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FAN'S CONDITION ON INDUCED SUBGRAPHS FOR CIRCUMFERENCE AND PANCYCLICITY

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Abstract. Let $\mathcal{H}$ be a family of simple graphs and k be a positive integer. We say that a graph G of order $n \geq k$ satisfies Fan's condition with respect to $\mathcal{H}$ with constant k, if for every induced subgraph H of G isomorphic to any of the graphs from $\mathcal{H}$ the following holds:

$$\forall u, v \in V(H)\colon d_H(u,v) = 2 \Rightarrow \max\{d_G(u), d_G(v)\} \geq k/2.$$

If G satisfies the above condition, we write $G \in \mathcal{F}(\mathcal{H}, k)$. In this paper we show that if G is 2-connected and $G \in \mathcal{F}(\{K_{1,3}, P_4\}, k)$, then G contains a cycle of length at least k, and that if $G \in \mathcal{F}(\{K_{1,3}, P_4\}, n)$, then G is pancyclic with some exceptions. As corollaries we obtain the previous results by Fan, Benhocine and Wojda, and Ning.

Keywords: Fan's condition, circumference, hamiltonian cycle, pancyclicity.

Mathematics Subject Classification: 05C38, 05C45.

1. INTRODUCTION

In the paper we consider only simple, finite and connected graphs. For basic terminology not defined here we use [6].

Let G be a graph of order n. G is said to be hamiltonian, if it contains a cycle C_n, and it is called pancyclic, if it contains cycles of all lengths k for $3 \leq k \leq n$. The length of a longest cycle in G is called the *circumference* of G and denoted $c(G)$.

For a family of graphs $\mathcal{H}$ we say that G is $\mathcal{H}$*-free* if for every $H \in \mathcal{H}$ there are no induced copies of H in G. If one demands G being $\mathcal{H}$-free, then $\mathcal{H}$ is *forbidden* in G. Pairs of forbidden subgraphs ensuring hamiltonicity or pancyclicity of 2-connected graphs were extensively examined by a number of researchers during the last forty years (e.g., [7,10,14,15]). These results were gathered by Bedrossian in his Ph.D. thesis. Together with the results he obtained, they can be formulated in the following way (graphs Z_i, B, W and N are represented on Figure 1; the "only if" parts of the below Theorems are due to Faudree and Gould).

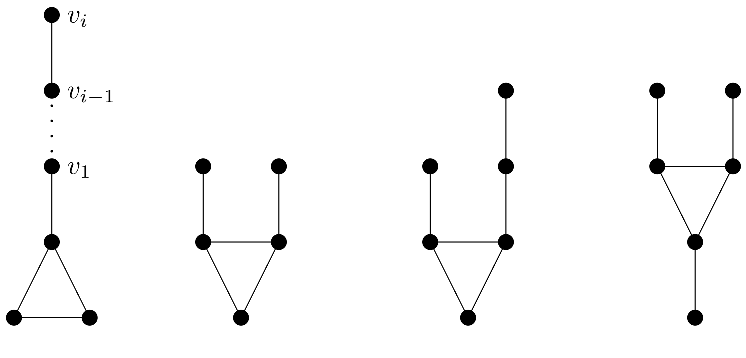

Fig. 1. Graphs Z_i, B, W and N

Theorem 1.1 (Bedrossian [1]; Faudree and Gould [12]). *Let R and S be connected graphs with R, $S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W.*

Theorem 1.2 (Bedrossian [1]; Faudree and Gould [12]). *Let R and S be connected graphs with R, $S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

The reason for the path P_3 being excluded from the assumptions of the above Theorems is that the only 2-connected P_3-free graph is a complete graph, which is obviously both hamiltonian and pancyclic.

Another popular approach to the problem of existence of cycles in graphs is the one involving degree conditions. A classical result in this field is due to Fan.

Theorem 1.3 (Fan [11]). *Let G be a 2-connected graph with n vertices and let $3 \leq k \leq n$. If*

$$d_G(u, v) = 2 \Rightarrow \max\{d_G(u), d_G(v)\} \geq k/2$$

for each pair of vertices u and v in G, then $c(G) \geq k$.

In the particular case when $k = n$, Fan's Theorem implies hamiltonicity of G. It was later shown by Benhocine and Wojda that this condition ensures in fact pancyclicity, besides three exceptional graphs (for the graph F_{4r} which consists of a clique of order $2r$ joined via perfect matching with r disjoint copies of a path with two vertices, see Figure 2).

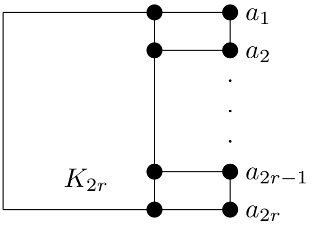

Fig. 2. Fan's graph F_{4r}

Theorem 1.4 (Benhocine and Wojda [3]). *Let G be a 2-connected graph with $n \geq 3$. If*

$$d_G(u,v) = 2 \Rightarrow \max\{d_G(u), d_G(v)\} \geq n/2$$

for each pair of vertices u and v in G, then G is pancyclic unless $n = 4r$, $r \geq 2$ and G is F_{4r}, or n is even and $G = K_{n/2,n/2}$ or else $n \geq 6$ and $G = K_{n/2,n/2} - e$.

A natural way of relaxing forbidden subgraphs conditions is to allow these subgraphs to be present in a graph but with a Fan-type degree conditions imposed on them. This idea was explored by many researchers, using various terminology and notations. Before we state their results, we introduce a notion that encapsulates these different notations.

Definition 1.5. Let $\mathcal{H}$ be a family of graphs and k be a positive integer. We say that a graph G satisfies Fan's condition with respect to $\mathcal{H}$ with constant k, if for every induced subgraph H of G isomorphic to any of the graphs from $\mathcal{H}$ the following holds:

$$\forall\, u, v \in V(H)\colon\, d_H(u,v) = 2 \;\Rightarrow \max\{d_G(u), d_G(v)\} \geq k/2.$$

By $\mathcal{F}(\mathcal{H}, k)$ we denote the family of graphs satisfying the Fan's condition with respect to $\mathcal{H}$ with constant k. If $\mathcal{H}$ consists of one element, say H, we write $\mathcal{F}(H, k)$ instead of $\mathcal{F}(\{H\}, k)$.

Given a family of graphs $\mathcal{H}$ and a constant k, every $\mathcal{H}$-free graph satisfies Fan's condition with respect to $\mathcal{H}$ with constant k. It is also clear, that if $G \in \mathcal{F}(P_3, k)$, then $G \in \mathcal{F}(\mathcal{H}, k)$. The authors of [2] imposed the Fan's condition on one of the pairs of subgraphs that appear in Theorems 1.1 and 1.2 and obtained the following results.

Theorem 1.6 (Bedrossian, Chen, and Schelp [2]). *Let G be a 2-connected graph with n vertices. If $3 \leq k \leq n$ and $G \in \mathcal{F}(\{K_{1,3}, Z_1\}, k)$, then $c(G) \geq k$.*

Theorem 1.7 (Bedrossian, Chen, and Schelp [2]). *Let G be a 2-connected graph of order $n \geq 3$ which is not a cycle. If $G \in \mathcal{F}(\{K_{1,3}, Z_1\}, n)$, then G is pancyclic unless $n = 4r$, $r \geq 2$ and G is F_{4r}, or n is even and $G = K_{n/2,n/2}$ or else $n \geq 6$ and $G = K_{n/2,n/2} - e$.*

Theorems 1.1 – 1.7 were the main motivation for our research. Similarly to Theorems 1.6 and 1.7, in this paper we prove the following.

Theorem 1.8. *Let G be a 2-connected graph with n vertices. If $3 \leq k \leq n$ and $G \in \mathcal{F}(\{K_{1,3}, P_4\}, k)$, then $c(G) \geq k$.*

Theorem 1.9. *Let G be a 2-connected graph of order $n \geq 3$. If $G \in \mathcal{F}(\{K_{1,3}, P_4\}, n)$, then G is pancyclic unless $n = 4r$, $r \geq 2$ and G is F_{4r}, or n is even and $G = K_{n/2,n/2}$ or else $n \geq 6$ and $G = K_{n/2,n/2} - e$.*

It is easy to see that Theorem 1.3 is a corollary from Theorem 1.8 and that Theorem 1.4 follows from Theorem 1.9. Note also that from the exceptional non-pancyclic graphs mentioned in Theorem 1.9 only the cycle $K_{2,2}$ satisfies Fan's condition with respect to $\{K_{1,3}, P_4\}$ with constant $n + 1$. Hence, the following result also can be deduced from Theorem 1.9.

Theorem 1.10 (Ning [18])**.** *Let G be a 2-connected graph with n vertices other than $K_{2,2}$. If $G \in \mathcal{F}(\{K_{1,3}, P_4\}, n + 1)$, then G is pancyclic.*

The above Theorem is one of the many results connecting the Fan-type condition with hamiltonicity and pancyclicity. Some of these results fully extend Theorems 1.1 and 1.2.

Theorem 1.11. *Let R and S be connected graphs with R, $S \neq P_3$ and let G be a 2-connected graph of order n. Then $G \in \mathcal{F}(\{R, S\}, n)$ implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and S is one of the following:*

- P_4, P_5, P_6 (Chen, Wei, and Zhang [8]),
- Z_1 (Bedrossian, Chen, and Schelp [2]),
- B (Li, Wei and Gao [17]),
- N (Chen, Wei and Zhang [9]),
- Z_2, W (Ning and Zhang [19]).

Theorem 1.12. *Let R and S be connected graphs with R, $S \neq P_3$ and let G be a 2-connected graph of order n which is not a cycle. Then $G \in \mathcal{F}(\{R, S\}, n+1)$ implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and S is one of the following:*

- Z_1 (Bedrossian, Chen, and Schelp [2]),
- Z_2, P_4 (Ning [18]),
- P_5 (Wideł [21]).

To close this section we propose the following two conjectures, which seem to be justified in view of the results presented so far.

Conjecture 1.13. *Let R and S be connected graphs with R, $S \neq P_3$ and let G be a 2-connected graph with n vertices. If $3 \leq k \leq n$, then $G \in \mathcal{F}(\{R, S\}, k)$ implies $c(G) \geq k$ if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W.*

Conjecture 1.14. *Let R and S be connected graphs with R, $S \neq P_3$ and let G be a 2-connected graph of order n with $G \notin \{C_n, F_{4(n/4)}, K_{n/2,n/2}\, K_{n/2,n/2} - e\}$. Then $G \in \mathcal{F}(\{R,S\},n)$ implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

In the next section we introduce terminology and notation used throughout the rest of the paper. The proofs of Theorems 1.8 and 1.9 are presented in Sections 3 and 4, respectively.

