

POSITIVE SOLUTIONS OF A SINGULAR FRACTIONAL BOUNDARY VALUE PROBLEM WITH A FRACTIONAL BOUNDARY CONDITION

Jeffrey W. Lyons and Jeffrey T. Neugebauer

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Abstract. For $\alpha \in (1, 2]$, the singular fractional boundary value problem

$$D_{0+}^{\alpha}x + f(t, x, D_{0+}^{\mu}x) = 0, \quad 0 < t < 1,$$

satisfying the boundary conditions $x(0) = D_{0+}^{\beta}x(1) = 0$, where $\beta \in (0, \alpha - 1]$, $\mu \in (0, \alpha - 1]$, and D_{0+}^{α} , D_{0+}^{β} and D_{0+}^{μ} are Riemann-Liouville derivatives of order α , β and μ respectively, is considered. Here f satisfies a local Carathéodory condition, and $f(t, x, y)$ may be singular at the value 0 in its space variable x . Using regularization and sequential techniques and Krasnosel'skii's fixed point theorem, it is shown this boundary value problem has a positive solution. An example is given.

Keywords: fractional differential equation, singular problem, fixed point.

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1. INTRODUCTION

For $\alpha \in (1, 2]$, we consider the singular fractional boundary value problem

$$D_{0+}^{\alpha}x + f(t, x, D_{0+}^{\mu}x) = 0, \quad 0 < t < 1, \tag{1.1}$$

satisfying the boundary conditions

$$x(0) = D_{0+}^{\beta}x(1) = 0, \tag{1.2}$$

where $\beta \in (0, \alpha - 1]$, $\mu \in (0, \alpha - 1]$, and D_{0+}^{α} , D_{0+}^{β} and D_{0+}^{μ} are Riemann-Liouville derivatives of order α , β and μ respectively. Here f satisfies the local Carathéodory condition on $[0, 1] \times \mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, ($f \in \text{Car}([0, 1] \times \mathcal{D})$) and $f(t, x, y)$ may be singular at

the value 0 in its space variable x . By a positive solution, we mean x satisfies (1.1), (1.2) and $x(t) > 0$ for $t \in (0, 1]$.

The study of fractional boundary value problems has seen a tremendous expansion in recent years motivated by both general theory and physical representations and applications. For the reader interested in such works, we refer to [2, 4, 7, 8]. Of interest to the work presented, we point to research investigating the existence of solutions to fractional boundary value problems [1, 6, 9–12].

In [1], the authors proved the existence of at least one positive solution to the Dirichlet boundary value problem

$$\begin{aligned} D_{0+}^{\alpha}x + f(t, x, D_{0+}^{\mu}x) &= 0, \\ x(0) = x(1) &= 0 \end{aligned}$$

with $\alpha \in (1, 2)$, $\mu > 0$ and $\alpha - \mu \geq 1$ using Green's functions and the Krasnosel'skii fixed point theorem after placing certain conditions upon f .

Our aim in this work is to use the same differential equation, but instead of Dirichlet boundary conditions, we incorporate fractional boundary conditions, $x(0) = D_{0+}^{\beta}x(1) = 0$ with $\beta \in (0, \alpha - 1]$. Recently, the Green's function for (1.1), (1.2) was found in [3] which affords us the opportunity to utilize operators and an application of Krasnosel'skii's fixed point theorem. Since f might have a singularity in the function space at $x = 0$, we must also use regularization and sequential techniques.

In section 2, we introduce definitions, assumptions, and define a sequence of functions, $\{f_n\}$, to handle the possible singularity at $x = 0$. Section 3 is where one will find the Green's function and its associated properties along with the Krasnosel'skii fixed point theorem. Additionally, we prove the existence of a sequence of positive solutions, $\{x_n(t)\}$, to the auxiliary problem. Finally, in section 4, we make the jump from a sequence of auxiliary solutions to a positive solution $x(t)$ of (1.1), (1.2). We conclude with an example.

2. PRELIMINARY DEFINITIONS AND ASSUMPTIONS

We start with the definition of the Riemann-Liouville fractional integral and fractional derivative. Let $\nu > 0$. The Riemann-Liouville fractional integral of a function x of order ν , denoted $I_{0+}^{\nu}u$, is defined as

$$I_{0+}^{\nu}x(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1}x(s)ds,$$

provided the right-hand side exists. Moreover, let n denote a positive integer and assume $n - 1 < \alpha \leq n$. The Riemann-Liouville fractional derivative of order α of the function $x : [0, 1] \rightarrow \mathbb{R}$, denoted $D_{0+}^{\alpha}x$, is defined as

$$D_{0+}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1}x(s)ds = D^n I_{0+}^{n-\alpha}x(t),$$

provided the right-hand side exists.

