi_1(\mathrm{Diff}^r_G(M)_0, 1) \to \pi_1(\mathrm{Diff}^r(B)_0, 1)$$

is finite. Then $\mathrm{Diff}^r_G(M)_0$ *is uniformly perfect if* $\mathrm{Diff}^r(B)_0$ *is uniformly perfect.*

Proof. The fibration map $P : \mathrm{Diff}^r_G(M)_0 \to \mathrm{Diff}^r(B)_0$ induces the following exact sequence of homotopy groups:

$$\ldots \to \pi_1(\mathrm{Diff}^r_G(M)_0, 1) \to \pi_1(\mathrm{Diff}^r(B)_0, 1) \to \pi_0(\ker P) \to \pi_0(\mathrm{Diff}^r_G(M)_0) = 1.$$

From the assumption $P_* : \pi_1(\mathrm{Diff}^r_G(M)_0, 1) \to \pi_1(\mathrm{Diff}^r(B)_0, 1)$ has finite cokernel. Thus $\pi_0(\ker P)$ is finite, that is, the connected components of $\ker P$ is finite. The proof follows from Theorem 3.1(2). □

4. UNIFORM PERFECTNESS OF $\mathrm{Diff}^r_{T^n}(M)_0$

In this section we study the uniform perfectness of $\mathrm{Diff}^r_{T^n}(M)_0$ for principal T^n-bundles over closed manifolds B. Then we have the following.

Theorem 4.1. *Suppose that* $\dim B \ge 3$. *Then* $\mathrm{Diff}^r_{T^n}(M)_0$ *is uniformly perfect if* $\mathrm{Diff}^r(B)_0$ *is uniformly perfect.*

Proof. Take any $f \in \mathrm{Diff}^r_{T^n}(M)_0$. Then from the assumption, we have $\bar{f} = P(f) = \prod_{j=1}^k [\bar{g}_j, \bar{h}_j]$, where $\bar{g}_j, \bar{h}_j \in \mathrm{Diff}^r(B)_0$ and k is a bounded number. By using the result of Curtis ([5]) again, we can take g_j and h_j in $\mathrm{Diff}^r_{T^n}(M)_0$ satisfying $P(g_j) = \bar{g}_j$ and $P(h_j) = \bar{h}_j$. Then we have $\psi = (\prod_{j=1}^k [g_j, h_j])^{-1} \circ f \in \ker P$.

Let $\{U_i\}_{i=1}^{\ell+1}$ and $\{V_i\}_{i=1}^{\ell+1}$ be open coverings of B such that each U_i and V_i are disjoint unions of open balls and $U_i \subset V_i$, where ℓ is the category number of B ($\ell \leq m$).

Since π is trivial over V_1, $\pi_j(T^n) = 1$ ($j \geq 2$) and $m \geq 3$, we can deform ψ over V_1 to $\psi_1 \in \ker P$ satisfying that $\psi_1 = \psi$ on U_1 and ψ_1 is the identity near the boundary of $\overline{V}_1$. For, $\overline{V}_1 - U_1$ is homeomorphic to $S^{m-1} \times [0,1]$ and $\psi\mid_{\partial(\overline{U}_1)}(x,\cdot) : \partial(\overline{U}_1) \to T^n$ is homotopic to the constant map e because $\pi_j(T^n) = 1$ ($j \geq 2$) and $m \geq 3$. Hence ψ can be deformed in V_1 to the identity near the boundary of $\overline{V}_1$ fixing ψ on $\overline{U}_1$(see the proof of Theorem 3.1(2)).

Next we get $\psi_2 \in \ker P$ satisfying that $\psi_2 = \psi_1$ on U_2 and ψ_2 is the identity near the boundary of $\overline{V}_2$ by performing the same procedure as above for $(\psi_1)^{-1} \circ \psi$ and V_2. After $\ell + 1$ times procedures, we have $\psi_1, \ldots, \psi_{\ell+1}(\in \ker P)$ satisfying that $\psi = \psi_1 \circ \ldots \circ \psi_{\ell+1}$ and each ψ_i is supported in V_i. Since each ψ_i is in $\ker P$, we have from Theorem 2(2) that ψ_i can be represented by a product of two commutators of elements in $\ker P$ and $\mathrm{Diff}^r_{T^n,c}(\mathbf{R}^m \times T^n)_0$ if $1 \leq r \leq \infty$. Thus ψ can be represented by a product of $2(\ell+1)$ commutators of elements in $\ker P$ and $\mathrm{Diff}^r_{T^n}(M)_0$. Hence f can be represented by a product of $k + 2(\ell+1)$ commutators of elements in $\mathrm{Diff}^r_{T^n}(M)_0$, where k and ℓ are bounded numbers. This completes the proof. □

Since $\pi_2(G) = 0$ for any Lie group G and $\mathrm{Diff}^r(B)_0 (r \neq 4)$ is uniformly perfect when B is a 3 dimensional closed manifold ([4, 10]), the above proof induces the following.

Corollary 4.2. *Suppose that B is a 3 dimensional closed manifold. Then $\mathrm{Diff}^r_G(M)_0$ is uniformly perfect for $r \neq 4$.*

We say that a manifold B belongs to a class $\mathcal{C}$ if B is one of the following:

1. an m dimensional closed manifold ($m \neq 2, 4$) and
2. an m dimensional closed manifold which has a handle decomposition without handles of the middle index ($m = 2, 4$).

Then Tuboi ([10, 11]) proved the following.