2. PRELIMINARIES

For a vertex $v \in V(G)$, we denote by $N_G(v)$ the neighbourhood of v, i.e., the set of vertices adjacent to v. For $A \subseteq V(G)$, we denote $G[A]$ the subgraph of G induced by the vertex set A. The *neighbourhood of v in $G[A]$*, namely $N_G(v) \cap A$, is denoted by $N_A(v)$ and the *closed neighbourhood of v in $G[A]$*, namely $N_A(v) \cup \{v\}$, is denoted by $N_A[v]$.

The complete bipartite graph $K_{1,3}$ is called a *claw*. The vertex of degree three of a claw is called its *center vertex* and the other vertices are its *end vertices*. Let $A = \{v_1, v_2, v_3, v_4\}$. If $G[A]$ is isomorphic to $K_{1,3}$, with v_1 being a center vertex of the claw and v_2, v_3 and v_4 being its end vertices, we say that $\{v_1; v_2, v_3, v_4\}$ induces $K_{1,3}$ (or induces a claw).

For a cycle C we distinguish one of two possible orientations of C. We write xC^+y for the path from $x \in V(C)$ to $y \in V(C)$ following the orientation of C, and xC^-y denotes the path from x to y opposite to the direction of C. By $d_C(x,y)$ we denote the length of the shorter of the paths xC^+y and xC^-y. Similarly, for a path $P = v_1 \dots v_m$ and two vertices v_i, $v_j \in V(P)$ with $i < j$, we write $v_iP^+v_j$ for the path $v_iv_{i+1} \dots v_{j-1}v_j$ and $v_jP^-v_i$ for the path $v_jv_{j-1} \dots v_{i+1}v_i$. For two positive integers k and m, where $k \leq m$, we say that G contains $[k,m]$-cycles if there are cycles C_k, C_{k+1}, $\dots$, C_m in G.

Let G be a graph of order n. Vertex $v \in V(G)$ is called *heavy* if $d_G(v) \geq n/2$.

Let $A, B \subset V(G)$ be subsets of vertices of G. By

$$e(A,B) = |\{e = uv \in E(G) \colon u \in A, v \in B\}|$$

we denote the total number of edges between A and B. If both A and B consist of one element, say $A = \{v_A\}$ and $B = \{v_B\}$, we write $e(v_A, v_B)$ instead of $e(\{v_A\},\{v_B\})$.

3. PROOF OF THEOREM 1.8

We utilize the general idea of the proof of Theorem 1.6. Before we begin the proof, we need to state the following.

Theorem 3.1 (Bondy [4])**.** *Let G be a 2-connected graph of order $|V(G)| \geq k$ and let $P = v_1 \dots v_m$ be a path of maximum length in G. If $d_G(v_1) + d_G(v_m) \geq k$, then $c(G) \geq k$.*

For the convenience of the reader, we restate Theorem 1.8 below.

Theorem 1.8. *Let G be a 2-connected graph with n vertices. If $3 \leq k \leq n$ and $G \in \mathcal{F}(\{K_{1,3}, P_4\}, k)$, then $c(G) \geq k$.*

Proof of Theorem 1.8. Suppose $c(G)$ is less than k. It will be shown that this leads to the existence of a longest path $P = v_1 \dots v_m$ in G such that $d_G(v_1) + d_G(v_m) \geq k$, a contradiction to Theorem 3.1.

For a given longest path $P = v_1 \dots v_m$ in G let v_{l_P} be the last neighbour of v_1 along P, i.e., $l_P = \max\{i \colon v_1 v_i \in E(G)\}$, and let v_{n_P} be the last nonneighbour of v_1 preceding v_{l_P}, that is $n_P = \max\{i \colon i < l_P \text{ and } v_1 v_{n_P} \notin E(G)\}$.

Clearly, $l_P > 2$. Furthermore, it follows from 2-connectivity of G that $l_P < m$, since otherwise there would be either a hamiltonian cycle or a path longer than P in G. Next observe that there exists a longest path P with $n_P > 2$. If this is not the case and $n_P = 1$, let Q be a path from v_i to v_j, $i \leq l_P - 1$, $j \geq l_P + 1$, such that $V(P) \cap V(Q) = \{v_i, v_j\}$. Then form the path $P' = v_{j-1} P^- v_{i+1} v_1 P^+ v_i Q^+ v_j P^+ v_m$, which is a longest path with $l_{P'} \geq j > l_P$, a contradiction when P is chosen to have the largest l_P value.

Fix a longest path $P = v_1 \dots v_m$ with n_P of largest possible value. With the above observations it will next be shown that there exists a longest path with one of its endvertices being v_m and the other having degree at least $k/2$. To do this, suppose that $d_G(v_1) < k/2$. Note that, since $n_P > 2$, we have $d_G(v_{n_P}) < k/2$, since otherwise the path $v_{n_P} P^- v_1 v_{n_P+1} P^+ v_m$ is a longest path with $d_G(v_{n_P}) \geq k/2$. Since $G \in \mathcal{F}(K_{1,3}, k)$, it follows that $\{v_{n_P+1}; v_1, v_{n_P}, v_{n_P+2}\}$ can not induce a claw. Thus v_{n_P+2} is adjacent to at least one of the vertices v_1 and v_{n_P}. Before the proof divides into subcases, we note that $d_G(v_{n_P+1}) < k/2$, since by the previous observation at least one of the paths $v_{n_P+1} P^- v_1 v_{n_P+2} P^+ v_m$ and $v_{n_P+1} v_1 P^+ v_{n_P} v_{n_P+2} P^+ v_m$ is a longest path in G beginning with v_{n_P+1}.

Throughout the proof, whenever we declare a contradiction due to a discovered induced subgraph of G isomorphic to the claw or the path P_4, it is because the subgraph does not satisfy Fan's condition with constant k.

Case 1. $v_1 v_{n_P+2} \in E(G), v_{n_P} v_{n_P+2} \notin E(G)$.

Note that under the assumptions of this case we have $m \geq n_P + 3$. We begin with crucial pieces of information regarding the degree of the vertex v_{n_P+2} and the adjacency structure of its neighbourhood.

Claim 3.2. *$v_{n_P+3} v_{n_P}, v_{n_P+3} v_{n_P+1} \notin E(G)$ and $d_G(v_{n_P+2}) \geq k/2$.*

Proof. Note that if v_{n_P+3} is adjacent to v_{n_P}, then under the assumptions of this case the path $P' = v_{n_P} P^- v_1 v_{n_P+1} P^+ v_m$ is a longest path in G with $n_{P'} \geq n_P + 2$, contradicting the choice of P. Similarly, if $v_{n_P+3} v_{n_P+1} \in E(G)$, then $P' = v_{n_P} P^- v_1 v_{n_P+2} v_{n_P+1} v_{n_P+3} P^+ v_m$ is a longest path with $n_{P'} \geq n_P + 1$. Thus v_{n_P+3} is adjacent neither to v_{n_P}, nor to v_{n_P+1}, and so $v_{n_P} v_{n_P+1} v_{n_P+2} v_{n_P+3}$ is an induced path P_4 in G. Since $G \in \mathcal{F}(P_4, k)$ and $d_G(v_{n_P}) < k/2$, it follows that $d_G(v_{n_P+2}) \geq k/2$. □

Claim 3.3. *$d_G(v_2) < k/2$ and $v_2v_{n_P+3} \notin E(G), v_2v_{n_P+1} \in E(G)$.*

Proof. Clearly, if $d_G(v_2) \geq k/2$, then $v_2P^+v_{n_P+1}v_1v_{n_P+2}P^+v_m$ is a longest path with $d_G(v_2) \geq k/2$, and if $v_2v_{n_P+3} \in E(G)$, then, by Claim 3.2, $v_{n_P+2}v_1v_{n_P+1}P^-v_2v_{n_P+3}P^+v_m$ is such path.

Now suppose that $v_2v_{n_P+1} \notin E(G)$. Since $d_G(v_2), d_G(v_{n_P+1}) < k/2$ and $G \in \mathcal{F}(\{K_{1,3}, P_4\}, k)$, the set $\{v_2, v_1, v_{n_P+1}, v_{n_P}\}$ can not induce P_4 and the set $\{v_{n_P+2}; v_2, v_{n_P+1}, v_{n_P+3}\}$ can not induce a claw. It follows from Claim 3.2 that $v_2v_{n_P} \in E(G)$ and $v_2v_{n_P+2} \notin E(G)$. But now $v_2v_{n_P}v_{n_P+1}v_{n_P+2}$ is an induced P_4 in G, a contradiction. □

Claim 3.4. *There are no edges between the vertices v_1, v_{n_P}, v_{n_P+1} and the vertices from the set $\{v_{n_P+3}, \ldots, v_m\}$.*

Proof. From the definition of n_P it follows that to prove that v_1 is not adjacent to any of the vertices from the set $\{v_{n_P+3}, \ldots, v_m\}$ it suffices to show that it is not adjacent to v_{n_P+3}. This is clearly true, since otherwise $v_{n_P+2}P^-v_1v_{n_P+3}P^+v_m$ would be a longest path with $d_G(v_{n_P+2}) \geq k/2$, by Claim 3.2.

Recall that $v_{n_P}v_{n_P+3} \notin E(G)$ and $v_{n_P+1}v_{n_P+3} \notin E(G)$, by Claim 3.2, and $v_2v_{n_P+1} \in E(G)$, by Claim 3.3. Suppose that v_{n_P} is adjacent to v_{n_P+j} for some $3 < j \leq m - n_P$. Then the path $P' = v_{n_P}P^-v_2v_{n_P+1}v_1v_{n_P+2}P^+v_m$ is a longest path in G with $n_{P'} \geq n_P + 3$, contradicting the choice of P.

From the observations made so far it follows that if v_{n_P+1} is adjacent to some vertex v_{n_P+j} with $3 < j \leq m - n_P$, then $\{v_{n_P+1}; v_1, v_{n_P}, v_{n_P+j}\}$ induces a claw. Since $d_G(v_1), d_G(v_{n_P}) < k/2$, this contradicts G being a graph from the family $\mathcal{F}(K_{1,3}, k)$. □

Next claim provides a characterization of properties of the vertices that lie on P between v_1 and v_{n_P}.