We will make use of the power rule, which states that [2]

$$D_{0+}^{\nu_2} t^{\nu_1} = \frac{\Gamma(\nu_1 + 1)}{\Gamma(\nu_1 + 1 - \nu_2)} t^{\nu_1 - \nu_2}, \quad \nu_1 > -1, \nu_2 \geq 0, \tag{2.1}$$

where it is assumed that $\nu_2 - \nu_1$ is not a positive integer. If $\nu_2 - \nu_1$ is a positive integer, then the right hand side of (2.1) vanishes. To see this, one can appeal to the convention that $\frac{1}{\Gamma(\nu_1 + 1 - \nu_2)} = 0$ if $\nu_2 - \nu_1$ is a positive integer, or one can perform the calculation on the left hand side and calculate

$$D^n t^{n - (\nu_2 - \nu_1)} = 0.$$

We say that f satisfies the local Carathéodory condition on $[0, 1] \times \mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, if

1. $f(\cdot, x, y) : [0, 1] \rightarrow \mathbb{R}$ is measurable for all $(x, y) \in \mathcal{D}$;
2. $f(t, \cdot, \cdot) : \mathcal{D} \rightarrow \mathbb{R}$ is continuous for a.e. $t \in [0, 1]$; and
3. for each compact set $\mathcal{H} \subset \mathcal{D}$, there is a function $\varphi_{\mathcal{H}} \in L^1[0, 1]$ such that

$$|f(t, x, y)| \leq \varphi_{\mathcal{H}}(t),$$

for a.e. $t \in [0, 1]$ and all $(x, y) \in \mathcal{H}$.

Throughout the paper,

$$\|x\|_L = \int_0^1 |x(t)| dt, \quad \|x\|_0 = \max_{t \in [0, 1]} |x(t)|,$$

and

$$\|x\| = \max\{\|x\|_0, \|D_{0+}^{\mu} x\|_0\}.$$

We assume the following conditions on f .

(H1) $f \in \text{Car}([0, 1] \times \mathcal{D})$, $\mathcal{D} = (0, \infty) \times \mathbb{R}$,

$$\lim_{x \rightarrow 0^+} f(t, x, y) = \infty,$$

for a.e. $t \in [0, 1]$ and all $y \in \mathbb{R}$, and there exists a positive constant m such that, for a.e. $t \in [0, 1]$ and all $(x, y) \in \mathcal{D}$,

$$f(t, x, y) \geq m.$$

(H2) f satisfies the estimate for a.e. $t \in [0, 1]$ and all $(x, y) \in \mathcal{D}$,

$$f(t, x, y) \leq \gamma(t) (q(x) + p(x) + \omega(|y|)),$$

where $\gamma \in L^1[0, 1]$, $q \in C(0, \infty)$, and $p, \omega \in C[0, \infty)$ are positive, q is nonincreasing, p and ω are nondecreasing, and

$$\int_0^1 \gamma(t) q(Mt^{\alpha-1}) dt < \infty, \quad M = \frac{m\beta}{(\alpha - \beta)\Gamma(\alpha + 1)},$$

$$\lim_{x \rightarrow \infty} \frac{p(x) + \omega(x)}{x} = 0.$$

We use regularization and sequential techniques to show the existence of solutions of (1.1), (1.2). Thus, for $n \in \mathbb{N}$, define f_n by

$$f_n(t, x, y) = \begin{cases} f(t, x, y), & x \geq 1/n, \\ f(t, \frac{1}{n}, y) & x < 1/n, \end{cases}$$

for a.e. $t \in [0, 1]$ and for all $(x, y) \in \mathcal{D}_* := [0, \infty) \times \mathbb{R}$. Then $f_n \in \text{Car}([0, 1] \times \mathcal{D}_*)$,

$$f_n(t, x, y) \geq m,$$

for a.e. $t \in [0, 1]$ and all $(x, y) \in \mathcal{D}_*$,

$$f_n(t, x, y) \leq \gamma(t)(q(1/n) + p(x) + p(1) + \omega(|y|)),$$

for a.e. $t \in [0, 1]$ and all $(x, y) \in \mathcal{D}_*$, and

$$f_n(t, x, y) \leq \gamma(t)(q(x) + p(x) + p(1) + \omega(|y|)),$$

for a.e. $t \in [0, 1]$ and all $(x, y) \in \mathcal{D}$.