Theorem 4.3. *If $B \in \mathcal{C}$ and $1 \leq r \leq \infty$, $r \neq \dim B + 1$, then $\mathrm{Diff}^r_c(B)_0$ is uniformly perfect.*

Corollary 4.4. *Let $\pi : M \to B$ be a principal T^n bundle over an m-dimensional closed manifold B. Suppose that $m \geq 3$ and B belongs to the class $\mathcal{C}$. If $1 \leq r \leq \infty$, $r \neq m + 1$, then $\mathrm{Diff}^r_{T^n}(M)_0$ is uniformly perfect.*

Proof. The proof follows from Theorem 4.1 and Theorem 4.3. □

Corollary 4.5. *Let M be a closed T^n-manifold with one orbit type. Suppose that the orbit manifold M/G belongs to the class $\mathcal{C}$. If $1 \leq r \leq \infty$, $r \neq \dim M - \dim G + 1$ and $\dim M - \dim G \geq 3$, then $\mathrm{Diff}^r_{T^n}(M)_0$ is uniformly perfect.*

Proof. The proof follows from Corollary of [1] and Corollary 4.4. □

5. UNIFORM PERFECTNESS OF $\mathrm{Diff}^r_G(M)_0$ FOR PRINCIPAL G-BUNDLES OVER LOW DIMENSIONAL CLOSED MANIFOLDS

In this section we consider the uniform perfectness of $\mathrm{Diff}^r_G(M)_0$ for principal G-bundles over closed manifolds B of dimension ≤ 2.

First we consider the case of $\mathrm{Diff}^r_G(M)_0$ for principal G-bundles over S^1. Since any principal G-bundle over S^1 is trivial, $\ker P$ is connected for a compact connected Lie group G. Furthermore, since $\mathrm{Diff}^r(S^1)_0$ is uniformly perfect($r \neq 2$), we have the following from Theorem 3.1(2).

Theorem 5.1. *Let $\pi : M \to S^1$ be a principal G bundle over S^1. Then $\mathrm{Diff}^r_G(M)_0$ is uniformly perfect for $r \neq 2$.*

Next we study the uniform perfectness of $\mathrm{Diff}^r_G(M)_0$ for principal G-bundles over closed orientable surfaces not homeomorphic to T^2. Then we have the following.

Theorem 5.2. *Let $\pi : M \to B$ be a principal G bundle over a 2 dimensional closed orientable manifold B.*

1. *When B is the 2-sphere S^2, $\mathrm{Diff}^r_G(M)_0$ is uniformly perfect for $r \neq 3$.*
2. *When B is a closed orientable surface not homeomorphic to S^2, T^2, $\mathrm{Diff}^r_G(M)_0$ is uniformly perfect if and only if $\mathrm{Diff}^r(B)_0$ is uniformly perfect.*

Proof. (1) For $B = S^2$, we have $\pi_1(\mathrm{Diff}^r(B)_0, 1) \cong \pi_1(SO(3), 1) \cong \mathbf{Z}_2$. Then the connected components of $\ker P$ are at most two. Thus (1) follows from Theorem 3.1(2) and the uniform perfectness of $\mathrm{Diff}^r(S^2)_0$ $(r \neq 3)$.

(2) When B is a closed surface not homeomorphic to S^2, T^2, $\mathrm{Diff}^r(B)_0$ is contractible. Thus the fibration $P : \mathrm{Diff}^r_G(M)_0 \to \mathrm{Diff}^r(B)_0$ is trivial. Then $\ker P$ is connected. Hence (2) follows from Theorem 3.1(2). □

Finally we have the following problem.

Problem 5.3. Discuss the uniform perfectness for the case $B = T^2$.

REFERENCES

[1] K. Abe, K. Fukui, *On commutators of equivariant diffeomorphisms*, Proc. Japan Acad. **54** (1978), 52–54.

[2] K. Abe, K. Fukui, *Erratum and addendum to "Commutators of C^∞-diffeomorphisms preserving a submanifold"*, J. Math. Soc. Japan **65** (2013) 4, 1–8.

[3] A. Banyaga, *The Structure of Classical Diffeomorphisms*, Kluwer Akademic Publisher, Dordrecht, 1997.

[4] D. Burago, S. Ivanov, L. Polterovich, *Conjugation-invariant norms on groups of geometric origin*, Advanced Studies in Pure Math. **52**, Groups of Diffeomorphisms (2008), 221–250.

[5] W.D. Curtis, *The automorphism group of a compact group action*, Trans. Amer. Math. Soc. **203** (1975), 45–54.

[6] K. Fukui, *Commutator length of leaf preserving diffeomorphisms*, Publ. Res. Inst. Math. Sci. Kyoto Univ. **48** (2012) 3, 615–622.

[7] J.M. Gambaudo, É. Ghys, *Commutators and diffeomorphisms of surfaces*, Ergod. Th. & Dynam. Sys. **24** (1980) 5, 1591–1617.

[8] J.N. Mather, *Commutators of diffeomorphisms I and II*, Comment. Math. Helv. **49** (1974), 512–528; **50** (1975), 33–40.

[9] W. Thurston, *Foliations and groups of diffeomorphisms*, Bull. Amer. Math. Soc. **80** (1974), 304–307.

[10] T. Tsuboi, *On the uniform perfectness of diffeomorphism groups*, Advanced Studies in Pure Math. **52**, Groups of Diffeomorphisms (2008), 505–524.

[11] T. Tsuboi, *On the uniform perfectness of the groups of diffeomorphisms of even-dimensional manifolds*, Comment. Math. Helv. **87** (2012), 141–185.

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