Claim 3.5. *For $i \in \{2, \ldots, n_P\}$ the following holds:*

(i) *$d_G(v_i) < k/2$,*
(ii) *$v_iv_{n_P+3} \notin E(G)$,*
(iii) *$v_iv_{n_P+1} \in E(G)$,*
(iv) *either v_i is adjacent to both v_1 and v_{n_P+2} or else it is not adjacent to any of them,*
(v) *v_i is not adjacent to any of the vertices from the set $\{v_{n_P+3}, \ldots, v_m\}$.*

Proof. The proof is by induction on i. For $i = 2$ the statements (i), (ii) and (iii) are true by Claim 3.3. To show that the condition (iv) holds, we first observe that v_2 is adjacent to v_1. Suppose $v_2v_{n_P+2} \notin E(G)$. Then under the assumptions of the case and depending on the existence of the edge $v_2v_{n_P}$, either $v_{n_P}v_2v_1v_{n_P+2}$ is an induced path or the set $\{v_{n_P+1}; v_2, v_{n_P}, v_{n_P+2}\}$ induces a claw. Since the degrees of v_1, v_2 and v_{n_P} are strictly less than $k/2$, this contradicts G being a member of the family $\mathcal{F}(\{K_{1,3}, P_4\}, k)$.

For the proof of (v) suppose that v_2 is adjacent to some vertex $v \in \{v_{n_P+3}, \ldots, v_m\}$. The path $vv_2v_{n_P+1}v_{n_P}$ can not be an induced one, since $d_G(v_2)$, $d_G(v_{n_P}) < k/2$. Thus it follows from Claim 3.4 that v_2 is adjacent to v_{n_P}. But now $\{v_2; v, v_{n_P}, v_1\}$ induces a claw with $d_G(v_2), d_G(v_{n_P}) < k/2$, a contradiction.

Now assume that for some $i < n_P$ the conditions (i)-(v) hold for the vertices $v_2, \ldots, v_i$. It will be shown that they hold also for v_{i+1}.

First observe that $d_G(v_{i+1}) < k/2$, since otherwise, by the condition (iii) for v_i, the path $v_{i+1}P^+v_{n_P+1}v_iP^-v_1v_{n_P+2}P^+v_m$ is a longest path in G with its first vertex having degree at least $k/2$. The validity of the condition (ii) is also straightforward: if $v_{i+1}v_{n_P+3} \in E(G)$, then a longest path with its first vertex having degree not less than $k/2$ is the path $v_{n_P+2}v_1P^+v_iv_{n_P+1}P^-v_{i+1}v_{n_P+3}P^+v_m$, by Claim 3.2.

Now suppose that the condition (iii) is not true, i.e., that $v_{i+1}v_{n_P+1}$ is not an edge in G. It follows that v_{i+1} is not adjacent to v_{n_P+2}, since otherwise, by (ii) for v_{i+1} and by Claim 3.4, the set $\{v_{n_P+2}; v_{i+1}, v_{n_P+1}, v_{n_P+3}\}$ induces a claw with $d_G(v_{i+1}), d_G(v_{n_P+1}) < k/2$.

If $v_iv_{n_P}$ is not an edge in G, then by (iii) for v_i, the vertex v_{i+1} is adjacent to v_{n_P} to avoid induced path $v_{i+1}v_iv_{n_P+1}v_{n_P}$ with $d_G(v_i), d_G(v_{n_P}) < k/2$. But now $v_{i+1}v_{n_P}v_{n_P+1}v_{n_P+2}$ is an induced P_4 with none of the vertices v_{i+1} and v_{n_P+1} having degree not less than $k/2$, a contradiction. Thus assume $v_iv_{n_P} \in E(G)$.

Note that v_i can not be adjacent to v_{n_P+2}. If this is not the case, then, depending on the existence of the edge $v_{i+1}v_{n_P}$, either $\{v_i; v_{n_P}, v_{n_P+2}, v_{i+1}\}$ is an induced claw in G or else $v_{i+1}v_{n_P}v_{n_P+1}v_{n_P+2}$ is an induced path P_4 that does not satisfy the Fan's condition.

From the fact that $v_iv_{n_P+2}$ is not an edge in G and from the condition (iv) for v_i it follows that $v_iv_1 \notin E(G)$. This implies that v_{i+1} is adjacent to v_1, since otherwise the path $v_{i+1}v_iv_{n_P+1}v_1$ is an induced P_4 with $d_G(v_1)$, $d_G(v_i) < k/2$. But now $v_iv_{i+1}v_1v_{n_P+2}$ is an induced P_4 with $d_G(v_1)$, $d_G(v_i) < k/2$, a contradiction. Thus the condition (iii) holds for v_{i+1}.

To show that the condition (iv) is satisfied by v_{i+1}, first suppose that $v_{i+1}v_{n_P} \notin E(G)$. Then v_{i+1} is adjacent to both v_1 and v_{n_P+2} to avoid induced claws $\{v_{n_P+1}; v_{i+1}, v_{n_P}, v_1\}$ and $\{v_{n_P+1}; v_{i+1}, v_{n_P}, v_{n_P+2}\}$ with both v_{i+1} and v_{n_P} having degrees less than $k/2$.

Now suppose that v_{i+1} is adjacent to v_{n_P}. If v_1 is a neighbour of v_{i+1}, then the same is true for v_{n_P+2}, since otherwise $v_{n_P}v_{i+1}v_1v_{n_P+2}$ is an induced P_4 with $d_G(v_1)$, $d_G(v_{n_P}) < k/2$. Similarly, $v_{i+1}v_{n_P+2} \in E(G)$ implies that v_{i+1} is adjacent to v_1, to avoid induced path $v_{n_P}v_{i+1}v_{n_P+2}v_1$. This proves (iv).

Finally, suppose that v_{i+1} is adjacent to some vertex $v \in \{v_{n_P+3}, \ldots, v_m\}$. By Claim 3.4 we can assume that $i+1 < n_P$. If $v_{i+1}v_1 \notin E(G)$, then $v_1v_{n_P+1}v_{i+1}v$ is an induced path P_4, by Claim 3.4. Since the degrees of both v_1 and v_{i+1} are less than $k/2$, this contradicts G belonging to the family $\mathcal{F}(P_4, k)$. Now suppose that v_{i+1} is adjacent to v_1. Then $v_{i+1}v_{n_P} \notin E(G)$ to avoid induced claw $\{v_{i+1}; v_1, v_{n_P}, v\}$. But now $v_{n_P}v_{n_P+1}v_{i+1}v$ is an induced path P_4, by Claim 3.4. This final contradiction shows that the property (v) holds for v_{i+1}. By mathematical induction the claim is true. □

Claim 3.6. *For every $i \in \{1, \ldots, n_P + 1\}$ the neighbourhood $N_G(v_i)$ of the vertex v_i is a subset of the set $\{v_1, v_2, \ldots, v_{n_P+2}\}$.*

Proof. Note that by Claims 3.4 and 3.5 the vertex v_i, with $1 \le i \le n_P + 1$, has no neighbours in the set $\{v_{n_P+3}, \dots, v_m\}$. Thus to prove the claim it suffices to show that v_i is not adjacent to any $v \in V(G) \setminus V(P)$. Clearly, if one of the vertices v_1, v_2 and v_{n_P+1} was adjacent to some vertex $v \notin V(P)$, this would create a path in G longer than P, i.e., one of the paths $vv_1P^+v_m$, $vv_2P^+v_{n_P+1}v_1v_{n_P+2}P^+v_m$ or $vv_{n_P+1}P^-v_1v_{n_P+2}P^+v_m$. Hence, the claim is true for $i \in \{1, 2, n_P + 1\}$.

For a proof by induction assume that the claim holds for the values from the set $\{1, 2, \dots, i\}$, where $2 \le i \le n_P - 1$. It will be shown that this implies the validity of the claim for $i + 1$.

Suppose that there is a vertex $v \in V(G) \setminus V(P)$ adjacent to v_{i+1}. Then v is not adjacent to any of v_i and v_{i+2}, since such an edge would create a path in G longer than P. Recall that $d_G(v_i), d_G(v_{i+2}) < k/2$, by Claim 3.5, and so $\{v_{i+1}; v_i, v, v_{i+2}\}$ can not induce a claw in G. Thus $v_iv_{i+2} \in E(G)$. We observe that if v_{i+1} is not adjacent to some vertex v_k with $1 \le k \le i-1$, then choosing k of largest possible value gives an induced path $v_kv_{k+1}v_{i+1}v$, by the induction hypothesis. This contradicts G being a member of the family $\mathcal{F}(P_4, k)$, by Claim 3.5. Thus v_{i+1} is adjacent to every vertex preceding it on the path P, in particular $v_1v_{i+1} \in E(G)$. But now $vv_{i+1}v_1P^+v_iv_{i+2}P^+v_m$ is a path longer than P, a contradiction.

□

Now it follows from Claim 3.6 that $G - v_{n_P+2}$ is not connected. This contradicts G being 2-connected and completes the proof of this case.

Case 2. $v_1v_{n_P+2} \notin E(G), v_{n_P}v_{n_P+2} \in E(G)$.

We begin the proof of this case with a counterpart of Claim 3.4.

Claim 3.7. *There are no edges between the vertices v_1, v_{n_P}, v_{n_P+1} and the vertices from the set $\{v_{n_P+3}, \dots, v_m\}$.*

Proof. The validity of the claim for v_1 follows immediately from the definition of n_P and the assumptions of this case. For v_{n_P+1} we first observe that $v_{n_P+1}v_{n_P+3} \notin E(G)$, since otherwise the path $P' = v_{n_P+2}v_{n_P}P^-v_1v_{n_P+1}v_{n_P+3}P^+v_m$ is a longest path in G with $n_{P'} \ge n_P + 1$, contradicting the choice of P. With this observation it is easy to see that if $v_{n_P+1}v \in E(G)$ for some vertex $v \in \{v_{n_P+4}, \dots, v_m\}$, then the path $P' = v_{n_P+1}v_1P^+v_{n_P}v_{n_P+2}P^+v_m$ is a longest path with $n_{P'} \ge n_P + 3$. Finally, if v_{n_P} has a neighbour in the set $\{v_{n_P+3}, \dots, v_m\}$, say v, then $v_1v_{n_P+1}v_{n_P}v$ is an induced P_4 in G with $d_G(v_{n_P})$, $d_G(v_1) < k/2$. A contradiction. □

Next we establish some properties of the vertices that preceed v_{n_P} on P.