3. POSITIVE SOLUTIONS OF THE AUXILIARY PROBLEM

To use these techniques, we first discuss solutions of the fractional differential equation

$$D_{0+}^\alpha x + f_n(t, x, D_{0+}^\mu x) = 0, \quad 0 < t < 1, \tag{3.1}$$

satisfying boundary conditions (1.2).

The Green's function for $-D_{0+}^\alpha u = 0$ satisfying the boundary conditions (1.2) is given by (see [3])

$$G(t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t < 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & 0 \leq t \leq s < 1. \end{cases} \tag{3.2}$$

Therefore, x is a solution of (3.1), (1.2) if and only if

$$x(t) = \int_0^1 G(t, s) f_n(s, x(s), D_{0+}^\mu x(s)) ds, \quad 0 \leq t \leq 1.$$

Lemma 3.1. *Let G be defined as in (3.2). Then*

1. $G(t, s) \in C([0, 1] \times [0, 1])$ and $G(t, s) > 0$ for $(t, s) \in (0, 1) \times (0, 1)$;
2. $G(t, s) \leq \frac{1}{\Gamma(\alpha)}$ for $(t, s) \in [0, 1] \times [0, 1]$; and
3. $\int_0^1 G(t, s) ds \geq \frac{\beta t^{\alpha-1}}{(\alpha - \beta)\Gamma(\alpha + 1)}$ for $t \in [0, 1]$.

Proof.

1. G is continuous by definition. The proof that $G(t, s) > 0$ for $(t, s) \in (0, 1) \times (0, 1)$ can be found in [3].
2. Next, we remark that since $0 \leq t \leq 1$ and $\alpha > 1$, $t^{\alpha-1} \leq 1$. Also, notice that since $0 \leq \beta \leq \alpha - 1$ and $0 \leq s \leq 1$, $(1 - s)^{\alpha-1-\beta} \leq 1$. So $G(t, s) \leq \frac{1}{\Gamma(\alpha)}$ for $(t, s) \in [0, 1] \times [0, 1]$.
3. Now, for $t \in [0, 1]$,

$$\begin{aligned} \int_0^1 G(t, s) ds &= \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} ds \\ &= \frac{1}{\Gamma(\alpha)} \left(t^{\alpha-1} \int_0^1 (1-s)^{\alpha-1-\beta} ds - \int_0^t (t-s)^{\alpha-1} ds \right) \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} \frac{\alpha - t(\alpha - \beta)}{\alpha(\alpha - \beta)}. \end{aligned}$$

But for $t \in [0, 1]$, $\alpha - (t\alpha - \beta) > \beta$. Therefore,

$$\begin{aligned} \int_0^1 G(t, s) ds &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} \frac{\alpha - t(\alpha - \beta)}{\alpha(\alpha - \beta)} \\ &\geq \frac{\beta t^{\alpha-1}}{(\alpha - \beta)\Gamma(\alpha + 1)}, \end{aligned}$$

for $t \in [0, 1]$. □

Define

$$Q_n x(t) = \int_0^1 G(t, s) f_n(s, x(s), D_{0+}^\mu x(s)) ds, \quad 0 \leq t \leq 1.$$

Let $X = \{x \in C[0, 1] : D_{0+}^\mu x \in C[0, 1]\}$ with norm $\|\cdot\|$ defined earlier. Notice X is a Banach space. Define a cone \mathcal{P} in X as

$$\mathcal{P} = \{x \in X : x(t) \geq 0 \text{ for } t \in [0, 1]\}.$$

Note if $x \in \mathcal{P}$ is a fixed point of Q_n , then x is a positive solution of (3.1), (1.2). To that end, we will use the well-known Krasnosel'skii Fixed Point Theorem, which is stated below, to show the existence of positive solutions of (3.1), (1.2).