Claim 3.8. *For $i \in \{1, 2, \dots, n_P - 2\}$ the following holds:*

(i) $d_G(v_{n_P-i}) < k/2$,
(ii) *v_{n_P-i} is adjacent to at least one of the vertices v_1 and v_{n_P+1},*
(iii) *$v_{n_P-i}v_{n_P+2} \in E(G)$ or else v_{n_P-i} is adjacent to v_1,*
(iv) *v_{n_P-i} is not adjacent to any of the vertices from the set $\{v_{n_P+3}, \dots, v_m\}$.*

Proof. We use induction on i. For $i = 1$ it is clear that $d_G(v_{n_P-1}) < k/2$, since the path $v_{n_P-1}P^-v_1v_{n_P+1}v_{n_P}v_{n_P+2}P^+v_m$ is a longest path in G beginning with v_{n_P-1}. Thus (i) holds. Recall that the degrees of both v_1 and v_{n_P} are less than $k/2$, and so the path $v_{n_P-1}v_{n_P}v_{n_P+1}v_1$ can not be an induced one. This implies (ii).

To show (iii) assume that v_{n_P-1} is not adjacent to v_{n_P+2} and suppose $v_1v_{n_P-1} \notin E(G)$. Then v_{n_P-1} is adjacent to v_{n_P+1} to avoid induced path $v_{n_P-1}v_{n_P}v_{n_P+1}v_1$ with $d_G(v_{n_P})$, $d_G(v_1) < k/2$. But now $\{v_{n_P+1}; v_1, v_{n_P-1}, v_{n_P+2}\}$ induces a claw. By (i), this contradicts G belonging to the family $\mathcal{F}(K_{1,3}, k)$.

For the proof of (iv) suppose that v_{n_P-1} has a neighbour, say v, in the set $\{v_{n_P+3}, \ldots, v_m\}$. Then v_{n_P-1} is not adjacent to v_1, since otherwise $\{v_{n_P-1}; v_1, v_{n_P}, v\}$ induces a claw, by Claim 3.7. It follows from (ii) that $v_{n_P-1}v_{n_P+1} \in E(G)$. But now $v_1v_{n_P+1}v_{n_P-1}v$ is an induced path P_4 with $d_G(v_1)$, $d_G(v_{n_P-1}) < k/2$, a contradiction. This proves (iv) for $i = 1$.

Now assume that the claim holds for the values from the set $\{1, 2, \ldots, i\}$, where $1 \leq i \leq n_P - 3$. It will be shown that this implies the validity of the claim for $i + 1$.

By the condition (iii) for v_{n_P-i} there is a longest path in G beginning with v_{n_P-i-1}, namely $v_{n_P-i-1}P^-v_1v_{n_P+1}P^-v_{n_P-i}v_{n_P+2}P^+v_m$ or $v_{n_P-i-1}P^-v_1v_{n_P-i}P^+v_m$. Thus $d_G(v_{n_P-i-1}) < k/2$, proving (i). For the proof of (ii) suppose that v_{n_P-i-1} is not adjacent neither to v_1 nor to v_{n_P+1}. This implies that both v_1 and v_{n_P+1} are neighbours of v_{n_P-i}, since otherwise, by (ii), one of the paths $v_{n_P-i-1}v_{n_P-i}v_1v_{n_P+1}$ and $v_{n_P-i-1}v_{n_P-i}v_{n_P+1}v_1$ would be an induced P_4 in G. Furthermore, v_{n_P} is not adjacent to v_{n_P-i-1}, to avoid induced path $v_{n_P-i-1}v_{n_P}v_{n_P+1}v_1$. It is also not adjacent to v_{n_P-i}, since otherwise $\{v_{n_P-i}; v_1, v_{n_P-i-1}, v_{n_P}\}$ induces a claw. But now $v_{n_P-i-1}v_{n_P-i}v_{n_P+1}v_{n_P}$ is an induced path with four vertices. Since the degrees of the vertices of this path are less than $k/2$, this contradicts G belonging to the family $\mathcal{F}(P_4, k)$ and proves (ii).

Now assume $v_{n_P-i-1}v_{n_P+2} \notin E(G)$ and suppose that v_{n_P-i-1} is not adjacent to v_1. From the condition (ii) for v_{n_P-i-1} it follows that $\{v_{n_P+1}; v_1, v_{n_P-i-1}, v_{n_P+2}\}$ induces a claw. Since the degrees of both v_1 and v_{n_P-i-1} are strictly less than $k/2$, this is a contradiction. Thus (iii) holds.

Finally, suppose that v_{n_P-i-1} is adjacent to some vertex $v \in \{v_{n_P+3}, \ldots, v_m\}$. Claim 3.7 implies that $n_P - i - 1 > 1$. If $v_{n_P-i-1}v_1 \notin E(G)$, then it follows from the condition (ii) and Claim 3.7 that the path $v_1v_{n_P+1}v_{n_P-i-1}v$ is an induced P_4 in G with the degrees of both v_1 and v_{n_P-i-1} being less than $k/2$. Thus $v_{n_P-i-1}v_1 \in E(G)$. This implies that v_{n_P-i-1} is not adjacent to v_{n_P}, since otherwise $\{v_{n_P-i-1}; v_1, v_{n_P}, v\}$ induces a claw, by Claim 3.7. Furthermore, in order to avoid induced path $v_{n_P}v_{n_P+1}v_{n_P-i-1}v$, the vertex v_{n_P-i-1} can not be adjacent to v_{n_P+1}. But now we obtain an induced path $v_{n_P-i-1}v_1v_{n_P+1}v_{n_P}$, a contradiction. By mathematical induction the claim is true. □

Claim 3.9. *For every $i \in \{0, 1, \ldots, n_P\}$ the neighbourhood $N_G(v_{n_P-i+1})$ of the vertex v_{n_P-i+1} is a subset of the set $\{v_1, \ldots, v_{n_P+2}\}$.*

Proof. Note that by Claims 3.7 and 3.8 the vertex v_{n_P-i+1}, with $0 \leq i \leq n_P$, has no neighbours in the set $\{v_{n_P+3}, \ldots, v_m\}$. Thus to prove the claim it suffices to show that v_{n_P-i+1} is not adjacent to any $v \in V(G) \setminus V(P)$. Clearly, if one of the vertices v_1, v_{n_P}

and v_{n_P+1} was adjacent to some vertex v lying outside the path P, this would create a path in G longer than P, i.e., one of the paths $vv_1P^+v_m$, $vv_{n_P}P^-v_1v_{n_P+1}P^+v_m$ or $vv_{n_P+1}v_1P^+v_{n_P}v_{n_P+2}P^+v_m$. Hence, the claim is true for $i \in \{0, 1, n_P\}$.

For a proof by induction assume that the claim holds for the values from the set $\{0, 1, \ldots, i\}$, where $1 \leq i \leq n_P - 2$. It will be shown that this implies the validity of the claim for $i + 1$.

Suppose that there is a vertex $v \in V(G) \setminus V(P)$ adjacent to v_{n_P-i}. Then v is not adjacent to any of v_{n_P-i-1} and v_{n_P-i+1}, to avoid creating a path in G longer than P. Recall that $d_G(v_{n_P-i-1}), d_G(v_{n_P-i+1}) < k/2$, by Claim 3.8 and by the fact that $d_G(v_1) < k/2$, and so $\{v_{n_P-i}; v_{n_P-i-1}, v, v_{n_P-i+1}\}$ can not induce a claw in G. Thus $v_{n_P-i-1}v_{n_P-i+1} \in E(G)$. Next we note that if v_{n_P-i} is not adjacent to some vertex v_k with $n_P - i < k \leq n_P$, then choosing k of smallest possible value gives an induced path $vv_{n_P-i}v_{k-1}v_k$, by the induction hypothesis. This contradicts G being a member of the family $\mathcal{F}(P_4, k)$, by Claim 3.8 and by the fact that $d_G(v_{n_P}) < k/2$. Thus v_{n_P-i} is adjacent to every vertex from the set $\{v_{n_P-i+1}, \ldots, v_{n_P+1}\}$. But now the path $vv_{n_P-i}v_{n_P+1}v_1P^+v_{n_P-i-1}v_{n_P-i+1}P^+v_{n_P}v_{n_P+2}P^+v_m$ is a path longer than P, a contradiction. □

Similarly to the previous case of the proof, now it follows from Claim 3.9 that $G - v_{n_P+2}$ is not connected, a contradiction with the assumption of 2-connectivity of G.

Case 3. $v_1v_{n_P+2} \in E(G)$, $v_{n_P}v_{n_P+2} \in E(G)$.

Recall that the degrees of the vertices v_1, v_{n_P} and v_{n_P+1} are less than $k/2$. Keeping that in mind, we first establish some basic facts regarding the vertex v_{n_P-1}.

Claim 3.10. $d_G(v_{n_P-1}) < k/2$, $v_{n_P-1}v_{n_P+1} \notin E(G)$, $v_{n_P-1}v_1 \in E(G)$.

Proof. Note that under the assumptions of this case the path $v_{n_P-1}P^-v_1v_{n_P+1}v_{n_P}v_{n_P+2}P^+v_m$ is a longest path in G. Thus $d_G(v_{n_P-1}) < k/2$. Now suppose that v_{n_P-1} is adjacent to v_{n_P+1}. Then the path $P' = v_1P^+v_{n_P-1}v_{n_P+1}v_{n_P}v_{n_P+2}P^+v_m$ is a longest path in G with $n_{P'} \geq n_p + 1$, contradicting the choice of P. Hence, $v_{n_P-1}v_{n_P+1} \notin E(G)$. This implies that $v_{n_P-1}v_1 \in E(G)$, since otherwise the path $v_1v_{n_P+1}v_{n_P}v_{n_P-1}$ would be an induced P_4 in G. □

Claim 3.11. *Every neighbour of v_1 in G is adjacent to at least one of the vertices v_{n_P-1}, v_{n_P+1}.*

Proof. If this is not the case, then there exists a neighbour v of v_1 such that $\{v_1; v_{n_P-1}, v_{n_P+1}, v\}$ induces a claw, by Claim 3.10. A contradiction follows from Claim 3.10. □

Now we focus our attention on the edges $v_{n_P-1}v_{n_P+2}$, $v_{n_P-1}v_{n_P+3}$ and $v_{n_P+1}v_{n_P+3}$. We begin with the following observation.

Claim 3.12. *v_{n_P+3} is adjacent to exactly one of the vertices v_{n_P-1} and v_{n_P+1}.*

Proof. Suppose the contrary. If the vertex v_{n_P+3} is not adjacent to any of the vertices v_{n_P-1}, v_{n_P+1}, then it follows from Claim 3.11 that $v_1v_{n_P+3} \notin E(G)$. Now, depending on the existence of the edge $v_{n_P}v_{n_P+3}$, we obtain induced path $v_{n_P+3}v_{n_P}v_{n_P+1}v_1$ or induced claw $\{v_{n_P+2}; v_{n_P}, v_1, v_{n_P+3}\}$, a contradiction.

If both $v_{n_P+3}v_{n_P-1}$ and $v_{n_P+3}v_{n_P+1}$ are edges in G, then the path $P' = v_{n_P-1}P^-v_1v_{n_P+2}v_{n_P}v_{n_P+1}v_{n_P+3}P^+v_m$ is a longest path in G with $n_{P'} \geq n_P + 2$, by Claim 3.10. This contradicts the choice of P. □

Claim 3.13. *$v_{n_P-1}v_{n_P+3}$ is not an edge in G.*

Proof. Suppose that the opposite holds. If $v_{n_P-1}v_{n_P+2}$ is not an edge in G, then the path $P' = v_{n_P-1}P^-v_1v_{n_P+1}v_{n_P}v_{n_P+2}v_{n_P+3}P^+v_m$ is a longest path in G with $n_{P'} \geq n_P + 2$, contradicting the choice of P. Thus $v_{n_P-1}v_{n_P+2} \in E(G)$.

It follows from Claim 3.12 that v_{n_P+1} is not adjacent to v_{n_P+3}. Since $d_G(v_{n_P+1}) < k/2$ and, by Claim 3.10, $d_G(v_{n_P-1}) < k/2$, the path $v_{n_P+1}v_{n_P}v_{n_P-1}v_{n_P+3}$ can not be an induced one. Thus it follows from Claim 3.10 that $v_{n_P}v_{n_P+3}$ is an edge in G. Now to avoid induced path $v_1v_{n_P+1}v_{n_P}v_{n_P+3}$, $v_1v_{n_P+3} \in E(G)$. But then the path $P' = v_1P^+v_{n_P-1}v_{n_P+2}v_{n_P+1}v_{n_P}v_{n_P+3}P^+v_m$ is a longest path in G with $n_{P'} \geq n_p + 2$, a contradiction. □

From Claims 3.12 and 3.13 it follows that the vertex v_{n_P+3} is not adjacent to v_{n_P-1} and that it is adjacent to v_{n_P+1}. Next we observe that to avoid induced path $v_{n_P-1}v_{n_P}v_{n_P+1}v_{n_P+3}$ the vertex v_{n_P+3} is adjacent to v_{n_P}, by Claim 3.10. It follows that v_{n_P+3} is adjacent also to v_1, since otherwise the path $v_1v_{n_P-1}v_{n_P}v_{n_P+3}$ is an induced one, also by Claim 3.10. But now, depending on the existence of the edge $v_{n_P-1}v_{n_P+2}$, one of the paths $P' = v_1P^+v_{n_P-1}v_{n_P+2}P^-v_{n_P}v_{n_P+3}P^+v_m$ and $P'' = v_{n_P-1}P^-v_1v_{n_P+1}v_{n_P+2}v_{n_P}v_{n_P+3}P^+v_m$ is a longest path in G. Since $n_{P'} \geq n_P + 2$ and $n_{P''} \geq n_P + 1$, this contradicts the choice of P. This final contradiction completes the proof of this case and shows that there exists a longest path in G with end vertex v_m and with the other end vertex of degree at least $k/2$.

In the above argument, each longest path considered has v_m as one of the end vertices. Thus, since one of the end vertices of P has degree $\geq k/2$, it could have been initially assumed that P is a longest path with $d_G(v_m) \geq k/2$ and with n_P of largest possible value. The above argument then shows that there exists a longest path P with both endvertices of degree $\geq k/2$. This contradiction with Theorem 3.1 completes the proof of the theorem. □

4. FAN'S CONDITION AND PANCYLICITY

4.1. PRELIMINARIES

Lemma 4.1 (Benhocine and Wojda [3]). *If a graph G of order $n \geq 4$ has a cycle C of length $n - 1$, such that the vertex not in $V(C)$ has degree at least $n/2$, then G is pancyclic.*

The next four lemmas provide a description of the cycle structure of hamiltonian graphs with two vertices that lie close (i.e., with distance one or two along the cycle) to each other on some hamiltonian cycle and have large degree sum.

Lemma 4.2 (Bondy [5]). *Let G be a graph of order n with a hamiltonian cycle C. If there exist two vertices x, $y \in V(G)$ such that $d_C(x,y) = 1$ and $d_G(x) + d_G(y) \geq n+1$, then G is pancyclic.*

Lemma 4.3 (Schmeichel and Hakimi [20]). *Let G be a graph of order n with a hamiltonian cycle $C = v_1v_2 \dots v_nv_1$. If $d_G(v_1) + d_G(v_n) \geq n$, then G is pancyclic unless G is bipartite or else G is missing only $(n-1)$-cycles.*

Furthermore, when G is missing only $(n-1)$-cycles and $d_G(v_1) = d_G(v_2) = n/2$, then the adjacency structure near v_1 and v_2 is the following: the path $v_{n-2}v_{n-1}v_nv_1v_2v_3$ is an induced one, and v_nv_{n-3}, v_nv_{n-4}, v_1v_4, v_1v_5 are edges in G.

Lemma 4.4 (Ferrara, Jacobson, and Harris [13]). *Let G be a graph of order n with a hamiltonian cycle C. If there exist two vertices x, $y \in V(G)$ such that $d_C(x,y) = 2$ and $d_G(x) + d_G(y) \geq n+1$, then G is pancyclic.*

Lemma 4.5 (Han [16]). *Let G be a graph of order n with a hamiltonian cycle C. If there exist two non-adjacent vertices x, $y \in V(G)$ such that $d_C(x,y) = 2$ and $d_G(x) + d_G(y) \geq n$, then G is pancyclic, unless G is bipartite or else G is missing only the $(n-1)$-cycle, or the 3-cycle.*

4.2. PROOF OF THEOREM 1.9

For the convenience of the reader we restate Theorem 1.9 below.

Theorem 1.9. *Let G be a 2-connected graph of order $n \geq 3$. If $G \in \mathcal{F}(\{K_{1,3}, P_4\}, n)$, then G is pancyclic unless $n = 4r$, $r \geq 2$ and G is F_{4r}, or n is even and $G = K_{n/2,n/2}$ or else $n \geq 6$ and $G = K_{n/2,n/2} - e$.*

We first prove three auxiliary lemmas that deal with the exceptional non-pancyclic graphs and establish the existence of short cycles in a graph satisfying the assumptions of Theorem 1.9.

Lemma 4.6. *Let G be a 2-connected, bipartite graph of order $n \geq 3$. If $G \in \mathcal{F}(\{K_{1,3}, P_4\}, n)$, then $G = K_{n/2,n/2}$ or else $n \geq 6$ and $G = K_{n/2,n/2} - e$.*

Proof. First suppose that G is $\{K_{1,3}, P_4\}$-free. Then it follows from Theorem 1.2 that G is a cycle. Since there are no induced paths with four vertices in G, G is a cycle $K_{2,2}$.

Now assume that G contains an induced claw or an induced path P_4. Let (X, Y) be a bipartition of $V(G)$. It follows from the assumptions that there is a vertex, say u, in G with $d_G(u) \geq n/2$. Clearly, if $|V(G)| = 4$, then G is isomorphic to $K_{2,2}$. Thus assume $|V(G)| \geq 6$. Without loss of generality let X be the set of bipartition containing u. It follows that $|Y| \geq n/2 \geq 3$. Note that since $G \in \mathcal{F}(K_{1,3}, n)$ and u together with any three of its neighbours induce a claw, at most one neighbour of u has degree less than $n/2$. This implies $|X| = |Y| = n/2$. By the symmetry, at most one vertex in X might have less neighbours than $n/2$. Let $x \in X$ and $y \in Y$ be those only vertices in G, the degrees of which are not necessarily equal to $n/2$. Clearly, every vertex of Y other than y is adjacent to x and every vertex from X other than x is adjacent to y. Thus, depending on the existence of the edge xy in G, G is isomorphic either to $K_{n/2,n/2}$ or else to $K_{n/2,n/2} - e$. □

Lemma 4.7. *Let G be a 2-connected, non-bipartite graph of order n. If $G \in \mathcal{F}(\{K_{1,3}, P_4\}, n)$ and there are no cycles of length $n-1$ in G, then G is isomorphic to F_{4r}, with $r > 2$.*

Proof. Suppose that G is $\{K_{1,3}, P_4\}$-free. Similarly to the previous Lemma, this implies that G is a cycle $K_{2,2}$, by Theorem 1.2. This contradicts the assumption of G not being bipartite. Hence, we can assume that G contains an induced claw or an induced path P_4, and so there are at least two heavy vertices in G.

Note that by Theorem 1.8 G is hamiltonian. It is easy to check that if G has no more than five vertices, then it is pancyclic. Thus we assume $|V(G)| \geq 6$. Let $C = v_0 \ldots v_{n-1} v_0$ be a hamiltonian cycle in G. Clearly, under the assumptions of the Lemma there are no edges of the form $v_i v_{i+2}$ in G. In the following any arithmetic involving the subscripts of the vertices of C is modulo n. We begin the proof with an observation regarding heavy vertices of G.