Theorem 3.2 (Krasnosel'skii's Fixed Point Theorem [5]). *Let \mathcal{B} be a Banach space, and let $\mathcal{P} \subset X$ be a cone in \mathcal{P} . Assume that Ω_1, Ω_2 are open sets with $0 \in \Omega_1$, and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow \mathcal{P}$ be a completely continuous operator such that*

$$\|Tu\| \geq \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_1, \quad \text{and} \quad \|Tu\| \leq \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_2.$$

Then T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 3.3. *Let (H1) and (H2) hold. Then $Q_n : \mathcal{P} \rightarrow \mathcal{P}$ and Q_n is a completely continuous operator.*

Proof. Suppose that $x \in \mathcal{P}$. Then,

$$Q_n x(t) = \int_0^1 G(t, s) f_n(s, x(s), D_{0+}^\mu x(s)) ds.$$

From Lemma 3.1 (1.), $G(t, s)$ is continuous and nonnegative on $[0, 1] \times [0, 1]$. So $Q_n x \in C[0, 1]$. Also, by using (2.1),

$$\begin{aligned} (D_{0+}^\mu Q_n)x(t) &= \frac{1}{\Gamma(\alpha - \mu)} \left(t^{\alpha - \mu - 1} \int_0^1 (1 - s)^{\alpha - \beta - 1} f_n(s, x(s), D_{0+}^\mu x(s)) ds \right. \\ &\quad \left. - \int_0^t (t - s)^{\alpha - \mu - 1} f_n(s, x(s), D_{0+}^\mu x(s)) ds \right), \end{aligned}$$

and so $D_{0+}^\mu Q_n x \in C[0, 1]$. So $Q_n : X \rightarrow X$. By (H1) and the definition of $f_n(t, x, y)$, we have $f_n(s, x(s), D_{0+}^\mu x(s)) \geq m > 0$ for a.e. $t \in [0, 1]$. Therefore, for $x \in \mathcal{P}$, Lemma 3.1 (1.) gives that $Q_n x(t) \geq 0$ for $t \in [0, 1]$. Thus, $Q_n : \mathcal{P} \rightarrow \mathcal{P}$.

Next, we show that Q_n is a continuous operator. To that end, let $\{x_k\} \subset \mathcal{P}$ be a convergent sequence such that $\lim_{k \rightarrow \infty} \|x_k - x\| = 0$. Then, $\lim_{k \rightarrow \infty} x_k(t) = x(t)$ uniformly on $[0, 1]$ and $\lim_{k \rightarrow \infty} D_{0+}^\mu x_k(t) = D_{0+}^\mu x(t)$ uniformly on $[0, 1]$. Also, $x \in \mathcal{P}$.

Let

$$\rho_k(t) = f_n(t, x_k(t), D_{0+}^\mu x_k(t)), \quad \rho(t) = f_n(t, x(t), D_{0+}^\mu x(t)).$$

Then, $\lim_{k \rightarrow \infty} \rho_k(t) = \rho(t)$ for a.e. $t \in [0, 1]$. Since $f_n \in \text{Car}([0, 1] \times \mathbb{R}^2)$ and $\{x_k\}$ and $\{D_{0+}^\mu x_k\}$ are bounded in $C[0, 1]$, there exists $\varphi \in L^1[0, 1]$ such that $m \leq \rho_k(t) \leq \varphi(t)$ for a.e. $t \in [0, 1]$ and all $k \in \mathbb{N}$. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{k \rightarrow \infty} \int_0^1 |\rho_k(s) - \rho(s)| ds = 0.$$

By Lemma 3.1 (2.),

$$|(Q_n x_k)(t) - (Q_n x)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^1 |\rho_k(s) - \rho(s)| ds.$$

Therefore, $\lim_{k \rightarrow \infty} (Q_n x_k)(t) = (Q_n x)(t)$ uniformly for $t \in [0, 1]$. Also,

$$\begin{aligned} |(D_{0+}^\mu Q_n x_k)(t) - (D_{0+}^\mu Q_n x)(t)| &\leq \frac{1}{\Gamma(\alpha - \mu)} \left(t^{\alpha - \mu - 1} \int_0^1 (1 - s)^{\alpha - \beta - 1} |\rho_k(s) - \rho(s)| ds \right. \\ &\quad \left. + \int_0^t (t - s)^{\alpha - \mu - 1} |\rho_k(s) - \rho(s)| ds \right) \\ &\leq \frac{2}{\Gamma(\alpha - \mu)} \int_0^1 |\rho_k(s) - \rho(s)| ds. \end{aligned}$$

So, $\lim_{k \rightarrow \infty} (D_{0+}^\mu Q_n x_k)(t) = (D_{0+}^\mu Q_n x)(t)$ uniformly for $t \in [0, 1]$. Thus, $\|Q_n x_k - Q_n x\| \rightarrow 0$ and hence, Q_n is a continuous operator.