Claim 4.8. *If v_i is a heavy vertex in G, then at least one of the vertices v_{i-1} and v_{i+1} is also heavy.*

Proof. Suppose to the contrary that neither v_{i-1} nor v_{i+1} is heavy. Since $G \in \mathcal{F}(P_4, n)$, this implies that none of the paths $v_{i-2}v_{i-1}v_i v_{i+1}$ and $v_{i-1}v_i v_{i+1}v_{i+2}$ can be an induced one. Since there are no cycles of length $n-1$ in G, $v_{i-2}v_i$, $v_{i-1}v_{i+1}$, $v_i v_{i+2} \notin E(G)$, implying that $v_{i-2}v_{i+1}$ and $v_{i-1}v_{i+2}$ are edges in G. Now consider the path $P = v_{i+3}C^+ v_{i-3}$. Clearly, $d_P(v_i) \geq n/2 - 2 = (|V(P)| + 1)/2$. If v_i is adjacent to two consecutive vertices of the path, say v_k and v_{k+1}, then the cycle $v_{i+1}C^+ v_k v_i v_{k+1} C^+ v_{i-2} v_{i+1}$ is a cycle of length $n-1$, a contradiction. This implies that $|V(P)|$ is odd and that the neighbourhood of v_i in P is $N_P(v_i) = \{v_{i+3}, v_{i+5}, \ldots, v_{i-5}, v_{i-3}\}$. Clearly, if $v_{i-1}v_{i+3} \in E(G)$, then there is a cycle of length $n-1$ in G, namely $v_{i+1}v_{i+2}v_{i-1}v_{i+3}C^+ v_{i-2}v_{i+1}$. Thus $v_{i-1}v_{i+3} \notin E(G)$. But now $\{v_i; v_{i-1}, v_{i+1}, v_{i+3}\}$ induces a claw in G. Since neither v_{i-1} nor v_{i+1} is heavy, this contradicts G being a member of the family $\mathcal{F}(K_{1,3}, n)$. □

Claim 4.9. *If v_i is a heavy vertex in G, then $d_G(v_i) = n/2$.*

Proof. By Claim 4.8, assume that both v_i and v_{i+1} are heavy. If the degree of v_i is strictly greater than $n/2$, then $d_G(v_i) + d_G(v_{i+1}) \geq n + 1$ and so G is pancyclic by Lemma 4.2. This contradicts the assumption of G missing the $(n-1)$-cycle. □

Claim 4.10. *If v_i and v_{i+1} are heavy vertices in G, then none of the vertices v_{i-2}, v_{i-1}, v_{i+2} and v_{i+3} is heavy and the vertices v_{i+4} and v_{i+5} are both heavy. Furthermore, the path $v_{i-2}v_{i-1}v_i v_{i+1}v_{i+2}v_{i+3}$ is an induced one, and $v_i v_{i-3}$, $v_i v_{i-4}$, $v_{i+1}v_{i+4}$, $v_{i+1}v_{i+5}$ are edges in G.*

Proof. Since G is missing the $(n-1)$-cycle, it is not bipartite and, by Claim 4.9, the degrees of both v_i and v_{i+1} are equal to $n/2$, it follows from Lemma 4.3 that $v_{i-2}v_{i-1}v_iv_{i+1}v_{i+2}v_{i+3}$ is an induced path P_6 in G and that v_i is adjacent to v_{i-3} and v_{i-4}, and v_{i+1} is adjacent to v_{i+4} and v_{i+5}. Suppose v_{i-1} is heavy. Then applying Lemma 4.3 to the pair v_{i-1}, v_i leads to a contradiction with the adjacency structure it provides, since $v_iv_{i+3} \notin E(G)$. Similar contradiction arises if we suppose that v_{i+2} is heavy and apply Lemma 4.3 to the pair v_{i+1}, v_{i+2}. Thus neither v_{i-1} nor v_{i+2} is heavy. Now suppose that v_{i-2} is heavy. From the previous observation and from Claim 4.8 it follows that v_{i-3} is also heavy. Since $v_{i-2}v_{i+1} \notin E(G)$ this again leads to a contradiction with the structure described by Lemma 4.3, when applied to the pair v_{i-2}, v_{i-3}. Similar contradiction is obtained if one assumes that v_{i+3} is heavy. Thus the first part of the Claim holds.

Now it will be shown that v_{i+4} is heavy. Suppose to the contrary that $d_G(v_{i+4}) < n/2$. Since the degree of v_{i+2} is also less than $n/2$, the path $v_{i+2}v_{i+3}v_{i+4}v_{i+5}$ can not be an induced one. This implies that $v_{i+2}v_{i+5} \in E(G)$. Similarly, to avoid induced claw $\{v_{i+1}; v_i, v_{i+2}, v_{i+4}\}$, v_i is adjacent to v_{i+4}. But these two edges create in G a cycle of length $n-1$, namely $v_{i+2}v_{i+5}C^+v_iv_{i+4}v_{i+3}v_{i+2}$, a contradiction. Thus v_{i+4} is heavy. Since $d_G(v_{i+3}) < n/2$, the heaviness of v_{i+5} follows from Claim 4.8. □

Since there is a heavy vertex in G, we can assume without loss of generality that the vertices v_0 and v_1 are heavy, by Claim 4.8. It follows from Claim 4.10 that v_4 and v_5 are also heavy. Applying Claim 4.10 to the pair v_4, v_5 we obtain the heaviness of the vertices v_8 and v_9, and so on, i.e., every vertex v of G with $d_C(v_0, v) \in \{4k, 4k+1\}$ for some non-negative integer k, is heavy. Similarly, every $v \in V(G)$ with $d_C(v_0, v) \in \{4k+2, 4k+3\}$ is not heavy. Thus the number of vertices of G is divisible by four. Let $n = 4r$, with $r \geq 2$. Then the set of heavy vertices of G is $\{v_0, v_1, v_4, v_5, \ldots, v_{4r-4}, v_{4r-3}\}$ and the remaining vertices are not heavy. With the following claim we establish the existence of a perfect matching between the sets of heavy and non-heavy vertices of G.

Claim 4.11. *Every heavy vertex of G is adjacent to exactly one non-heavy vertex.*

Proof. Suppose the contrary. Let v_i be a heavy vertex of G with at least two non-heavy neighbours. From Claims 4.8, 4.9 and 4.10 it follows that at least one of these neighbours, say v_k, satisfies $d_C(v_i, v_k) \geq 5$. Claims 4.8 and 4.10 imply that exactly one of the vertices v_{i-1} and v_{i+1} is also not heavy. Thus $\{v_i; v_{i-1}, v_{i+1}, v_k\}$ can not induce a claw, since $G \in \mathcal{F}(K_{1,3}, n)$. Since there are no cycles of length $n-1$ in G, it follows that v_kv_{i-1} or v_kv_{i+1} is an edge in G.

Depending on which of the vertices v_{k-1} and v_{k+1} is heavy, either $v_{k-1}v_{k+2}$ or else $v_{k-2}v_{k+1}$ is an edge in G, by Claim 4.10. Denote this edge w_1w_2. This, together with the previous observations, implies that either $v_iC^+w_1w_2C^+v_{i-1}v_kv_i$ or $v_iC^-w_2w_1C^-v_{i+1}v_kv_i$ is a cycle in G. Since the length of this cycle is $n-1$, this contradicts the assumption of G missing the $(n-1)$-cycle. □

Claim 4.11 implies that, since there are $2r$ heavy vertices and $2r$ non-heavy vertices in G, in order for the heavy vertices to be indeed heavy, every two of them are adjacent. Thus the heavy vertices induce a clique in G and there is a perfect matching between this clique and the set of non-heavy vertices, since every heavy vertex has a non-heavy neighbour that lies next to it on the cycle C. Clearly, every non-heavy vertex v has a unique non-heavy neighbour u with $d_C(v,u) = 1$. To complete the proof it suffices to show that every non-heavy vertex is adjacent to exactly one non-heavy vertex.

Suppose this is not the case. Let v_k be a non-heavy vertex with v_{k+1} being also not heavy. Suppose v_k has a neighbour in a pair of non-heavy vertices $\{v_m, v_{m+1}\}$. From Claim 4.10 it follows that $d_C(v_k, v_m) \geq 7$. Since the heavy vertices of G induce a clique, either $v_k v_m v_{m-1} v_{m+2} C^+ v_{k-1} v_{m-2} C^- v_k$ or $v_k v_{m+1} C^+ v_{k-1} v_{m-1} C^- v_k$ is a cycle in G. The length of this cycle is $n-1$. This final contradiction completes the proof of Lemma 4.7. □

Lemma 4.12. *Let G be a 2-connected graph of order $n \geq 3$ and let u and v be heavy vertices in G. If $G \in \mathcal{F}(\{K_{1,3}, P_4\}, n)$, then*

(i) *if G is not bipartite, then G contains a triangle,*
(ii) *there is a cycle of length four in G.*

Proof. For the proof of (i) assume that G is not bipartite. As the statement is easy to verify for $n \leq 4$, we further assume that $n \geq 5$. Clearly, if there is an edge in the subgraph of G induced by the neighbourhood $N_G(u)$ of u, then there is a triangle in G. Suppose that $G[N_G(u)]$ is edgeless. Since $G \in \mathcal{F}(K_{1,3}, n)$, it follows that at most one of the neighbours of u is not heavy. Observe that G is hamiltonian by Theorem 1.8. Let $C = v_1 \ldots v_n v_1$ be a hamiltonian cycle in G with $v_1 = u$. Since at least one of the vertices v_2 and v_n is heavy, Lemma 4.3 implies that there is a triangle in G.

Now it will be shown that (ii) holds. Clearly, if u and v have at least two common neighbours, then G contains C_4. Thus suppose they have at most one common neighbour. Since both u and v are heavy, it follows that $uv \in E(G)$. If u and v have no common neighbours, then $V(G) = A \cup B \cup \{u,v\}$, where $N_G(u) = A \cup \{v\}$, $N_G(v) = B \cup \{u\}$ and $A \cap B = \emptyset$. Since G is 2-connected, there is an edge ab in G for some $a \in A$ and $b \in B$. This edge creates the cycle $uabvu$ of length four.

Assume that there is exactly one common neighbour of u and v in G, say w. Let $N_G(u) = A \cup \{v,w\}$ and $N_G(v) = B \cup \{u,w\}$, where $A \cap B = \emptyset$. Furthermore, assume that $N_G[w] \cap (A \cup B) = \emptyset$ and that there are no edges between the sets A and B, since otherwise there is a cycle of length four in G. From the 2-connectivity of G it follows that there is a path connecting A and B that is disjoint with the vertices u and v. Hence, there is a vertex in $V(G)$ that does not belong to $A \cup B \cup \{u,v,w\}$. This implies that

$$|A| + |B| + 3 < n.$$

On the other hand, since u and v are heavy, both A and B contain at least $n/2 - 2$ vertices. Thus

$$|A| + |B| + 4 \geq n.$$

Hence, $|A| + |B| + 4 = n$, $|A| = |B| = n/2 - 2$, and there is exactly one vertex, say x, in the set $V(G) \setminus (A \cup B \cup \{u,v,w\})$. In order to create a path between A and B with the set of vertices disjoint with both u and v, the vertex x is as adjacent to some $a \in A$ and some $b \in B$. Clearly, none of the vertices from $A \cup B$ is heavy. Since

$G \in \mathcal{F}(P_4, n)$, it follows that neither the path $uaxb$ nor the path $vbxa$ can be induced in G. Thus xu, $xv \in E(G)$. These edges create a cycle of demanded length, namely $uxvwu$. □

Now we are ready to prove Theorem 1.9.

Proof of Theorem 1.9. Let G be a graph satisfying the assumptions of the Theorem. Assume that G is not one of $K_{n/2,n/2}$, $K_{n/2,n/2} - e$ and F_{4r}. Lemmas 4.6 and 4.7 imply that G is neither bipartite nor missing the $(n-1)$-cycle. Furthermore, there is a hamiltonian cycle in G, by Theorem 1.8.

Toward a contradiction, suppose that G is not pancyclic. Then it follows from Theorem 1.2 that G is not $\{K_{1,3}, P_4\}$-free and so there are at least two heavy vertices in G. The following Claim gathers the pieces of information regarding cycles in G that we have obtained so far. □

Claim 4.13. *G contains cycles of lengths three, four, $n-1$ and n.*

Proof. The existence of the long cycles is clear. The fact that there are cycles C_3 and C_4 in G follows from Lemma 4.12. □

By Claim 4.13, if $n \leq 6$, then G is pancyclic. So we assume that $n \geq 7$.

Claim 4.14. *If x, $y \in V(G)$ are heavy in G, then for every hamiltonian cycle C in G holds $d_C(x,y) \geq 2$. Furthermore, if $d_C(x,y) = 2$, then $d_G(x) = d_G(y) = n/2$ and $xy \in E(G)$.*

Proof. Clearly, if $d_C(x,y) = 1$, then G is pancyclic by Lemma 4.3. If $d_C(x,y) = 2$ and the degree of at least one of x and y is strictly greater than $n/2$, then G is pancyclic by Lemma 4.4. Finally, if $d_C(x,y) = 2$ and x is not adjacent to y, pancyclicity of G follows from Claim 4.13 and Lemma 4.5. □

Let u be a vertex in G with $d_G(u) \geq n/2$.

Case 1. $G - u$ is not 2-connected.

Under the assumptions of this case there is a vertex in G, say v, such that $G - \{u, v\}$ is not connected. Since G is hamiltonian, we can set $C = uy_1 \ldots y_{h_2} v x_{h_1} \ldots x_1 u$ to be a hamiltonian cycle with $H_1 = \{x_1, \ldots, x_{h_1}\}$ and $H_2 = \{y_1, \ldots, y_{h_2}\}$ being the components of $G - \{u, v\}$. The following simple observation is crucial for the further reasoning.

Claim 4.15. *There are no heavy vertices in at least one of the sets H_1 and H_2.*

Proof. Suppose this is not the case. Then $h_1 = h_2 = (n-2)/2$ and there are vertices $x \in H_1$, $y \in H_2$ such that $N_G(x) = H_1 \cup \{u, v\}$ and $N_G(y) = H_2 \cup \{u, v\}$. Thus $uyvxu$ is a cycle of length four in G. To this cycle we can append all vertices from H_2, one-by-one, creating cycles uy_1yvxu, uy_1y_2yvxu, …, $uC^+yy_{h_2}vxu$, $uC^+yy_{h_2-1}y_{h_2}vxu$, …, uC^+vxu. The vertices from H_1 can be appended to the longest of these cycles in a similar manner. In this way we obtain $[4, n]$-cycles in G. Since G contains a triangle, by Claim 4.13, it is pancyclic. A contradiction. □

It follows from Claim 4.15 that for the rest of the proof of this case we may assume a lack of heavy vertices in H_1. We also assume that y_1 is not heavy, since the opposite yields a contradiction with Claim 4.14.

The next three claims describe the neighbourhood $N_G(u)$ of the vertex u.

Claim 4.16. $N_{H_2}[u] \subset N_G[y_1]$.

Proof. Otherwise u is adjacent to some vertex $y \in H_2 \setminus N_G[y_1]$. Then $\{u; y, y_1, x_1\}$ induces a claw in G. Since neither x_1 nor y_1 is heavy, this contradicts G being a member of the family $\mathcal{F}(K_{1,3}, n)$. □

Claim 4.17. $N_{H_1}(u) = H_1$ *and* $N_{H_1}[u]$ *induces a clique.*

Proof. Since the statement is clearly true for $h_1 = 1$, assume that there are at least two vertices in H_1. Suppose that there is a vertex $x_i \in H_1$ such that $ux_i \notin E(G)$. Choose minimal i with this property. Then the path $y_1ux_{i-1}x_i$ is induced in G. Since there are no heavy vertices in H_1 and y_1 is not heavy, this is a contradiction with G belonging to the family $\mathcal{F}(P_4, n)$.

Now suppose that there are two nonadjacent neighbours of u in H_1, say x and x'. Then $\{u; x, x', y_1\}$ induces a claw, with none of its endvertices being heavy. Since $G \in \mathcal{F}(K_{1,3}, n)$, this is a contradiction. □

Claim 4.18. $N_{H_2}(u) \neq H_2$.

Proof. Suppose the contrary. Then $uy_{h_2}vx_{h_1}u$ is a cycle in G, by Claim 4.17. To this cycle we can append the vertices from H_1, one-by-one, also by Claim 4.17. To the longest of the cycles obtained the vertices from H_2 can be appended in a similar way. With this procedure we obtain $[4, n]$-cycles in G. The pancyclicity of G follows from Claim 4.13. □

It follows from Claim 4.18 that there is a vertex y_k in $N_{H_2}(u)$ such that $y_{k+1} \in H_2$ and u is not adjacent to y_{k+1}. Choose minimal k satisfying these conditions.

Claim 4.19. y_k *is heavy. In consequence,* $k \geq 2$*, both* y_{k-1} *and* y_{k+1} *are not heavy, and* $y_{k-1}y_{k+1} \notin E(G)$.

Proof. Clearly, the path $x_1uy_ky_{k+1}$ is an induced one. Since $G \in \mathcal{F}(P_4, n)$ and x_1 is not heavy, it follows that y_k is heavy. Now Claim 4.14 implies that $k \geq 2$ and that neither y_{k-1} nor y_{k+1} is heavy. The fact that y_{k-1} is not adjacent to y_{k+1} follows from Lemma 4.1. □

Claim 4.20. *There are* $[n - h_1, n]$*-cycles in* G.

Proof. Claim 4.19 implies that u is adjacent to y_2. Thus $C' = uy_2C^+x_{h_1}u$ is a cycle of length $n - h_1$, by Claim 4.17. Since u is adjacent to every vertex of H_1, all these vertices can be appended to C', one-by-one, creating cycles of demanded lengths. □

Claim 4.21. $uv \in E(G)$.

Proof. Suppose the contrary. Then $y_1v \in E(G)$ to avoid induced path $y_1ux_{h_1}v$ with neither y_1 nor x_{h_1} being heavy. Now it follows from Claims 4.16 and 4.17 that $d_G(y_1) \geq n/2 - h_1 + 1$. Set $G' = G - \{x_1, \dots, x_{h_1-1}\}$ if $h_1 > 1$ or $G' = G$ otherwise. Since $y_1y_k \in E(G)$, by Claim 4.16, G' is hamiltonian, with $C' = uy_{k-1}C^-y_1y_kC^+x_{h_1}u$ being its hamiltonian cycle. Note that $d_{G'}(y_1) + d_{G'}(y_k) \geq n/2 - h_1 + 1 + n/2 = |G'|$, by Claim 4.19, and that uy_1y_2u is a triangle in G'. Thus it follows from Lemma 4.3 that G' is either pancyclic or else missing only $(n-h_1)$-cycle. In either case Claim 4.20 implies pancyclicity of G. □

The next claim provides a full description of the neighbourhood of the vertex y_1.

Claim 4.22. $N_G[y_1] = N_{H_2}[u]$.

Proof. Suppose that the Claim is not true. Then it follows from Claims 4.16, 4.17 and 4.21 that $d_G(y_1) \geq n/2 - h_1 - 1 + 1 = n/2 - h_1$. By Claim 4.21 the cycle $uy_{k-1}C^-y_1y_kC^+vu$ is a hamiltonian cycle in $G' = G - H_1$. Since $d_{G'}(y_1) + d_{G'}(y_k) \geq |G'|$ and uy_1y_2u is a triangle in G', it follows from Lemma 4.3 that G' is either pancyclic or else missing only $(n - h_1 - 1)$-cycle. By Claim 4.20, the same is true for G. Since uy_2C^+vu is a cycle of length $n - h_1 - 1$, G is pancyclic. □

Claim 4.23. *y_{h_2} is adjacent neither to u nor to y_1.*

Proof. Suppose this is not the case. Then, by Claim 4.22, y_{h_2} is adjacent to both u and y_1. If $vy_k \notin E(G)$, then set $G' = G - (H_1 \cup \{v\})$. Note that the cycle $uy_{k-1}C^-y_1y_kC^+y_{h_2}u$ is a hamiltonian cycle in G' and $uy_2C^+y_{h_2}u$ is a cycle of length $|G'| - 1$. Since $d_{G'}(y_1) + d_{G'}(y_k) \geq |G'|$, Lemma 4.3 implies that G' is pancyclic. Together with Claim 4.20 this implies pancyclicity of G.

Now assume $vy_k \in E(G)$. If $vy_{k-1} \in E(G)$, then consider $G' = G - H_1$. Again, G' is a graph with both $|G'|$- and $(|G'| - 1)$-cycles, namely, $vy_{k-1}C^-uy_kC^+v$ and uy_2C^+vu. Clearly, $d_{G'}(u) + d_{G'}(y_k) \geq |G'|$, by Claim 4.19, and G' is not bipartite. Thus G' is pancyclic, by Lemma 4.3, and so G is pancyclic, by Claim 4.20.