For $W \in \mathbb{R}^+$, define $\mathcal{W} = \{x \in \mathcal{P} : \|x\| \leq W\}$ to be a bounded subset of \mathcal{P} . Let ρ be as before. Then there exists a $\varphi \in L^1[0, 1]$ with $m \leq \rho(t) \leq \varphi(t)$ for a.e. $t \in [0, 1]$ as before. Since, for $x \in \mathcal{W}$,

$$|(Q_n x)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^1 \varphi(s) ds = \frac{\|\varphi\|_1}{\Gamma(\alpha)},$$

and

$$|(D_{0+}^\mu Q_n x)(t)| \leq \frac{2}{\Gamma(\alpha - \mu)} \int_0^1 \varphi(s) ds = \frac{2\|\varphi\|_1}{\Gamma(\alpha - \mu)},$$

it follows that $\{Q_n x : x \in \mathcal{W}\}$ and $\{D_{0+}^\mu Q_n x : x \in \mathcal{W}\}$ are uniformly bounded. Next, let $0 \leq t_1 < t_2 \leq 1$. Then for $x \in \mathcal{W}$,

$$\begin{aligned} |Q_n x(t_2) - Q_n x(t_1)| &\leq \frac{1}{\Gamma(\alpha)} \left((t_2^{\alpha - 1} - t_1^{\alpha - 1}) \int_0^1 (1 - s)^{\alpha - 1 - \beta} \varphi(s) ds \right. \\ &\quad \left. + \int_0^{t_1} ((t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1}) \varphi(s) ds \right. \\ &\quad \left. + (t_2 - t_1)^{\alpha - 1} \int_{t_1}^{t_2} \varphi(s) ds \right) \end{aligned}$$

and

$$\begin{aligned} & |(D_{0+}^\mu Q_n x)(t_2) - (D_{0+}^\mu Q_n x)(t_1)| \\ & \leq \frac{1}{\Gamma(\alpha - \mu)} \left(\left(t_2^{\alpha - \mu - 1} - t_1^{\alpha - \mu - 1} \right) \int_0^1 (1 - s)^{\alpha - \beta - 1} \varphi(s) ds \right. \\ & \quad \left. + \int_0^{t_1} \left((t_2 - s)^{\alpha - \mu - 1} - (t_1 - s)^{\alpha - \mu - 1} \right) \varphi(s) ds + (t_2 - t_1)^{\alpha - \mu - 1} \int_{t_1}^{t_2} \varphi(s) ds \right). \end{aligned}$$

Thus, with the appropriate choice of δ , it can be shown that for $\epsilon > 0$, if $t_2 - t_1 < \delta$, $|Q_n x(t_2) - Q_n x(t_1)| < \epsilon$ and $|(D_{0+}^\mu Q_n x)(t_2) - (D_{0+}^\mu Q_n x)(t_1)| < \epsilon$. Therefore, $\{Q_n x : x \in \mathcal{W}\}$ and $\{D_{0+}^\mu Q_n x : x \in \mathcal{W}\}$ are equicontinuous, and by the Arzelà-Ascoli theorem, Q_n is a completely continuous operator. \square

Lemma 3.4. *Let (H1) and (H2) hold. Then (3.1), (1.2) has a positive solution x^* with $x^*(t) \geq Mt^{\alpha-1}$ for $t \in [0, 1]$.*

Proof. Define $\Omega_1 = \{x \in X : \|x\| < M\}$. Then for $x \in P \cap \partial\Omega_1$ and $t \in [0, 1]$,

$$(Q_n x)(t) = \int_0^1 G(t, s) f_n(s, x(s), D_{0+}^\mu x(s)) \geq m \int_0^1 G(t, s) \geq Mt^{\alpha-1}.$$

So $\|Q_n x\|_0 \geq M$. Consequently, $\|Q_n x\| \geq \|x\|$ for $x \in P \cap \partial\Omega_1$.

Next, notice that for $x \in \mathcal{P}$ and $t \in [0, 1]$,

$$\begin{aligned} |(Q_n x)(t)| & \leq \frac{1}{\Gamma(\alpha)} \int_0^1 \gamma(s) (q(1/n) + p(x(s)) + p(1) + \omega(|D_{0+}^\mu x(s)|)) \\ & \leq \frac{1}{\Gamma(\alpha)} (q(1/n) + p(\|x\|_0) + p(1) + \omega(\|D_{0+}^\mu x\|_0)) \|\gamma\|_L. \end{aligned}$$

Also, for $x \in \mathcal{P}$,

$$\begin{aligned} |D_{0+}^\mu (Q_n x)(t)| & = \left| \frac{1}{\Gamma(\alpha - \mu)} \left(t^{\alpha - \mu - 1} \int_0^1 (1 - s)^{\alpha - \beta - 1} f_n(s, x(s), D_{0+}^\mu x(s)) \right. \right. \\ & \quad \left. \left. - \int_0^t (t - s)^{\alpha - \mu - 1} f_n(s, x(s), D_{0+}^\mu x(s)) \right) \right| \\ & \leq \frac{2}{\Gamma(\alpha - \mu)} (q(1/n) + p(\|x\|_0) + p(1) + \omega(\|D_{0+}^\mu x\|_0)) \|\gamma\|_L. \end{aligned}$$

So for $K = \max \left\{ \frac{1}{\Gamma(\alpha)}, \frac{2}{\Gamma(\alpha - \mu)} \right\}$,

$$\|Q_n x\| \leq K (q(1/n) + p(\|x\|) + p(1) + \omega(\|x\|)) \|\gamma\|_L$$

for $x \in \mathcal{P}$. Since $\lim_{x \rightarrow \infty} \frac{p(x) + \omega(x)}{x} = 0$, there exists an $S > 0$ such that

$$K (q(1/n) + p(S) + p(1) + \omega(S)) \|\gamma\|_L < S.$$

Let $\Omega_2 = \{x \in X : \|x\| < S\}$. Then $\|Q_n x\| \leq \|x\|$ for $x \in \mathcal{P} \cap \partial\Omega_2$.

It follows from Theorem 3.2 that Q_n has a fixed point $x^* \in \mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$. Consequently, (3.1), (1.2) has a solution x^* with $\|x^*\| \geq M$. □

4. POSITIVE SOLUTIONS OF THE SINGULAR PROBLEM

Lemma 4.1. *Let (H1) and (H2) hold. Let x_n be a solution to (3.1), (1.2). Then the sequences $\{x_n\}$ and $\{D_{0+}^\mu x_n\}$ are relatively compact in $C[0, 1]$.*

Proof. Similar to the proof of Lemma 3.3, we use Arzelà-Ascoli to show these sequences are relatively compact. Note that

$$x_n(t) = \int_0^1 G(t, s) f_n(s, x_n(s), D_{0+}^\mu x_n(s)) ds$$

and

$$D_{0+}^\mu x_n(t) = \frac{1}{\Gamma(\alpha - \mu)} \left(t^{\alpha - \mu - 1} \int_0^1 (1 - s)^{\alpha - \beta - 1} f_n(s, x_n(s), D_{0+}^\mu x_n(s)) ds - \int_0^t (t - s)^{\alpha - \mu - 1} f_n(s, x_n(s), D_{0+}^\mu x_n(s)) ds \right)$$

for $t \in [0, 1]$ and $n \in \mathbb{N}$. It follows from the proof of Lemma 3.4 that $x_n(t) \geq Mt^{\alpha-1}$ for all $t \in [0, 1]$, $n \in \mathbb{N}$. But

$$f_n(t, x_n(t), D_{0+}^\mu x_n(t)) \leq \gamma(t) (q(x_n(t)) + p(x_n(t)) + p(1) + \omega(\|D_{0+}^\mu x_n(t)\|)).$$

It was assumed that q is nonincreasing and p and ω are nondecreasing. Therefore,

$$f_n(t, x_n(t), D_{0+}^\mu x_n(t)) \leq \gamma(t)(q(Mt^{\alpha-1}) + p(\|x_n\|_0) + p(1) + \omega(\|D_{0+}^\mu x_n\|_0)).$$

This implies

$$x_n(t) \leq \frac{1}{\Gamma(\alpha)} \left[\int_0^1 \gamma(t)q(Mt^{\alpha-1})dt + (p(\|x_n\|_0) + p(1) + \omega(\|D_{0+}^\mu x_n\|_0))\|\gamma\|_L \right],$$

and

$$D_{0+}^\mu x_n(t) \leq \frac{2}{\Gamma(\alpha - \mu)} \left[\int_0^1 \gamma(t)q(Mt^{\alpha-1})dt + (p(\|x_n\|_0) + p(1) + \omega(\|D_{0+}^\mu x_n\|_0))\|\gamma\|_L \right],$$

for all $t \in [0, 1]$ and $n \in \mathbb{N}$. Note it was assumed that $\int_0^1 \gamma(t)q(Mt^{\alpha-1})dt < \infty$.