Hence, v is adjacent to y_k and not adjacent to y_{k-1}. Now to avoid $\{y_k;\, y_{k-1}, v, y_{k+1}\}$ inducing a claw with neither y_{k-1} nor y_{k+1} being heavy, v is adjacent to y_{k+1}. But then $y_{k+1}vC^+uy_kC^-y_1y_{h_2}C^-y_{k+1}$ is a hamiltonian cycle in G with both u and y_k being heavy. This contradicts Claim 4.14. □

Observe that, by Claims 4.21, 4.22 and 4.23, the path $y_1uvy_{h_2}$ is an induced one. Since y_1 is not heavy, it follows that v is heavy. In consequence, y_{h_2} is not heavy, by Claim 4.14.

Claim 4.24. *y_{h_2} is adjacent to both y_k and y_{k+1}.*

Proof. We first observe that $vy_{h_2-1} \in E(G)$. Clearly, otherwise the path $x_{h_1}vy_{h_2}y_{h_2-1}$ is an induced one. Since neither x_{h_1} nor y_{h_2} is heavy, this contradicts G belonging to the family $\mathcal{F}(P_4, n)$.

Now suppose that y_{h_2} is not adjacent to y_k. Set $G' = G - (H_1 \cup y_{h_2})$. It follows from Claims 4.19, 4.21, 4.22 and 4.23 that $d_{G'}(y_1) + d_{G'}(y_k) \geq |G'|$. Since

$y_1y_kC^+y_{h_2-1}vuy_{k-1}C^-y_1$ is a hamiltonian cycle and $uy_2C^+y_{h_2-1}vu$ is a cycle of length $|G'|-1$ in G', Lemma 4.3 implies pancyclicity of G'. Thus there are $[3, n-h_1-1]$-cycles in G and so G is pancyclic, by Claim 4.20.

Hence, $y_{h_2}y_k \in E(G)$. Suppose $y_{h_2}y_{k+1} \notin E(G)$. It follows from Claims 4.22 and 4.23 and the choice of k that $\{y_k; y_1, y_{h_2}, y_{k+1}\}$ induces a claw. Since none of the endvertices of this claw is heavy, this is a contradiction. Thus y_{h_2} is adjacent to y_{k+1}. □

Claim 4.25. *v is adjacent to every vertex from the set $\{y_k, y_{k+1}, \dots, y_{h_2}\}$.*

Proof. Suppose that the above statement is not true. Then there exists a vertex $y_m \in N_{H_2}(v)$ such that $y_{m-1} \in \{y_k, y_{k+1}, \dots, y_{h_2-1}\}$ and $vy_{m-1} \notin E(G)$. Choose maximal m satisfying these conditions. Note that, since v is heavy and $G-v$ is not 2-connected, we could change u with v in the beginning of the proof of this case and repeat the reasoning conducted so far, obtaining in particular that $N_{H_1}(v) = H_1$, and $N_{H_2}(v) \neq H_2$. Then y_m would be an equivalent of y_k for u, and thus we could show that y_m is heavy, and so on. Finally, similarly to Claim 4.24, i.e., the existence of the edge $y_{h_2}y_{k+1}$, by symmetry we would obtain the existence of the edge y_1y_{m-1}. But then the cycle $uy_kC^-y_1y_{m-1}C^-y_{k+1}y_{h_2}C^-y_mvC^+u$ is a hamiltonian cycle in G with $d_G(u) + d_G(y_k) \geq n$, a contradiction with Claim 4.14. □

Now it follows from Claim 4.25 that uy_kvu is a triangle in G. Since $\{y_1, \dots, y_{k-1}\} \subset N_G(u)$ and $\{y_{k+1}, \dots, y_{h_2}\} \subset N_G(v)$, we can append the vertices from H_2 to this triangle, one-by-one, obtaining cycles of all lengths from three up to $h_2 + 2 = n - h_1$. Since there are also $[n-h_1, n]$-cycles in G, by Claim 4.20, this implies that G is pancyclic. This final contradiction completes the proof of this case.

Case 2. $G-u$ is 2-connected.

Set $G' = G-u$. Note that G' is not hamiltonian, by Lemma 4.1, and that for every heavy vertex v of G other than u we have $d_{G'}(v) \geq n/2-1 = (n-2)/2$. Thus $G' \in \mathcal{F}(\{K_{1,3}, P_4\}, n-2)$. It follows from Theorem 1.8 that there is a cycle of length $n-2$ in G', say $C' = w_0w_1 \dots w_{n-3}w_0$. In the following any arithmetic involving the subscripts of the vertices of C' is modulo $n-2$.

Let x be the vertex of G' such that $x \notin V(C')$. Lemma 4.1 implies that $d_{G'}(x) \leq (n-2)/2$. Next we will show that this inequality is in fact strict.

Claim 4.26. *$d_{G'}(x) < (n-2)/2$.*

Proof. Suppose that the above statement is not true, i.e., that $d_{G'}(x) = (n-2)/2$. Since G' is not hamiltonian, we can assume $N_{C'}(x) = \{w_0, w_2, \dots, w_{n-4}\}$. It is not difficult to see, that if u is joined by an edge with two consecutive vertices of C', then G is pancyclic. Thus

$$n/2 \leq d_G(u) = d_{C'}(u) + e(u,x) \leq (n-2)/2 + 1 = n/2,$$

implying that $ux \in E(G)$ and u is joined to either each vertex of the set $\{w_0, w_2, \dots, w_{n-4}\}$ or else to each vertex of the set $\{w_1, w_3, \dots, w_{n-3}\}$. If the first case occurs, then G is clearly pancyclic. Thus assume the latter is true. Since G is not bipartite, there is a chord in C' joining two vertices whose indices have the same parity. One can easily check that G is pancyclic. □

Claim 4.27. *$ux \in E(G)$ and $d_G(u) = n/2$.*

Proof. If at least one of the above conditions is not satisfied, then $d_{C'}(u) \geq (n-1)/2$, implying pancyclicity of $G - x$, by Lemma 4.1, and, in consequence, pancyclicity of G. □

Fix k for which there are no k-cycles in G. It follows from Claim 4.13 and the existence of C' that $k \in \{5, 6, \ldots, n-3\}$. Furthermore, for every i from the set $\{0, 1, \ldots, n-3\}$ we have $e(u, w_i) + e(u, w_{i+k-2}) \leq 1$, since otherwise $uw_i{C'}^{+}w_{i+k-2}u$ is a cycle C_k. This implies, together with Claim 4.27, that

$$n - 2 \leq 2d_{C'}(u) = \sum_{i=0}^{n-3}[e(u, w_i) + e(u, w_{i+k-2})] \leq n - 2.$$

Thus $d_{C'}(u) = (n-2)/2$ and the following holds:

$$\forall i \in \{0, 1, \ldots, n-3\}\colon e(u, w_i) + e(u, w_{i+k-2}) = 1. \tag{4.1}$$

We also note that in order to avoid the cycle $xw_i{C'}^{+}w_{i+k-3}ux$ of length k, for every i with $0 \leq i \leq n-3$ the following inequality holds:

$$e(x, w_i) + e(u, w_{i+k-3}) \leq 1. \tag{4.2}$$

Now we examine relations between the vertices u and x and the vertices of the cycle C'.

Claim 4.28. *Let l be an integer satisfying $1 \leq l \leq k-4$. If w_i is a neighbour of x in $V(C')$, then*

(i) $xw_{i-l} \notin E(G)$,
(ii) $uw_{i-l} \in E(G)$,
(iii) w_{i-l} *is not heavy in* G,
(iv) $w_{i-l}w_{i+1} \in E(G)$.

Proof. The proof is by induction on l. Clearly, $xw_{i-1}, xw_{i+1} \notin E(G)$ to avoid hamiltonian cycle in G'. Since x is adjacent to w_i, it follows from (4.2) that $uw_{i+k-3} \notin E(G)$. Thus, by (4.1), u is adjacent to w_{i-1}. Note that $uxw_i{C'}^{+}w_{i-1}u$ is a hamiltonian cycle in G. Since u is heavy, Claim 4.14 implies that w_{i-1} is not heavy. To prove (iv) observe that if $w_{i-1}w_{i+1}$ is not an edge in G, then $\{w_i; w_{i-1}, x, w_{i+1}\}$ induces a claw. Since neither w_{i-1} nor x, by Claim 4.26, is heavy, this contradicts G being a member of the family $\mathcal{F}(K_{1,3}, n)$.

Assume that the Claim holds for the values from the set $\{1, 2, \ldots, l\}$ with l satisfying $l < k-4$. We will show that this implies the validity of the claim for $l+1$.

Suppose $xw_{i-l-1} \in E(G)$. Then, by the condition (iv) for l, there is a hamiltonian cycle in G', namely $xw_{i-l-1}{C'}^{-}w_{i+1}w_{i-l}{C'}^{+}w_ix$. This contradiction proves (i).

By the conditions (i) and (ii) the vertex w_{i-l} is adjacent to u and not adjacent to x. Thus $uw_{i-l-1} \in E(G)$ to avoid induced path $xuw_{i-l}w_{i-l-1}$ with neither x nor w_{i-l} being heavy. This proves (ii). Now, since $uw_{i-l-1} \in E(G)$ and, by (iv), w_{i-l} is adjacent to w_{i+1}, the cycle $uw_{i-l-1}C'^{-}w_{i+1}w_{i-l}C'^{+}w_ixu$ is a hamiltonian cycle in G. Since u is heavy, Claim 4.14 implies that w_{i-l-1} is not heavy.

For the proof of (iv) suppose that w_{i-l-1} is not adjacent to w_{i+1}. Note that $uw_{i+1} \in E(G)$ to avoid induced path $xuw_{i-1}w_{i+1}$ with neither x nor w_{i-1} being heavy in G. But this implies that $\{u; x, w_{i-l-1}, w_{i+1}\}$ induces a claw in G. Since none of the vertices x and w_{i-l-1} is heavy, this contradicts G belonging to the family $\mathcal{F}(K_{1,3}, n)$. By mathematical induction the Claim is true. □

Since G is 2-connected, there is a vertex $w_i \in V(C')$ adjacent to x. From Claim 4.28 it follows that $uxw_iw_{i+1}w_{i-1}C'^{-}w_{i-k+4}u$ is a cycle in G. Since the length of this cycle is k, this contradicts the choice of k. This final contradiction completes the proof.

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