Therefore, by again setting $K = \max \left\{ \frac{1}{\Gamma(\alpha)}, \frac{2}{\Gamma(\alpha-\mu)} \right\}$,

$$\|x_n\| \leq K \left[\int_0^1 \gamma(t)q(Mt^{\alpha-1})dt + (p(\|x_n\|_0) + p(1) + \omega(\|Dux_n\|_0))\|\gamma\|_L \right],$$

for $n \in \mathbb{N}$. Since $\lim_{x \rightarrow \infty} \frac{p(x) + \omega(x)}{x} = 0$, there exists an $S > 0$ such that

$$K \left[\int_0^1 \gamma(t)q(Mt^{\alpha-1})dt + (p(v) + p(1) + \omega(v))\|\gamma\|_L \right] < S,$$

for each $v \geq S$. Thus $\|x_n\| < S$ for $n \in \mathbb{N}$ and the sequences $\{x_n\}$ and $\{D_{0+}^\mu x_n\}$ are uniformly bounded in $C[0, 1]$.

Now, we show the sequences $\{x_n\}$ and $\{D_{0+}^\mu x_n\}$ are equicontinuous in $C[0, 1]$. Let $0 \leq t_1 < t_2 \leq 1$. Using the fact that

$$0 < f_n(t, x_n(t), D_{0+}^\mu x_n(t)) \leq \gamma(t)(q(Mt^{\alpha-1}) + p(S) + p(1) + \omega(S)),$$

we have

$$\begin{aligned} & |x_n(t_2) - x_n(t_1)| \\ & \leq \Gamma(\alpha) \left((t_2^{\alpha-1} - t_1^{\alpha-1}) \int_0^1 (1-s)^{\alpha-1-\beta} (\gamma(s)(q(Ms^{\alpha-1}) + p(S) + p(1) + \omega(S)))ds \right. \\ & \quad + \int_0^{t_1} ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}) (\gamma(s)(q(Ms^{\alpha-1}) + p(S) + p(1) + \omega(S)))ds \\ & \quad \left. + (t_2 - t_1)^{\alpha-1} \int_{t_1}^{t_2} (\gamma(s)(q(Ms^{\alpha-1}) + p(S) + p(1) + \omega(S)))ds \right), \end{aligned}$$

and

$$\begin{aligned}
 & |(D_{0+}^\mu x_n)(t_2) - (D_{0+}^\mu x_n)(t_1)| \\
 & \leq \frac{1}{\Gamma(\alpha - \mu)} \left((t_2^{\alpha - \mu - 1} - t_1^{\alpha - \mu - 1}) \times \right. \\
 & \int_0^1 (1 - s)^{\alpha - \beta - 1} (\gamma(s)(q(Ms^{\alpha - 1}) + p(S) + p(1) + \omega(S))) ds \\
 & + \int_0^{t_1} ((t_2 - s)^{\alpha - \mu - 1} - (t_1 - s)^{\alpha - \mu - 1}) (\gamma(s)(q(Ms^{\alpha - 1}) + p(S) + p(1) + \omega(S))) ds \\
 & \left. + (t_2 - t_1)^{\alpha - \mu - 1} \int_{t_1}^{t_2} (\gamma(s)(q(Ms^{\alpha - 1}) + p(S) + p(1) + \omega(S))) ds \right).
 \end{aligned}$$

Thus, with the appropriate choice of δ , it can be shown that for $\epsilon > 0$, if $t_2 - t_1 < \delta$, $|x_n(t_2) - x_n(t_1)| < \epsilon$ and $|(D_{0+}^\mu x_n)(t_2) - (D_{0+}^\mu x_n)(t_1)| < \epsilon$. Therefore, $\{x_n\}$ and $\{D_{0+}^\mu x_n\}$ are equicontinuous in $C[0, 1]$. So $\{x_n\}$ and $\{D_{0+}^\mu x_n\}$ are relatively compact in $C[0, 1]$. \square

Theorem 4.2. *Let (H1) and (H2) hold. Then (1.1), (1.2) has a positive solution x with $x(t) \geq Mt^{\alpha - 1}$ for $t \in [0, 1]$.*

Proof. From Lemma 3.4, (3.1), (1.2) has a positive solution for each $n \in \mathbb{N}$. Call these solutions x_n . From Lemma 4.1, the sequence $\{x_n\}$ is relatively compact in X . Therefore, without loss of generality, there exists an $x \in X$ with $\lim_{n \rightarrow \infty} x_n = x$ uniformly in X . Consequently, $x \in P$, $x(t) \geq Mt^{\alpha - 1}$ for $t \in [0, 1]$ and

$$\lim_{n \rightarrow \infty} f_n(t, x_n(t), D_{0+}^\mu x_n(t)) = f(t, x(t), D_{0+}^\mu x(t)),$$

for a.e. $t \in [0, 1]$. Since

$$0 \leq G(t, s) f_n(x_n(s), D_{0+}^\mu x_n(s)) \leq \frac{1}{\Gamma(\alpha)} \gamma(s)(q(Ms^{\alpha - 1}) + p(S) + p(1) + \omega(S)) \in L^1[0, 1]$$

for a.e. $s \in [0, 1]$ and all $t \in [0, 1]$, $n \in \mathbb{N}$, it follows from the Lebesgue Dominated Convergence Theorem that

$$\lim_{n \rightarrow \infty} \int_0^1 G(t, s) f_n(x_n(s), D_{0+}^\mu x_n(s)) ds = \int_0^1 G(t, s) f(t, x(t), D_{0+}^\mu x(t)) ds.$$

Since

$$x_n(t) = \int_0^1 G(t, s) f_n(s, x_n(s), D_{0+}^\mu x_n(s)) ds,$$

for $t \in [0, 1]$,

$$x(t) = \int_0^1 G(t, s) f(t, x(t), D_{0+}^\mu x(t)) ds,$$

for $t \in [0, 1]$. Thus, x is a positive solution of (1.1), (1.2). □

5. EXAMPLE

Example 5.1. Fix $\alpha \in (1, 2]$, $\beta \in (0, \alpha - 1]$, $\mu \in (0, \alpha - 1]$. Let $i, k \in (0, 1)$, $j \in (0, \frac{1}{\alpha-1})$. Define

$$f(t, x, y) = \frac{1}{\sqrt{|2t-1|}} \left(x^i + \frac{1}{x^j} + |y|^k \right).$$

Additionally, set $\gamma(t) = \frac{1}{\sqrt{|2t-1|}}$, $q(x) = \frac{1}{x^j}$, $p(x) = x^i$, $\omega(y) = y^k$, $m = 1$ and $M = \frac{\beta}{(\alpha-\beta)\Gamma(\alpha+1)}$.

Notice that for $t \in [0, 1] \setminus \{\frac{1}{2}\}$ and $(x, y) \in (0, \infty) \times \mathbb{R}$,

$$f(t, x, y) \geq \frac{1}{\sqrt{|2t-1|}} \geq 1 = m.$$

Hence f satisfies condition (H1). Also, $f(t, x, y) = \gamma(t)(q(x) + p(x) + \omega(|y|))$, $\gamma \in L^1[0, 1]$, $q \in C(0, \infty)$ is nonincreasing, and $p, \omega \in C[0, \infty)$ are nondecreasing. Last,

$$\int_0^1 \frac{M^{-j} t^{-j(\alpha-1)}}{\sqrt{|2t-1|}} dt < \infty,$$

since $j(\alpha - 1) < 1$, and

$$\lim_{x \rightarrow \infty} \frac{x^i + x^k}{x} = 0,$$

since $i, k \in (0, 1)$. So (H2) is also satisfied. Thus, Theorem 4.2 provides that there is at least one positive solution $x(t)$ to the fractional differential equation

$$D_{0+}^\alpha x + \frac{1}{\sqrt{|2t-1|}} \left(x^i + \frac{1}{x^j} + |D_{0+}^\mu x|^k \right) = 0,$$

satisfying

$$x(0) = D_{0+}^\beta x(1) = 0.$$

Further, for $t \in [0, 1]$,

$$x(t) \geq \frac{\beta t^{\alpha-1}}{(\alpha - \beta)\Gamma(\alpha + 1)}.$$

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Jeffrey W. Lyons
jlyons@nova.edu

Nova Southeastern University
Department of Mathematics
Fort Lauderdale, FL 33314 USA

Jeffrey T. Neugebauer
jeffrey.neugebauer@eku.edu

Eastern Kentucky University
Department of Mathematics and Statistics
Richmond, KY 40475 USA

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